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JUMPING FILTRATIONS AND MARTINGALES WITH FINITE VARIATION

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ABSTRACT: On a probability space $(\Omega,\mathcal{F},P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a *jumping filtration* if there is a sequence (T_n) of stopping times increasing to $+\infty$, such that on each set $\{T_n\leq t<T_{n+1}\}$ the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide up to null sets. The main result is that $(\mathcal{F}_t)$ is a jumping filtration iff all martingales have a.s. locally finite variation.

1 - INTRODUCTION

Let $(\Omega,\mathcal{F},P)$ be a probability space. By definition, a (right-continuous) filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a *jumping filtration* if there exists a *localizing sequence* $(T_n)_{n\in\mathbb{N}}$ (i.e. a sequence of stopping times increasing a.s. to $+\infty$) with $T_0=0$ and such that for all $n\in\mathbb{N}$, $t>0$:

$$\text{the } \sigma\text{-fields } \mathcal{F}_t \text{ and } \mathcal{F}_{T_n} \text{ coincide up to null sets on } \{T_n\leq t<T_{n+1}\}. \tag{1}$$

The sequence (T_n) is then called a *jumping sequence*. Note that it is by no means unique. Our aim is to prove the

THEOREM 1: *A filtration is a jumping filtration iff all its martingales are a.s. of locally finite variation* (here and throughout the paper, martingales are supposed to be càdlàg).

The necessary condition is easy (see Section 2) and not surprising, in view of the following known fact: consider a marked point process, that is an increasing sequence (T_n) of times, and associated marks X_n taking values in some measurable space $(E,\mathcal{E})$, and suppose that $T_n\uparrow\infty$ a.s. Let $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by some initial σ-field $\mathcal{G}$ and the marked point process (i.e. the smallest filtration such that $\mathcal{G} \subseteq \mathcal{F}_0$ and each T_n is a stopping time and X_n is $\mathcal{F}_{T_n}$-measurable). Then one knows (see [1]) that $(\mathcal{F}_t)$ is a jumping filtration with jump times (T_n), and if further $(E,\mathcal{E})$ is a Blackwell space all martingales have a.s. locally finite variation.

In fact, *any* jumping filtration is generated by some marked point process, with a "very large" set of marks: take times T_n as in (1), and $(E,\mathcal{E}) = \Pi_{n\in\mathbb{N}}(E_n,\mathcal{E}_n)$ where $E_n=\Omega\cup\{\Delta\}$ (Δ is an extra point) and $\mathcal{E}_n$ is the σ-field of E_n generated by $\mathcal{F}_{T_n}$, and $X_n(\omega)$ is the point of E with coordinates Δ, except the n^{th} coordinate which is ω.

So Theorem 1 implies that if all martingales are a.s. of locally finite variation, the filtration is indeed generated by an initial σ-field $\mathcal{F}_0$ and a marked point process.

When the filtration is quasi-left continuous, the sufficient condition is relatively simple to prove, and some additional results are available: this is done in Section 3. The general case needs a systematic use of stochastic integrals w.r.t. random measures: some auxiliary results about these are gathered in Section 4, and the proof is given in Section 5.

2 - THE NECESSARY CONDITION

Assume here that $(\mathcal{F}_t)$ is a jumping filtration, with jumping sequence (T_n). For the necessary part of Theorem 1 it suffices to prove that a uniformly integrable martingale M which is 0 on $[0,T_n]$ and constant on $[T_{n+1},\infty)$ for some n is a.s. of locally finite variation.

Set $T=T_n$ and $S=T_{n+1}$, and call G a regular version of the law of the pair (S,M_S), conditional on $\mathcal{F}_T$. By hypothesis, for each t there is an $\mathcal{F}_T$-measurable variable N_t such that $M_t = N_t$ a.s. on $\{T\le t<S\}$. We have the following string of a.s. equalities (the third one comes from the martingale property; further $(u,x) \to |x|$ is G-integrable for a.a. ω, because M is uniformly integrable, and $G'(t) = G((t,\infty]\times\mathbb{R})$):

$$N_t G'(t) 1_{\{T\le t\}} = E[N_t 1_{\{T\le t<S\}} | \mathcal{F}_T] = E[M_t 1_{\{T\le t<S\}} | \mathcal{F}_T]$$

$$= E[M_S 1_{\{T\le t<S\}} | \mathcal{F}_T] = 1_{\{T\le t\}} \int G(du,dx) \times 1_{\{u>t\}} \qquad (2)$$

The right-hand side of (2), which we denote by A_t, is a.s. càdlàg with locally finite variation, as a function of t; further, the left-hand side of (2) is a.s. equal to $M_tG'(t)$ on the set $\{T\le t<S\}$, so outside a null set we have $M_tG'(t) = A_t$ for all t with $t<S$. Since $M_t=M_S$ for $t\ge S$ and G' is non-increasing, it follows that $t \to M_t$ is a.s. of locally finite variation if $S=\infty$ or if $G'(S)>0$ or if $G'(S)=0$ and $G'(S-)>0$; by definition of G', at least one of these properties holds, hence the result.

3 - THE QUASI-LEFT CONTINUOUS CASE

Recall that the filtration is called *quasi-left continuous* if $\mathcal{F}_{T-}=\mathcal{F}_T$ (up to null sets) for all predictable times T, or equivalently if all martingales are quasi-left continuous. In this case, the proof of the sufficient condition in Theorem 1 is simple, and provides additional information about the existence of a minimal jumping sequence. More precisely, we have:

THEOREM 2: a) *If the filtration* $(\mathcal{F}_t)$ *is quasi-left continuous and all martingales are a.s. of locally finite variation, then* $(\mathcal{F}_t)$ *is a jumping filtration. Furthermore there is a jumping sequence* $(T_n)_{n\in\mathbb{N}}$ *such that*

(i) T_n *is totally inaccessible when* $n\ge1$ *and* $T_n<T_{n+1}$ *if* $T_n<\infty$.

(ii) *Every totally inaccessible time* T *satisfies* $[\![T]\!] \subseteq \cup_{n\ge1}[\![T_n]\!]$ *a.s.*

(iii) *Any other jumping sequence* (T'_n) *satisfies* $\cup[\![T'_n]\!] \supseteq \cup[\![T_n]\!]$ *a.s.*

(iv) *Local martingales jump only at the times* T_n.

b) *If* $(\mathcal{F}_t)$ *is a jumping filtration, with a jumping sequence consisting in totally inaccessible times, then the filtration is quasi-left continuous.*

(iii) means that (T_n) is the unique minimal jumping sequence, while (ii) means that it is the "maximal" sequence of totally inaccessible times.

