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From an Example of Lévy's

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The motive of this paper is to prove completely an assertion of P. Lévy [3], who claimed that for each positive integer n , there exists a polynomial F_n of degree n such that the Wiener integral with respect to Brownian motion $\{B(u); 0 \leq u\}$

$$X(t) = \int_0^t F_n \left(\frac{u}{t} \right) dB(u)$$

is again a Brownian motion and

$$\mathbf{B}(X; t) \subsetneq \mathbf{B}(B; t).$$

Here $\mathbf{B}(X; t)$ is the σ -field generated by $\{X(s); s \leq t\}$.

Over the last few years, the non-canonical representation of Brownian motion of this kind related has been of interest to many authors, in particular Th. Jeulin & M. Yor [4,5,6,7] and M. Hitsuda [2].

Let $X_0(t) = B(t), (t \geq 0)$ be a standard Brownian motion. In this paper, our precise purpose is to construct a sequence of Brownian motion $\{X_n(t); t \geq 0\} (n \geq 0)$ such that $X_n(t)$ can be represented as a Wiener integral

$$X_n(t) = \int_0^t F_n\left(\frac{u}{t}\right) dB(u), \quad (n \geq 1).$$

Here $F_n(t)$ is a polynomial of degree n in t . And if $\mathbf{M}(X_n; t)$ is a linear span of $\{X_n(u); 0 \leq u \leq t\}$ in $L^2(\Omega, P)$, then for all $t > 0$,

$$(1) \quad \mathbf{M}(X_{n+1}; t) \subsetneq \mathbf{M}(X_n; t), \quad (n \geq 0),$$

and further, for all $t > 0$ and $n \geq 1$,

$$(2) \quad \int_0^t F_n \left(\frac{u}{t} \right) u^k du = 0, \quad k = 1, 2, \dots, n.$$

Let $F_n(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$. We know by calculating the covariance that $X_n(t)$ is a Brownian motion if and only if the coefficients of $F_n(t)$ satisfy the following equations

[illegible]

In the simplest case

$$\begin{cases} a_n = \frac{2n+1}{n}, a_0 = -\frac{n+1}{n}, \\ a_1 = a_2 = \cdots = a_{n-1} = 0. \end{cases}$$

is a solution of equation (3).

Theorem 1 If $P_n(t) = \frac{2n+1}{n}t^n - \frac{n+1}{n}$ and if $F_n(t)$ are defined by the following recursive formula

$$\begin{cases} F_1(t) = P_1(t) \\ F_n\left(\frac{u}{t}\right) = F_{n-1}\left(\frac{u}{t}\right) - \int_u^t F_{n-1}\left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} P_n\left(\frac{\tau}{t}\right) d\tau, \quad (n \geq 2) \end{cases}$$

then $F_n(t)$ satisfies (2), the coefficients of $F_n(t)$ are given by

$$a_k = (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n}, \quad k = 0, 1, \dots, n$$

and

$$X_n(t) := \int_0^t F_n\left(\frac{u}{t}\right) dB(u), \quad (n \geq 1)$$

are Brownian motions satisfying condition (1). Further, $X_n(t)$ and $X_{n+1}(t)$ are related by

$$(4) \quad X_{n+1}(t) = \int_0^t P_{n+1}\left(\frac{u}{t}\right) dX_n(u), \quad (n \geq 0).$$

In order to prove the theorem, we prepare the following lemma.

Lemma 1 If $s < n$, we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{s} = 0.$$

To prove this, we note

$$\begin{aligned} \frac{1}{s!} \left(\frac{d}{dx}\right)^s ((1+x)^n x^n) &= \frac{1}{s!} \left(\frac{d}{dx}\right)^s \left(\sum_{k=0}^n \binom{n}{k} x^{n+k}\right) \\ &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{s} x^{n+k-s}. \end{aligned}$$

The result follows by letting $x = -1$.

The validity of the coefficients of $F_n(t)$ can be established by mathematical induction. The assertion is trivial for $n = 1$. Suppose the assertion holds for n . Now using lemma 1 and then noting

$$\begin{aligned} \binom{n}{k} \binom{n+1+k}{n} \left(1 - \frac{2(n+1)+1}{n+1-k}\right) &= -\binom{n+1}{k} \binom{n+2+k}{n+1}, \\ \binom{n}{k} \binom{n+1+k}{n} \frac{2(n+1)+1}{n+1-k} &= \frac{2(n+1)+1}{n+1} \binom{n+1}{k} \binom{n+1+k}{n}, \end{aligned}$$

we see that

$$\begin{aligned} F_{n+1}\left(\frac{u}{t}\right) &= F_n\left(\frac{u}{t}\right) - \int_u^t F_n\left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} P_{n+1}\left(\frac{\tau}{t}\right) d\tau \\ &= \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n} \left(1 - \frac{2(n+1)+1}{n+1-k}\right) \left(\frac{u}{t}\right)^k \\ &\quad + \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n} \frac{2(n+1)+1}{n+1-k} \left(\frac{u}{t}\right)^{n+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^n (-1)^{n+1+k} \binom{n+1}{k} \binom{n+2+k}{n+1} \left(\frac{u}{t}\right)^k + \binom{2n+3}{n+1} \left(\frac{u}{t}\right)^{n+1} \\
&= \sum_{k=0}^{n+1} (-1)^{n+1+k} \binom{n+1}{k} \binom{n+2+k}{n+1} \left(\frac{u}{t}\right)^k.
\end{aligned}$$

which shows the assertion holds for coefficients of $F_{n+1}(t)$.

