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YAO-ZHONG HU

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On the differentiability of functions of an operator

by YaoZhong HU

Introduction. Let f be a continuous function on $\mathbb{R}$. Then it is well known how to define $f(A)$ when A is a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, using the spectral decomposition of A. But if A, H are two non-commuting self-adjoint operators, no explicit computation of $f(A+tH)$ is known. Our problem here is to study the regularity of $f(A+tH)$ under some regularity assumptions on f. We will assume that $\mathcal{H}$ is finite-dimensional. This note is a complement to our article "Some operator inequalities" in volume XXVIII[1], and answers a question of P.A. Meyer.

Notation. Given real numbers $\lambda_i \neq \lambda_j$, we put (divided differences : see the first pages of Donoghue [1] for more detail)

$$\{\lambda_2, \lambda_1\}f = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}$$
$$\{\lambda_{k+1}, \dots, \lambda_1\}f = \{\lambda_{k+1}, \lambda_k\}\{\cdot, \lambda_{k-1} \dots, \lambda_1\}f$$

LEMMA. *Let f have continuous derivatives up to of order $n+1$. Let $R_{n+1}(a,b)$ be the corresponding Taylor remainder*

$$R_{n+1}(a,b)\, f = f(b) - f(a) - (b-a)f'(a) - \tfrac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(a)\ .$$

Then $\{\cdot, \lambda\}f$ has derivatives up to order n, and we have

$$\frac{d^n}{dt^n}\{t, \lambda\}f = \frac{n!}{(\lambda - x)^{n+1}} R_{n+1}(\lambda, x)\, f\ .$$

One can deduce that, if f is of class $\mathcal{C}^k$, the function $\{\lambda_{k+1}, \dots, \lambda_1\}f$ can be extended by continuity to $\mathbb{R}^{k+1}$ including the diagonals, and that

$$|\{\lambda_{k+1}, \dots, \lambda_1\}f| \leq \gamma(k)\, \| f \|_{k;T}$$

where the last norm is the $\mathcal{C}^k$ norm of f on any interval T containing $\lambda_1, \dots \lambda_{k+1}$. Then many results proved for polynomials $f(t) = t^d$ can be extended by density to $\mathcal{C}^k$ functions. In particular, the divided differences are symmetric in all their arguments from the following lemma, which is the crucial point of the calculation.

LEMMA. *When $f(t) = t^d$, we have*

$$\{\lambda_k, \dots, \lambda_1\}f = \sum_{m_k + \dots + m_1 = d-k+1} \lambda_k^{m_k} \dots \lambda_1^{m_1}\ .$$

PROOF. By induction on k.

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[1] La rédaction du Séminaire regrette ce retard de publication, dû à une erreur de transmission du manuscrit.

Computation of derivatives. When f is a polynomial, the operator function $f(A)$ is infinitely differentiable at every A and we can write its partial derivatives

$$\frac{\partial^k}{\partial t_1 \dots \partial t_k} f(A + t_1 H^1 + \dots + t_k H^k)\big|_{t_1 = \dots = t_k = 0} = \Phi(A\,;\, H^1, \dots, H^k)$$

where $\Phi(A\,;\,\cdot)$ is a symmetric k-linear functional. The problem is to give a uniform estimate of these derivatives knowing the $\mathcal{C}^k$ norm of f, which will allow us to extend the result from polynomials to $\mathcal{C}^k$ functions. Since Φ arises from polarization of the function $\Phi(H, \dots, H)$, it will be sufficient to estimate this function.

When $f(t) = t^d$ we have

$$\frac{d^k}{dt^k}(A + tH)^d\big|_{t=0} = \sum_{m_1 + \dots + m_{k+1} = d-k} A^{m_1} H \dots H A^{m_{k+1}}$$

Choose a basis in which A is diagonal with eigenvalues λ_i. Then the matrix of this operator D is

$$D_{ij} = \sum_{u_1, \dots, u_{k+1}} \delta_{iu_1} h_{u_1 u_2} \dots h_{u_k u_{k+1}} \delta_{u_{k+1} j} \sum_{m_1 + \dots + m_{k+1} = d-k} \lambda_{u_1}^{m_1} \dots \lambda_{u_{k+1}}^{m_{k+1}}$$

and this last coefficient is $\{\lambda_{u_{k+1}}, \dots, \lambda_{u_1}\} f$, in which the explicit form of f no longer appears. It follows that, if f is a polynomial, whenever the spectrum of A is contained in some interval T, we have a domination in Hilbert-Schmidt norm

$$\| \frac{d^k}{dt^k} f(A + tH)\big|_{t=0} \|_{HS} \leq C \| f \|_{k,T} \, \| H^k \|_{HS}$$

This is no longer basis dependent, and shows that, approximating locally a $\mathcal{C}^k$ function by polynomials in $\mathcal{C}^k$ norm, the function $f(A+tH)$ is k-times continuously differentiable in t.

This reasoning suggests that in infinite dimensions $f(A)$ is differentiable at A bounded, but along Hilbert-Schmidt directions.

REFERENCES

[1] DONOGHUE (W.F.). *Monotone Matrix Functions and Analytic Continuation,* Springer (Grundlehren 207), 1974.

[2] NÖRLUND (N.). Vorlesungen über Differenzenrechnung, Berlin 1924.

Institute of Math. Science, Academia Sinica,
Wuhan 430071, China
and
University of Oslo, POB 1053 Blindern N-0316 Oslo