Proof. We first suppose all the assumptions in (a).

α) Let $\mathcal{I}$ denote the class of all totally inaccessible times. We prove first that for any sequence $(S_n)_{n\ge1}$ in $\mathcal{I}$ and any $q\in\mathbb{N}$, we have

$$\text{the random set } U = [\![0,q]\!]\cap(\cup_{n\ge1}[\![S_n]\!]) \text{ is a.s. finite.} \tag{3}$$

Set $V = \{\omega$: there are infinitely many s with $(\omega,s)\in U\}$. Suppose that (3) fails, that is $\varepsilon := P(V)/2 > 0$. Call $\pi(A)$ the projection of a subset A of $\Omega\times\mathbb{R}_+$ on Ω. Define by induction optional subsets U_n of U and stopping times $T_n\in\mathcal{I}$ as such: set $U_1=U$; then if U_n is known the optional section theorem yields a stopping time T_n such that $[\![T_n]\!]\subseteq U_n$ (hence $T_n\in\mathcal{I}$) and $P(\pi(U_n)\cap\{T_n=\infty\}) \le \varepsilon 2^{-n}$; then set $U_{n+1} = U_n\setminus[\![T_n]\!]$. Clearly $V\subseteq\pi(U_n)$ for all n, hence $P(V\cap\{T_n=\infty\}) \le \varepsilon 2^{-n}$ and thus $A := \cap\{T_n<\infty\}$ satisfies $P(V\setminus A) \le \varepsilon$ and $P(A)\ge\varepsilon>0$.

Now call M^n the purely discontinuous martingale having a jump of size +1 at time T_n if $T_n<\infty$, and which is continuous elsewhere. The bracket of M^n is $[M^n,M^n]_t = 1_{\{T_n\le t\}}$, and the M^n's are pairwise orthogonal because they have no common jumps. Then the series $\sum \frac{1}{n} M^n$ converges in L^2 to a square-integrable martingale whose variation on $[0,q]$ is bigger than $\sum \frac{1}{n} 1_{\{T_n\le q\}}$. In particular this variation is infinite on the set A (since $T_n<\infty \Rightarrow T_n\le q$), so $P(A)=0$ by hypothesis, hence a contradiction and (3) is proved.

β) Next we construct the sequence $(T_n)_{n\in\mathbb{N}}$ by induction. Set $T_0=0$. Suppose that T_n is known, and call $\mathcal{I}_n$ the (non-empty) set of all $T\in\mathcal{I}$ with $T\ge T_n$, and $T>T_n$ if $T_n<\infty$. Then define T_{n+1} to be the essential infimum of all T in $\mathcal{I}_n$. Since $S,S'\in\mathcal{I}_n \Rightarrow S\wedge S'\in\mathcal{I}_n$, there is a decreasing sequence $(S_p)_{p\ge1}$ in $\mathcal{I}_n$, with limit T_{n+1}. In view of (3), we must have $S_p=T_{n+1}$ for p large enough (depending on ω), a.s.: hence $T_{n+1}\in\mathcal{I}_n$ and $T_{n+1}>T_n$ if $T_n<\infty$, and we have (i).

Using (3) once more, we get $\lim_n T = +\infty$ a.s. Since any $T\in\mathcal{I}$ has $T\ge T_{n+1}$ on the set $\{T>T_n\}$ by the definition of T_{n+1}, we have (ii). All local martingale having only totally inaccessible jumps, (iv) follows from (ii).

γ) Next we prove that (T_n) is a jumping sequence. Let $n\in\mathbb{N}$, $t\ge0$ and $A\in\mathcal{F}_t$, and set $T=T_n$, $S=T_{n+1}$. We consider the martingale $N^A_s = P(A\cap\{T\le t<S\}\,|\,\mathcal{F}_s)$, and also the point process $X_s = 1_{\{S\le s\}}$ with its compensator Y. Since $A\cap\{T\le t<S\}\in\mathcal{F}_{S\wedge t}$, we have

$$N^A_s = N^A_{s\wedge S\wedge t}, \qquad N^A_S = 1_B. \tag{4}$$

Then $M_s = N^A_s - N^A_{s\wedge T}$ is null on $[0,T]$ and constant on $[S\wedge t,\infty)$, and so by (iv) has only one jump at time S, which is $\Delta M_S = \Delta N^A_S 1_{\{S\le t\}} = -N^A_{S-}1_{\{t\le S\}}$. Thus, with the predictable process $H_u = -N^A_{u-}\,1_{\{u\ge t\}}$, we obtain $M = X'-Y'$ with $X'_s = \int_0^s H_u dX_u$ and $Y'_s = \int_0^s H_u dY_u$ being the compensator of X'. Hence

$$N^A_s = N^A_T + \int_0^{s\wedge t} N^A_{u-}\, dY_u \qquad \text{a.s. if } T\le s<S. \tag{5}$$

Now, observing that $Y_s=0$ for $s\le T$ and with $\mathcal{E}(Y)$ denoting the Doléans exponential of Y, we deduce $N_s^A = N_T^A\ \mathcal{E}(Y)_{s\wedge t}$ if $T\le s<S$. Similarly $N_t^\Omega = N_T^\Omega\ \mathcal{E}(Y)_{s\wedge t}$ if $T\le s<S$, hence

$$N_t^A\ N_T^\Omega \ =\ N_t^\Omega\ N_T^A \quad \text{a.s. on } \{T\le t<S\}. \tag{6}$$

Note that $A' = \{N_T^A=N_T^\Omega>0\}$ is $\mathcal{F}_T$-measurable, and $N_t^A = 1_{A\cap\{T\le t<S\}}$ and $N_t^\Omega = 1_{\{T\le t<S\}}$: we readily deduce $A\cap\{T\le t<S\} = A'\cap\{T\le t<S\}$ a.s., hence (1).