We next show (2). By the recursive formula of F_n , we obtain

$$\begin{aligned}
\int_0^t F_n\left(\frac{u}{t}\right) u^k du &= \int_0^t F_{n-1}\left(\frac{u}{t}\right) u^k du - \int_0^t \int_u^t F_{n-1}\left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} P_n\left(\frac{\tau}{t}\right) u^k d\tau du \\
&= \int_0^t F_{n-1}\left(\frac{u}{t}\right) u^k du - \int_0^t d\tau \frac{\partial}{\partial \tau} P_n\left(\frac{\tau}{t}\right) \int_0^\tau F_{n-1}\left(\frac{u}{\tau}\right) u^k du
\end{aligned}$$

This equals zero if $k < n$ by induction; and when $k = n$, this becomes

$$\begin{aligned}
&\int_0^t F_{n-1}\left(\frac{u}{t}\right) u^n du - \left[P_n\left(\frac{\tau}{t}\right) \int_0^\tau F_{n-1}\left(\frac{u}{\tau}\right) u^n du \right]_{\tau=0}^t - \int_0^t P_n\left(\frac{u}{t}\right) u^n du \\
&= \int_0^t P_n\left(\frac{u}{t}\right) u^n du = 0.
\end{aligned}$$

which is what we needed to prove.

Again we easily verify, by mathematical induction, that

$$\int_0^1 F_n(u) du = \frac{(-1)^n}{n+1}.$$

Thus we have proved, in combination with the previous equation, that the coefficients of F_n are another solution to equation (3).

Now if we write

$$X_n(t) = \int_0^t F_n\left(\frac{u}{t}\right) dB(u),$$

then by the above argument, $X_n(t)$ is again a Brownian motion. The differential of $X_n(t)$, by Itô's formula [1], is seen to be

$$dX_n(u) = dB(u) + \int_0^u \frac{\partial}{\partial u} F_n\left(\frac{\tau}{u}\right) dB(\tau) du.$$

Therefore

$$\begin{aligned}
&\int_0^t P_{n+1}\left(\frac{u}{t}\right) dX_n(u) \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right) dB(u) + \int_0^t \left\{ P_{n+1}\left(\frac{u}{t}\right) \int_0^u \frac{\partial}{\partial u} F_n\left(\frac{\tau}{u}\right) dB(\tau) \right\} du \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right) dB(u) + \int_0^t dB(\tau) \int_\tau^t P_{n+1}\left(\frac{u}{t}\right) \frac{\partial}{\partial u} F_n\left(\frac{\tau}{u}\right) du \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right) dB(u) + \\
&\quad \int_0^t dB(\tau) \left\{ F_n\left(\frac{\tau}{u}\right) P_{n+1}\left(\frac{u}{t}\right) \Big|_{u=\tau}^t - \int_\tau^t F_n\left(\frac{\tau}{u}\right) \frac{\partial}{\partial u} P_{n+1}\left(\frac{u}{t}\right) du \right\} \\
&= \int_0^t F_n\left(\frac{u}{t}\right) dB(u) - \int_0^t \left\{ \int_u^t F_n\left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} P_{n+1}\left(\frac{\tau}{t}\right) d\tau \right\} dB(u) \\
&= X_{n+1}(t)
\end{aligned}$$

This establishes (4).

To show (1), let us fix $t_0 > 0$ and let $z = \int_0^{t_0} u^{n+1} dX_n(u)$. Now $z \in \mathbf{M}(X_n; t)$ and note that for all t such that $0 < t \leq t_0$,

$$E[X_{n+1}(t) \cdot z] = \int_0^t P_{n+1}\left(\frac{u}{t}\right) u^{n+1} du = 0.$$

This verifies (1). The proof of theorem is thus completed.

Remark 1 *This construction was suggested by P. Lévy in his book [3] and $F_1(t)$ was given there.*

Remark 2 *Although we have (1), for all $n > 0$, we notice,*

$$(5) \quad \mathbf{M}(B; \infty) = \mathbf{M}(X_n; \infty).$$

This equation has the following interpretation. For each finite time t , as we have already seen, $\mathbf{B}(X_n; t)$ contains less information than $\mathbf{B}(B; t)$. Nevertheless, $\mathbf{B}(X_n; t)$ will “catch up” with $\mathbf{B}(B; t)$ by increasing time to infinity.

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References

- [1] K. Itô, Introduction to Probability Theory. vol.3. (in Japanese) Iwanami-Shoten, (1978).
- [2] M. Hitsuda, Gaussian innovations and integral operators. Proceedings of the Sixth USSR-Japan Symposium. Kiev, Aug. 5-10, 1991. World Scientific.
- [3] P. Lévy, *Processus Stochastiques et mouvement brownien*. Gauthier-Villars. (1965), p. 299–304.
- [4] Th. Jeulin, M. Yor, Filtration des ponts browniens et équations différentielles linéaires. Séminaire de Probabilités XXIV. Lect. Notes in Maths. 1426. Springer (1990), p. 227–265.
- [5] Th. Jeulin, M. Yor, Une décomposition non-canonique du drap brownien. Séminaire de Probabilités XXVI. Lect. Notes in Maths. 1526. Springer (1992), p. 322–347.
- [6] Th. Jeulin, M. Yor, Moyennes mobiles et semimartingales. Séminaire de Probabilités XXVII. Lect. Notes in Maths. 1557. Springer (1993), p. 53–77.
- [7] Th. Jeulin, M. Yor, Sur certaines décompositions non-canoniques des diffusions réelles. Manuscrit non publié (Décembre 1990).