δ) Now we prove (iii). Let (T'_n) be another jumping sequence. If (iii) were not true, there would exist a pair n,p of integers such that $P(T'_n<T_p<T'_{n+1})>0$. According to a (trivial) extension of Proposition (3.40) of [1], there is a $(0,\infty]$-valued $\mathcal{F}_{T'_n}$-measurable variable R such that $S = T'_n+R$ has $S=T_p<\infty$ on the set $\{T'_n<T_p<T'_{n+1}\}$. Now, S is clearly a predictable time, and $P(T_p=S<\infty) > 0$ contradicts the property $T_p\in\mathcal{I}$. ∎

ε) It remains to prove (b). So now we assume that $(\mathcal{F}_t)$ is a jumping filtration, with a jumping sequence (T_n) having $T_n\in\mathcal{I}$ for $n\ge1$. It is enough to show that if M is a bounded martingale and T is a finite predictable time, then $\Delta M_T=0$ a.s. on each set $A = \{T_n<T<T_{n+1}\}$. (1) implies $\mathcal{F}_T\cap A = \mathcal{F}_{T-}\cap A = \mathcal{F}_{T_n}\cap A$ up to null sets. Further, $P(B\backslash A)=0$ if $B = \{T_n<T\le T_{n+1}\}$, so $\mathcal{F}_T\cap B = \mathcal{F}_{T-}\cap B$ up to null sets as well. But $B\in\mathcal{F}_{T-}$, hence $\Delta M_T 1_B$ is measurable w.r.t. the completion of $\mathcal{F}_{T-}$, and $\Delta M_T 1_B = 1_B E(\Delta M_T|\mathcal{F}_{T-})$ a.s. Since $E(\Delta M_T|\mathcal{F}_{T-}) = 0$ a.s. (M is a martingale and T is predictable), we obtain $\Delta M_T=0$ a.s. on B. ∎

4 - RANDOM MEASURES AND MARTINGALES WITH FINITE VARIATION

1) Let us begin with two auxiliary results, which are more or less known. We consider two measurable spaces $(G,\mathcal{G})$ and $(H,\mathcal{H})$, and a positive transition measure $\eta(x;dy)$ from $(G,\mathcal{G})$ into $(H,\mathcal{H})$. The first lemma concerns the atoms of maximal mass of $\eta(x,.)$:

LEMMA 3: *Assume that* $(H,\mathcal{H})$ *is a Polish space with its Borel σ-field and that* $\eta(x,H)\le1$ *for all* $x\in G$. *Then if* $\alpha(x) = \sup(\nu(x,\{y\}): y\in H)$:

a) α *is* $\mathcal{G}$*-measurable.*

b) *There is a measurable function* ζ: $(G,\mathcal{G}) \to (H,\mathcal{H})$ *such that* $\alpha(x) = \eta(x,\{\zeta(x)\})$.

c) *There is a* $\mathcal{G}\otimes\mathcal{H}$*-measurable set* B *such that* $\frac{1}{2} \le \int\eta(x,dy)1_B(x,y) \le \frac{3}{4}$ *if* $\alpha(x)\le\frac{1}{4}$ *and* $\eta(x,E)>\frac{3}{4}$.

Proof. There is a bi-measurable bijection φ from H into a Borel subset H' of $[0,1)$ containing 0, and we set $\eta'(x,A) = \eta(x,\varphi^{-1}(A\cap H'))$ for every Borel subset of $[0,1)$. Then y' is an atom of $\eta'(x,.)$ iff $y'=\varphi(y)$ where y is an atom of $\eta(x,.)$, with the same mass. Set $A(n,m)=[m2^{-n},(m+1)2^{-n})$.

a) The functions $f_n(x) = \sup(\eta'(x,A(n,m)): 0\le m\le 2^n-1)$ are measurable and decreases to $\alpha(x)$, hence the result.

b) Set $M_n(x) = \inf(m: \eta'(x,A(n,m))\ge\alpha(x))$. If $\alpha(x)=0$ then $M_n(x)=0$ and $A(n,M_n(x))$ decreases to $\{0\}$. If $\alpha(x)>0$, for all n large enough we have for all m: either $A(n,m)$ contains exactly one atom of $\eta'(x,.)$ of mass $\alpha(x)$, or it contains no such atom and $\eta'(n,A(n,m))<\alpha(x)$; thus for n large enough we have $A(n+1,M_{n+1}(x))\subseteq A(n,M_n(x))$. Hence for all x the sequence $A(n,M_n(x))$ converges as $n\to\infty$ to a singleton, say $\{\zeta'(x)\}$, with $\zeta'(x)=0$ if $\alpha(x)=0\in H'$ and $\zeta'(x)\in H'$ otherwise (because $\eta'(x,\{\zeta'(x)\})=\alpha(x)$). Then $\zeta = \zeta'\circ\varphi^{-1}$ satisfies the requirements.

c) Set $U(x) = 1\wedge\inf(t\ge0:\eta'(x,[0,t])\ge1/2)$. If $\alpha(x)\le1/4$ and $\eta'(x,[0,1]) = \eta(x,H) > 3/4$ we have $1/2 \le \eta'(x,[0,U(x)]) \le 3/4$. Then $B = \{(x,y): \varphi(y)\in[0,U(x)]\}$ answers the question. ∎

The second lemma is a variation on the fact that if $L^2(\mu)\subseteq L^1(\mu)$ for a measure μ, then μ is of finite total mass, and it results from discussions with J. Azéma and Ph. Biane.

LEMMA 4: *Assume that there is a* $\mathcal{G}\otimes\mathcal{H}$*-measurable partition* $(F_n)_{n\ge1}$ *of* $G\times H$ *such that* $\int\eta(x,dy)1_{F_n}(x,y) \le 1$ *for all* n. *There is a* $\mathcal{G}\otimes\mathcal{F}$*-measurable function* U *with* $0<U\le1$ *and* $\int\eta(x,dy)U^2(x,y) \le 1$ *for all* $x\in G$, *and*

$$\int\eta(x,dy)U(x,y) = \infty \iff \eta(x,F) = \infty. \tag{7}$$

Proof. We define by induction the sequence $\gamma_n(x)$, with $\gamma_0(x)=0$ and

$$\gamma_{n+1}(x) = \inf(m: \textstyle\sum_{i:\gamma_n(x)<i\le m} \int\eta(x,dy)1_{F_i}(x,y) \ge 1).$$

Set $N(x) = \inf(n: \gamma_n(x)=\infty)$ and $K_n = \cup_{i\ge1}[F_i\cap\{(x,y):\gamma_{n-1}(x)<i\le\gamma_n(x)\}]$ if $n\ge1$. Finally set $\delta = \sum_{n\ge1} n^{-2}$ and $U = (1\vee2\delta)^{-1/2} \sum_{n\ge1} \frac{1}{n} 1_{K_n}$.

The K_n's constitute a measurable partition of $G\times F$, hence U is measurable and $0<U\le 1$. By construction $\int\eta(x,dy)1_{K_n}(x,y) \le 2$, so $\int\eta(x,dy)U^2(x,y) \le 1$. Further the integral $\int\eta(x,dy)1_{K_n}(x,y)$ is bigger than 1 if $n<N(x)$ and null if $n>N(x)$, while $\eta(x,F)=\infty \Leftrightarrow N(x)=\infty$, so (7) follows. ■

2) Now we turn to random measures. We fix a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, and $\mathcal{P}$ denotes the predictable σ-field on $\Omega\times\mathbb{R}_+$. Let E be a Polish space with its Borel σ-field $\mathcal{E}$, and $\tilde{\Omega}=\Omega\times\mathbb{R}_+\times E$, and $\tilde{\mathcal{P}}=\mathcal{P}\otimes\mathcal{E}$. An *integer-valued random measure* is a random measure μ on $\mathbb{R}_+\times E$ of the form

$$\mu(\omega;dt,dx) = \sum_{s>0,\gamma_s(\omega)\in E} \varepsilon_{(s,\gamma_s(\omega))}(dt,dx), \tag{8}$$

where γ is an optional process taking values in $E\cup\{\Delta\}$, and for which there is a $\tilde{\mathcal{P}}$-measurable partition (G_n) of $\tilde{\Omega}$ with $E[\sum_{s>0} 1_{G_n}(.,s,\gamma_s)] < \infty$. It is known that there is such a partition with $E[\sum_{s>0} 1_{G_n}(.,s,\gamma_s)] \le 1$ for all n.

We denote by ν the (predictable) *compensator* of μ, and we use all notation of [1], Chapter III: in particular if W is a $\tilde{\mathcal{P}}$-measurable function on $\tilde{\Omega}$ we set (with $+\infty$ whenever an integral is not well defined):

$$\hat{W}_t(\omega) = \int_E W(\omega,t,x)\nu(\omega,\{t\},dx), \qquad a_t = \hat{1}_t = \nu(.,\{t\}\times E), \tag{9}$$

$$W*\mu_t = \int_{[0,t]\times E} W(.,s,x)\mu(.;ds,dx), \text{ and similarly fo } W*\nu, \tag{10}$$

$$\left.\begin{aligned} C^{\infty}(W)_t &= (W-\hat{W})^2*\nu_t + \sum_{s\le t}(1-a_s)(\hat{W}_s)^2, \\ C^{0}(W)_t &= |W-\hat{W}|*\nu_t + \sum_{s\le t}(1-a_s)|\hat{W}_s|. \end{aligned}\right\} \tag{11}$$

Recall that one may define the stochastic integral process $W*(\mu-\nu)$ of W w.r.t. $\mu-\nu$ (for a $\tilde{\mathcal{P}}$-measurable W) iff one may write $W=W'+W''$, with W', W'' $\tilde{\mathcal{P}}$-measurable and $C^{\infty}(W')_t+C^0(W'')_t<\infty$ a.s. for all $t<\infty$. Further, if $\mathcal{L}^2$ (resp. $\mathcal{L}^1$) is the set of all $\tilde{\mathcal{P}}$-measurable functions W such that $C^{\infty}(W)_{\infty}$ (resp $C^0(W)_{\infty}$) is integrable, we have ([1], Proposition (3.71)):

$$\left.\begin{aligned} W\in\mathcal{L}^2 &\Leftrightarrow W*(\mu-\nu) \text{ is a square-integrable martingale} \\ W\in\mathcal{L}^1 &\Leftrightarrow W*(\mu-\nu) \text{ has integrable variation over } \mathbb{R}_+. \end{aligned}\right\} \tag{12}$$

Finally, we also set

$$\alpha_t(\omega) = \sup_{x\in E} \nu(\omega;\{t,x\}), \qquad J = \{a>0\}, \qquad K_{\varepsilon} = \{\alpha>\varepsilon\}. \tag{13}$$

By Lemma 3, α is a predictable process, and J, K_{ε} are predictable sets.

Note also that $0\le\alpha\le a\le 1$.

Our main aim in this section is to prove the following two theorems, in which $\mathcal{L}^i_b$ denotes the set of all bounded functions in $\mathcal{L}^i$ (i=1,2), and

$$F^\varepsilon_t = 1_{(K_\varepsilon)^c}*\nu_t + \sum_{s\in K_\varepsilon, s\le t} (1-\alpha_s). \tag{14}$$

THEOREM 5: *Let* $\varepsilon\in(0,1)$. *There is equivalence between:*

a) $\mathcal{L}^2_b\subseteq\mathcal{L}^1_b$.

b) $\mathcal{L}^1_b$ *equals the set of all bounded* $\tilde{\mathcal{P}}$*-measurable functions on* $\tilde{\Omega}$.

c) $\mathcal{L}^2_b$ *equals the set of all bounded* $\tilde{\mathcal{P}}$*-measurable functions on* $\tilde{\Omega}$.

d) $E(F^\varepsilon_\infty) < \infty$.

Proof. We begin with some remarks. If $\varepsilon<\eta$ then $K_\eta\subseteq K_\varepsilon$; if further $s\in K_\varepsilon\setminus K_\eta$, then $1-\alpha_s \le a_s/\varepsilon$ and $a_s \le (1-\alpha_s)/(1-\eta)$. Hence

$$F^\varepsilon \le (1 + \frac{1}{\varepsilon})F^\eta, \qquad F^\eta \le (1 + \frac{1}{1-\eta})F^\varepsilon, \tag{15}$$

and (d) does not depend on $\varepsilon\in(0,1)$. In the rest of the proof we take $\varepsilon=1/4$ and write $K=K_{1/4}$, $F=F^{1/4}$. Next, apply Lemma 3 to $(G,\mathcal{G}) = (\Omega\times\mathbb{R}_+,\mathcal{P})$ and $(H,\mathcal{H})=(E,\mathcal{E})$, with the measure $\eta((\omega,t),dx) = \nu(\omega,\{t\}\times dx)$: we obtain a predictable E-valued process ζ such that $\alpha_t = \nu(\{t,\zeta_t\})$ and a $\tilde{\mathcal{P}}$-measurable set B such that $1/2\le\hat{1}_B\le 3/4$ when $\alpha\le 1/4$ and $a>3/4$. Then the sets $A = \{(\omega,t,\zeta_t(\omega)) : (\omega,t)\in K\}$ and $C = [(K^c\cap\{0<a\le 3/4\})\times E]\cup[B\cap((K^c\cap\{a>3/4\})\times E)]$ are $\tilde{\mathcal{P}}$-measurable and satisfy for some predictable process β:

$$\left.\begin{array}{ll} A \subseteq K\times E, & \hat{1}_A = \alpha 1_K , \\ C \subseteq (J\setminus K)\times E, & \hat{1}_C = \beta 1_{J\setminus K} \quad \text{with } \beta=a \text{ if } a\le 3/4, \ 1/2\le\beta\le 3/4 \text{ if } a>3/4. \end{array}\right\}$$

This, (14) and the definition of $K=K_{1/4}$ yield

$$F_\infty \le 1_{J^c}*\nu_\infty + 4\sum_{s\in K} \alpha_s(1-\alpha_s) + 8\sum_{s\in J\setminus K} \beta_s(1-\beta_s). \tag{16}$$

(b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (d). Consider the measure η on $(\tilde{\Omega},\tilde{\mathcal{P}})$ defined by $\eta(W) = E[W1_{J^c}*\nu_\infty]$. Lemma 4 implies the existence of a $\tilde{\mathcal{P}}$-measurable function U with $0<U\le 1$ and $\eta(U^2)<\infty$ and such that $\eta(U)<\infty$ implies $\eta(\tilde{\Omega})<\infty$. But (11) yields $E[C^\infty(U1_{J^c})_\infty] = \eta(U^2)$, so $U1_{J^c}\in\mathcal{L}^2_b$ by definition of $\mathcal{L}^2_b$, hence (a) implies

$U1_{J^c} \in \mathcal{L}^1_b$ and by (11) again $\eta(U) = E[C^0(U1_{J^c})_\infty] < \infty$. Therefore

$$E(1_{J^c} * \nu_\infty) < \infty. \tag{17}$$

Next we consider the measure η on $(\Omega\times\mathbb{R}_+,\mathcal{P})$ defined by $\eta(H) = E[\sum_{s\in K} \alpha_s(1-\alpha_s)H_s]$. If H is a predictable process, (11) yields

$$C^\infty(H1_A)_t = \sum_{s\in K, s\le t}\alpha_s(1-\alpha_s)H_s^2, \qquad C^0(H1_A)_t = \sum_{s\in K, s\le t}2\alpha_s(1-\alpha_s)|H_s|, \tag{18}$$

hence by the same argument as above, (a) and Lemma 4 imply $\eta(\Omega\times\mathbb{R}_+)<\infty$, that is

$$E[\sum_{s\in K} \alpha_s(1-\alpha_s)] < \infty. \tag{19}$$

Finally, with the measure defined by $\eta(H) = E[\sum_{s\in J\setminus K} \beta_s(1-\beta_s)H_s]$ (and $(\beta,C,J\setminus K)$ instead of (α,A,K) in (18)) one obtains similarly

$$E[\sum_{s\in J\setminus K} \beta_s(1-\beta_s)] < \infty. \tag{20}$$

Putting together (17), (19) and (20), we deduce (d) from (16).

(d) $\Rightarrow$ (b): Let W be a $\tilde{\mathcal{P}}$-measurable function bounded by a constant δ. First $|\hat{W}|\le\delta a$, and since $\sum_s a_s(1-a_s) \le F_s$ (because $1-a\le 1-\alpha$) we have

$$E[\sum_s (1-a_s)|\hat{W}_s|] < \infty. \tag{21}$$

From the definition of A, W can be written as $W = U+H1_A$ with $U = W1_{A^c}$ and H a predictable process. We have $|W-\hat{W}|\le 2\delta$ and $|\hat{U}|\le\delta\hat{1}_{A^c}$, and $W-\hat{W} = H(1-\alpha)-\hat{U}$ on A, and $\hat{1}_{A^c}=1-\alpha$ on K, so that with $V = |W-\hat{W}|$:

$$\hat{V} \le 2\delta\hat{1}_{A^c} + \delta(1-\alpha) + |\hat{U}| \le 4\delta(1-\alpha) \text{ on } K \tag{22}$$

Hence with V as above $V*\nu_\infty = (V1_{K^c})*\nu_\infty + \sum_{s\in K} \hat{V}_t \le 4\delta F_\infty$, and adding this to (21) yields $E[C^0(W)_\infty] < \infty$, hence $W\in\mathcal{L}^1_b$.

(d) $\Rightarrow$ (c): This is proved as the previous implication, with $V = (W-\hat{W})^2$ satisfying $\hat{V} \le 8\delta^2(1-\alpha)$ on K instead of (22).

(c) $\Rightarrow$ (d): By hypothesis $E[C^\infty(1)_\infty] < \infty$, hence (17). We also have $E[C^\infty(1_A)_\infty] < \infty$ which gives (19), and $E[C^\infty(1_B)_\infty] < \infty$ which gives (20). ∎

THEOREM 6: *Let* $\varepsilon\in(0,1)$. *There is equivalence between:*

a) *All stochastic integrals* $W*(\mu-\nu)$ *are a.s. of locally finite variation.*

b) *We have* $F^{\varepsilon}_t < \infty$ *a.s. for all* $t<\infty$.

Proof. (b) ⇒ (a). By localization we can assume $E(F^{\varepsilon}_{\infty})<\infty$. According to (3.69), (3.70) and (3.71) of [1], any stochastic integral $W*(\mu-\nu)$ is the sum of a local martingale with locally finite variation and another stochastic integral $W'*(\mu-\nu)$ with W' bounded. Then (d) ⇒ (c) of Theorem 5 shows the result.

(a) ⇒ (b). If $C^{\infty}(W)_t<\infty$ (or $E[C^{\infty}(W)_t]<\infty$) for all t, (a) and (12) yield $C^0(W)_t<\infty$ a.s. for all t; there is even a localizing sequence (T_n) with $E[C^0(W)_{T_n}]<\infty$, so we have *locally* (a) of Theorem 5. However the localizing sequence *a priori* depends on W, so we cannot apply (a) ⇒ (d) of Theorem 5.

Below we assume (a), and we fix $t>0$. In order to prove $F^{\varepsilon}_t<\infty$ a.s. it is enough by (15) and (16) to prove, with $K=K_{1/4}$:

$$1_{J^c}*\nu_t < \infty, \quad \sum_{s\in K, s\le t} \alpha_s(1-\alpha_s) < \infty, \quad \sum_{s\in J\setminus K, s\le t} \beta_s(1-\beta_s) < \infty. \tag{23}$$

We can apply Lemma 4 with $(G,\mathcal{G}) = (\Omega,\mathcal{F})$, $(H,\mathcal{H}) = (\mathbb{R}_+\times E,\mathcal{R}_+\otimes\mathcal{E})$ and $\eta(\omega,.) = (1_{J^c\cap[0,t]}\cdot\nu)(\omega,.)$: there is an $\mathcal{F}\otimes\mathcal{R}_+\otimes\mathcal{E}$-measurable function U with $0<U\le 1$ and $\int\eta(\omega;ds,dx)U^2(\omega,s,x) \le 1$ and (7). If $\rho(d\omega,ds,dx) = P(d\omega)\eta(\omega;ds,dx)$ there is a $\tilde{\mathcal{P}}$-measurable partition (D_n) of $\tilde{\Omega}$ with $\rho(D_n)<\infty$, so $W = \rho(U|\tilde{\mathcal{P}})$ is well defined. We then have $0\le W\le 1$, $W^2 \le \rho(U^2|\tilde{\mathcal{P}})$, and $\rho(U^2) = E[\int\eta(.;ds,dx)U^2(.,s,x)] \le 1$, hence $E[C^{\infty}(W1_{J^c\cap[0,t]})_{\infty}] = \rho(W^2) \le 1$ and (a), (12) and a localization argument imply $C^0(W1_{J^c\cap[0,t]})_{\infty} < \infty$ a.s. Hence there is a localizing sequence (T_n) such that

$$\rho(U1_{[0,T_n]}) = \rho(W1_{[0,T_n]}) = E[(W1_{J^c})*\nu_{T_n\wedge t}] < \infty,$$

hence $\int\eta(ds,dx)U(s,x) < \infty$ a.s. on $\cup_n\{T_n\ge t\} = \Omega$. Then (7) yields $\eta(\mathbb{R}_+\times E)<\infty$ a.s., which gives the first part of (23).

We apply the same argument with $\eta(\omega,ds) = \sum_{r\in K, r\le t} \alpha_r(1-\alpha_r)\varepsilon_r(ds)$, and $(G,\mathcal{G}) = (\Omega,\mathcal{F})$ and $(D,\mathcal{D}) = (\mathbb{R}_+,\mathcal{R}_+)$. Let U be as in Lemma 4. Then set $\rho(d\omega,ds) = P(d\omega)\eta(\omega,ds)$ and $H = \rho(U|\mathcal{P})$, which has $0\le H\le 1$ and $\rho(H^2)\le 1$. Then $W = H1_A$ satisfies (18), so $E[C^{\infty}(W)_{\infty}] = \rho(H^2) \le 1$, hence we deduce exactly as above that $C^0(W)_{\infty}<\infty$ a.s., and that $\int\eta(ds)U(s) < \infty$ a.s., hence $\eta(\mathbb{R}_+)<\infty$ a.s. and this gives the second part of (23).

The third part of (23) is proved similarly, upon substituting $(\beta,J\setminus K,B)$ with (α,K,A). ■

3) So far the measure μ was fixed, but here we allow it to change. Denote by $\mathcal{A}$ the class of all integer-valued random measures, on all Polish spaces E, and by $\mathcal{A}_0$ the subclass of all measures in $\mathcal{A}$ such that the process α associated by (13) has $\alpha_t \le 1/2$ identically. If $\mu \in \mathcal{A}$ has the compensator ν, let $\mathcal{M}(\mu)$ be the space of all local martingales of the form $W*(\mu-\nu)$.

PROPOSITION 7: *If* $\mu \in \mathcal{A}$ *there exists* $\mu' \in \mathcal{A}_0$ *with* $\mathcal{M}(\mu') = \mathcal{M}(\mu)$.

Proof. a) We start with $\mu \in \mathcal{A}$, on the Polish space E, and with compensator ν. As in the proof of Theorem 5, there is a predictable E-valued process ζ with $\alpha_t = \nu(\{t,\zeta_t\})$. Recall the process γ in (8), and in (13) set $K=K_{1/2}$.

The measure μ' will be on $\mathbb{R}_+\times E'$, with $E' = E\times\{0\}+E_\Delta\times\{1\}$. We set $E'_\Delta = E'+\{\Delta'\}$, and

$$\gamma'_t = \begin{cases} (\gamma_t,0) & \text{if } t\notin K \text{ and } \gamma_t\in E \\ (\gamma_t,1) & \text{if } t\in K \text{ and } \gamma_t\in E_\Delta\setminus\{\zeta_t\} \\ \Delta' & \text{otherwise.} \end{cases} \tag{24}$$

This is an optional E'_Δ-valued process, with which one associates the random measure μ' by (8). Clearly $\mu' \in \mathcal{A}$, and we add a dash to all quantities related to μ': e.g. ν', α', etc...

b) We presently prove that $\mu' \in \mathcal{A}_0$. First, with any $\tilde{\mathcal{P}}'$-measurable function W' on $\tilde{\Omega}' = \Omega\times\mathbb{R}_+\times E'$, we associate the $\tilde{\mathcal{P}}$-measurable function $f(W')(\omega,t,x) = W'(\omega,t,(x,0))$. We have

$$W'1_{K^c}*\mu' = f(W')1_{K^c}*\mu, \qquad W'1_{K^c}*\nu' = f(W')1_{K^c}*\nu \tag{25}$$

(the first equality is obvious, and the second one follows by taking the compensators). Thus $\alpha'_t = \alpha_t \le 1/2$ and $a'_t = a_t$ if $t\in K^c$. Next, since for every finite predictable time T the measure $\nu(\{T\}\times.)$ is characterized by the property $\nu(\{T\}\times A) = P(\gamma_T\in A\,|\,\mathcal{F}_{T-})$, and similarly for ν', up to changing ν' on a null set we can assume that we have identically:

$$\left.\begin{aligned} \nu'(\{t\}\times(E\times\{0\})) &= 0, \qquad \nu'(\{t\}\times(C\times\{1\}))=\nu(\{t\}\times(C\setminus\{\zeta_t\})), \\ \nu'(\{t\}\times\{\Delta,1\}) &= 1-\alpha_t. \end{aligned}\right\} \text{ if } t\in K. \tag{26}$$

Then $\alpha'_t \leq a'_t = 1-\alpha_t < 1/2$ if $t\in K$. Hence $\alpha'_t\leq 1/2$ for all t, and $\mu'\in\mathcal{A}_0$.

c) It remains to prove $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. Since stochastic integrals w.r.t. random measures are characterized by their jumps, it suffices to prove that if W is $\tilde{\mathcal{P}}$-measurable (resp. W' is $\tilde{\mathcal{P}}'$-measurable), we can find a $\tilde{\mathcal{P}}'$-measurable W' (resp. a $\tilde{\mathcal{P}}$-measurable W) such that for all t:

$$W(t,\gamma_t)1_E(\gamma_t)-\int_E \nu(\{t\}\times dx)W(t,x) = W'(t,\gamma'_t)1_{E'}(\gamma'_t)-\int_{E'} \nu'(\{t\}\times dy)W'(t,y). \tag{27}$$

$\tilde{\mathcal{P}}$-measurable functions are functions of the form

$$W(t,x) = H_t 1_{\{\zeta_t\}}(x) + U(t,x), \quad \text{with } U(t,\zeta_t) = 0, \tag{28}$$

where H is predictable and U is $\tilde{\mathcal{P}}$-measurable, and the left-hand side of (27) is then

$$H_t 1_{\{\zeta_t\}}(\gamma_t) + U(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - \alpha_t H_t - \hat{U}_t. \tag{29}$$

$\tilde{\mathcal{P}}'$-measurable functions are functions of the form (with $(x,i)\in E'$):

$$W'(t,(x,i)) = G^i_t 1_{\{\zeta_t\}}(x) + G_t 1_{\{\Delta,1\}}(x,i) + U^i(t,x), \quad \text{with } U^i(t,\zeta_t) = 0, \tag{30}$$

where G, G^0, G^1 are predictable and U^0, U^1 are $\tilde{\mathcal{P}}$-measurable on $\tilde{\Omega}$. Due to (24), (25) and (26), the right-hand side of (27) is then

$$\left.\begin{array}{ll} G^0_t 1_{\{\zeta_t\}}(\gamma_t) + U^0(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - \alpha_t G^0_t - \hat{U}^0_t & \text{if } t\notin K \\ G_t 1_{\{\Delta\}}(\gamma_t) + U^1(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - (1-a_t)G_t - \hat{U}^1_t & \text{if } t\in K \end{array}\right\} \tag{31}$$

If we start with (28) and if we define W' by (30) with $G^1=0$, $G^0=H1_{K^c}$, $U^0=U1_{K^c}$, $G=-H1_K$, $U^1(t,x) = [U(t,x)-H_t 1_{\{x\neq\zeta_t\}}]1_K(t)$, a simple computation shows that (29) and (31) are equal. We also have equality between (29) and (31) if we start with (30) and define W by (28), with $H = H^0 1_{K^c}-G1_K$ and $U(t,x) = U^0(t,x)1_{K^c}(t) + [U^1(t,x)-G_t 1_{\{x\neq\zeta_t\}}]1_K(t)$. Therefore (27) holds. ∎

COROLLARY 8: *Let $\mu\in\mathcal{A}$. There is equivalence between:*

a) *All elements of $\mathcal{M}(\mu)$ are a.s. of locally finite variation.*

b) *For every $\mu'\in\mathcal{A}_0$ such that $\mathcal{M}(\mu')=\mathcal{M}(\mu)$ we have $1*\nu_t<\infty$ a.s. for all* $t<\infty$ (or equivalently $1*\mu'_t<\infty$ a.s. for all $t<\infty$).

Proof. (a) ⇒ (b): Let $\mu'\in\mathcal{A}_0$ with $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. Then (a) ⇒ (b) of Theorem 6 applied to μ' implies $F'^{1/2}_t<\infty$ a.s., where $F'^{1/2}$ is associated with μ' by (14). Since $\mu'\in\mathcal{A}_0$ we have $K'_{1/2}=\emptyset$, so $1*\nu'_t<\infty$ a.s. It is also well known that $1*\nu'_t<\infty$ a.s. for all $t<\infty$ is equivalent to $1*\mu'_t<\infty$ a.s. for all $t<\infty$.

(b) ⇒ (a): This readily follows from (b) ⇒ (a) of Theorem 6 and from the existence of $\mu'\in\mathcal{A}_0$ with $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. ■

5 - THE SUFFICIENT CONDITION OF THEOREM 1

For the proof of the sufficient condition in Theorem 1, we need two preliminary lemmas. If $\mu\in\mathcal{A}$ is given by (8), we set $D(\mu) = \{(\omega,t):\gamma_t(\omega)\in E\}$.

LEMMA 9: *For any sequence* (μ_n) *in* $\mathcal{A}_0$ *there exists* $\mu\in\mathcal{A}_0$ *such that* $D(\mu) = \cup_n D(\mu_n)$ *and* $\cup_n \mathcal{M}(\mu_n) \subseteq \mathcal{M}(\mu)$.

Proof. For each n, μ_n is a random measure on the Polish space E^n, with the associated process γ^n (see (8)). Set $E = \Pi E^n_\Delta$, $E_\Delta = E+\{\Delta\}$, and define an E_Δ-valued optional process γ by

$$\gamma_t = \begin{cases} (\gamma^1_t,\dots,\gamma^n_t,\dots) & \text{if } t\in\cup D(\mu_n) \\ \Delta & \text{otherwise.} \end{cases}$$

The associated measure μ belongs to $\mathcal{A}_0$ (indeed if $x=(x_1,x_2,..)$ is an atom of $\nu(\{t\}\times.)$, there is at least one n such that $x_n\neq\Delta$ and x_n is an atom of $\nu_n(\{t\}\times.)$, and $\nu(\{t,x\}) \le \nu_n(\{t,x_n\})$, so $\alpha_t \le \sup_n \alpha^n_t \le 1/2$). By construction $D(\mu) = \cup_n D(\mu_n)$. Finally if $M = W_n*(\mu_n-\nu_n)\in\mathcal{M}(\mu_n)$ one easily checks that $M = W*(\mu-\nu)\in\mathcal{M}(\mu)$ if $W(\omega,t,(x_1,\dots)) = W_n(\omega,t,x_n)1_{E^n}(x_n)$. ■

LEMMA 10: *Assume that all martingales are a.s. of locally finite variation.*

a) *If* $\mu\in\mathcal{A}_0$ *the set* $D(\mu)$ *has almost all its* $\mathbb{R}_+$*-sections locally finite.*

b) *There exists* $\mu\in\mathcal{A}_0$ *such that any other* $\mu'\in\mathcal{A}_0$ *has* $D(\mu')\subseteq D(\mu)$ *a.s.*

Proof. a) The $\mathbb{R}_+$-section of $D(\mu)$ through ω is locally finite iff $1*\mu_t(\omega)<\infty$ for all $t<\infty$, so the claim follows from Corollary 8.

b) We construct by induction an increasing sequence of stopping times$(T_n)_{n\ge0}$ and a sequence $(\mu_n)_{n\ge1}$ of elements of $\mathcal{A}_0$ with the following:

$$\left.\begin{array}{ll} T_n<\infty \;\Rightarrow\; T_n<T_{n+1} \quad \text{a.s.}, & [\![T_n]\!]\subseteq D(\mu_n) \quad \text{a.s.} \\ \text{for all} \quad \mu\in\mathcal{A}_0, \quad \text{we have} &]\!]T_{n-1},T_n[\![\cap D(\mu) = \emptyset \quad \text{a.s.} \end{array}\right\} \tag{32}$$

We start with $T_0=0$. Suppose that we know (T_n,μ_n) with (32) for $n\le p$. Set $S(\mu) = \inf(t>T_p\colon t\in D(\mu))$ if $\mu\in\mathcal{A}_0$, and $T_{p+1} = \text{ess}\inf(S(\mu)\colon \mu\in\mathcal{A}_0)$. (a) implies $S(\mu)>T_p$ a.s. on the set $A = \{T_p<\infty\}$. By Lemma 9 if $\mu,\mu'\in\mathcal{A}_0$ there is $\mu''\in\mathcal{A}_0$ with $D(\mu'') = D(\mu)\cup D(\mu')$, hence $S(\mu'') = S(\mu)\wedge S(\mu')$. Therefore there exists a sequence (ρ_n) in $\mathcal{A}_0$ such that $S(\rho_n)$ decreases a.s. to T_{p+1}. Applying again Lemma 9, we obtain $\mu_{p+1}\in\mathcal{A}_0$ with $D(\mu_{p+1}) = \cup_n D(\rho_n)$, so $T_{p+1} = S(\mu_{p+1})$ a.s. and thus $T_{p+1}>T_p$ a.s. on A. We have $[\![T_{p+1}]\!]\subseteq D(\mu_{p+1})$ a.s., and for any $\mu\in\mathcal{A}_0$ we have $T_{p+1}\le S(\mu)$ a.s., so the last property in (32) is satisfied for $n=p+1$.

So far, we have constructed the sequences (T_n), (μ_n) with (32). Taking the measure μ associated with the sequence (μ_n) in Lemma 9 gives (b). ∎

Proof of the sufficient condition of Theorem 1. We assume that all martingales are a.s. of locally finite variation. Let $\mu\in\mathcal{A}_0$ be the measure constructed in Lemma 10, and $T_n = \inf(t\colon 1*\mu_t=n)$. Since $1*\mu_t<\infty$ a.s. for all $t<\infty$, we have outside a null set: $T_0=0$, $T_n\uparrow\infty$, $T_n<T_{n+1}$ if $T_n<\infty$. We will prove that $(\mathcal{F}_t)$ is a jumping filtration with jumping sequence (T_n). To this effect, it suffices to prove that for $t\ge 0$, $A\in\mathcal{F}_t$, $n\in\mathbb{N}$ fixed, there exists $A'\in\mathcal{F}_{T_n}$ with

$$A\cap\{T_n\le t<T_{n+1}\} \;=\; A'\cap\{T_n\le t<T_{n+1}\} \quad \text{a.s.} \tag{33}$$

The proof is similar to part (γ) of the proof of Theorem 2. We set $T=T_n$, $S=T_{n+1}$, and consider the martingale $N^A_s = P(A\cap\{T\le t<S\}\,|\,\mathcal{F}_s)$. Let $\rho\in\mathcal{A}$ be the random measure associated with the jumps of the pair (N^A,N^Ω): it is given by (8) with $E=\mathbb{R}^2\backslash\{0\}$ and $\gamma_t = (\Delta N^A_t,\Delta N^\Omega_t)$, and we know that both N^A and N^Ω belong to $\mathcal{M}(\rho)$. By Proposition 7 there is $\rho'\in\mathcal{A}_0$ with $N^A,N^\Omega\in\mathcal{M}(\rho')$. By Lemma 9 there is $\mu'\in\mathcal{A}_0$ with $N^A,N^\Omega\in\mathcal{M}(\mu')$ and $D(\mu') = D(\mu)\cup D(\rho')$, while Lemma 10 yields $D(\mu')\subseteq D(\mu)$ a.s., so in fact $D(\mu')=D(\mu)$ a.s. Therefore we also have $T_n = \inf(t\colon 1*\mu'_t=n)$ a.s., so up to substituting μ with μ' we can and will assume that $N^A,N^\Omega\in\mathcal{M}(\mu)$.

Set $M_s = N^A_s - N^A_{s\wedge T}$, so $M\in\mathcal{M}(\mu)$. As for Theorem 2, we have (4) and $\Delta M_S = -N^A_{S-}1_{\{t\le S\}}$. Thus $D(\mu)\cap\{\Delta M\ne 0\} \subseteq [\![S]\!]$ and $\Delta M_s = -N^A_{s-}1_{\{T<s\le t\wedge S\}}$ for all $t\in D(\mu)$, outside a null set. In other words, if γ is associated with μ by (8) and if $U(\omega,s,x) = -N^A_{t-}(\omega)1_{\{T(\omega)<s\le t\wedge S(\omega)\}}$, outside a null set we have

$\Delta M_t(\omega) = U(\omega,t,\gamma_t(\omega))1_E(\gamma_t(\omega))$ for all $(\omega,t)\in D(\mu)$. Since U is $\tilde{\mathcal{P}}$-measurable, we deduce from Theorems (3.45) and (4.47) of [1] that, since $M\in\mathcal{M}(\mu)$:

$$M = W*(\mu-\nu), \quad \text{with} \quad W = U + \frac{\hat{U}}{1-a} 1_{\{a<1\}}.$$

A simple computation shows that $W(s,x) = -\frac{1}{1-a_s} 1_{\{a_s<1\}} N^A_{s-} 1_{(T,t\wedge S]}(s)$. If $F = 1*\nu$ we then deduce that

$$M_s = \int_T^{s\wedge t} N^A_{r-} \frac{1}{1-a_r} 1_{\{a_r<1\}} dF_r \quad \text{if} \quad T\le s<S.$$

The process $Y_s = \int_{s\wedge T}^{s\wedge t} \frac{1}{1-a_r} 1_{\{a_r<1\}} dF_r$ is increasing and finite-valued, and $N^A_s = N^A_T+M_s$ if $s\ge T$, hence (5) holds. Similarly (5) holds for N^Ω, so we deduce (6), and $A' = \{N^A_T=N^\Omega_T>0\}$ satisfies (33).

REMARK: When the σ-field $\mathcal{F}_\infty$ is separable, the proof is much simpler. Indeed, in this case there is a sequence $(M^n)_{n\in\mathbb{N}}$ of martingales which "generates" (in the stochastic integrals sense) the space of all local martingales. Therefore if μ is the integer-valued random measure on $E = \mathbb{R}^{\mathbb{N}}$ associated with the jumps of the infinite-dimensional process $(M^n)_{n\in\mathbb{N}}$, $\mathcal{M}(\mu)$ is the space of of local martingales, so we do not need Lemmas 9 and 10. ∎

REFERENCE

[1] J. JACOD: Calcul stochastique et problèmes de martingales. Lect. Notes In Math., 714. Springer Verlag: Berlin, 1979.

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