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Rate of explosion of the Amperean area of the planar Brownian loop

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ABSTRACT: We study the asymptotic behaviour of approximations of the Amperean area (i.e. the integral of the squared index function) of the Brownian loop, which is almost surely infinite.

AMS SUBJECT CLASSIFICATION. Primary 60J65, Secondary 60H05.

KEY WORDS. Brownian loop, Amperean area, winding numbers

Introduction

Let $\gamma = (\gamma_t,\ 0 \le t \le 1)$ be a continuous loop in the complex plane and define for all $z \in \mathbb{R}^2 \setminus \{\gamma_t, 0 \le t \le 1\}$, the index n_z of γ around z. The Amperean area $\Omega(\gamma)$ of γ is defined by:

$$\Omega(\gamma) = \int_{\mathbb{R}^2 \setminus \gamma} (n_z)^2 \, dz,$$

where dz denotes the Lebesgue measure in $\mathbb{R}^2$. When γ is not smooth enough, n_z may not be bounded as z varies and it may happen that $\Omega(\gamma) = \infty$ because of too many small windings of the loop. From now on in this paper, we will consider the case where γ is a standard Brownian loop with $\gamma_0 = \gamma_1 = 0$ for which it is known that $\Omega(\gamma) = \infty$ a.s. (see [L], page 245, $[W_1]$ and $[W_2]$) and we will estimate approximations of $\Omega(\gamma)$.

Problems related to the planar Brownian loop have been studied in several works. Lévy [L] has derived the exact law of the 'stochastic area' using the Fourier decomposition of the loop. The explicit law of the index n_z (with fixed z) has been derived in terms of Bessel functions by Yor [Y] (see also [E]), to which we refer for a rigorous definition of the Brownian loop. In $[W_2]$, we introduced approximations n_z^ε of n_z, which we will use in this paper. More precisely, we proved that, although

$\int_{\mathbb{R}^2} |n_z| dz = \infty$ almost surely,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} n_z^\varepsilon dz = \mathcal{A}(\gamma) \quad \text{in Probability,}$$

where $\mathcal{A}(\gamma) = (1/2) \int_0^1 (\gamma_s^1 d\gamma_s^2 - \gamma_s^2 d\gamma_s^1)$ denotes Lévy's stochastic area of $\gamma = \gamma^1 + i\gamma^2$ and where n_z^ε is defined by the following stochastic integral:

$$n_z^\varepsilon = \frac{1}{2\pi} \Im \left(\int_0^1 \frac{d\gamma_s}{\gamma_s - z} 1_{|\gamma_s - z| > \varepsilon} \right)$$

($\Im$ denotes the imaginary part of a complex number and we identify $\mathbb{R}^2$ with C). Intuitively, $2\pi n_z^\varepsilon$ corresponds to the windings around z made outside the small disc $\mathcal{D}(z, \varepsilon)$ centered at z and with radius ε. Note that $n_z^\varepsilon = n_z$ as soon as z is not in the Wiener sausage S_ε of radius ε (that is $S_\varepsilon = \cup_{t \leq 1} \mathcal{D}(\gamma_t, \varepsilon)$). The main result of the present paper is the following:

Theorem 1.

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^2} (n_z^\varepsilon)^2 dz = \frac{1}{2\pi} \quad \text{in Probability.}$$

The motivation for this work has been given by recent works of physicists ([CDO], [GWS], [WS]...). The Amperean area of planar stochastic loops appears naturally in modelizations of particle systems for which magnetic interaction plays an important role (type II-superconductors, anyon gas...); see e.g. equation (6) in [GWS], where Ω is exactly the Debye Action functional. Very loosely speaking, the coefficient ε in our approximation, corresponds to the biggest distance for which interactions other than the magnetic interaction between particles cannot be neglected.

Our proof is based on simple properties of stochastic integrals (second moment computations...) and it seems unlikely that it can be easily adapted to obtain similar results for random walks on a lattice. It may nevertheless be conjectured that analogous results hold, where ε corresponds to the size of the lattice.

1. The Brownian motion.

1.1 Preliminaries.

It is much easier to deal with stochastic integrals with respect to the Brownian motion than to the Brownian loop. In this section, we will only focus our attention on Brownian motion, and we will derive the results concerning the Brownian loop in the next section.

Let $Z = (Z_t, t \geq 0)$ be a complex Brownian motion started from $Z_0 = 0$. As in [W_2], section 7, we put

$$n_z^\varepsilon = \frac{1}{2\pi} \Im \left(\int_0^1 \frac{dZ_s}{Z_s - z} 1_{|Z_s - z| > \varepsilon} \right),$$

for all $\varepsilon > 0$. One should keep in mind the equality

$$n_z^\varepsilon = \frac{1}{2\pi}\int_0^1 1_{|Z_s - z| > \varepsilon} \frac{(X_s - x)dY_s - (Y_s - y)dX_s}{|Z_s - z|^2}, \tag{1}$$

where $z = x + iy$ and $Z_s = X_s + iY_s$. Nevertheless, we will mainly use the complex multiplicative notation for clarity reasons.

Since n_z^ε does not decrease fast enough as $|z| \to \infty$, it is obvious that $\int_{\mathbb{R}^2}(n_z^\varepsilon)^2 dz = \infty$ almost surely. More precisely, if $z(R) = (Z_1/|Z_1|)Re^{i\theta}$, then

$$n_{z(R)}^\varepsilon \sim n_{z(R)} \sim \frac{|Z_1|}{2\pi R}\sin\theta$$

as $R \to \infty$, and consequently,

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{\mathcal{D}(0,R)} (n_z^\varepsilon)^2 dz = \frac{|Z_1|^2}{4\pi} \quad \text{a.s.}$$

(this phenomenon does not occur for the loop, as $n_z = 0$ on the unbounded connected component of the complement of the loop). To avoid this problem, we introduce for all $\delta > \varepsilon$,

$$m_z^{\varepsilon,\delta} = n_z^\varepsilon - n_z^\delta = \frac{1}{2\pi}\Im\left(\int_0^1 \frac{dZ_s}{Z_s - z} 1_{|Z_s - z| \in]\varepsilon,\delta]}\right),$$

and more generally, for all time t,

$$m_z^{\varepsilon,\delta}(t) = \frac{1}{2\pi}\Im\left(\int_0^t \frac{dZ_s}{Z_s - z} 1_{|Z_s - z| \in]\varepsilon,\delta]}\right).$$

Note that $m_z^{\varepsilon,\delta}(t) = 0$ as soon as $z \notin S_\delta^t$, where S_δ^t denotes the Wiener sausage $\cup_{s\le t}\mathcal{D}(Z_s, \delta)$ of radius δ on the time-interval $[0, t]$. Let us put

$$X_{\varepsilon,\delta} = \int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta})^2 dz;$$

we will see that $X_{\varepsilon,\delta} < \infty$ a.s. (see e.g. Lemma 1-(i)).

Let us also fix, for all $k \ge 0$, $\delta_k = \exp(-k^2)$, and define the set Σ of all sequences $(\varepsilon_k, k \ge 0)$ of positive real numbers, such that for all $k \ge 0$, $\varepsilon_k \le \exp(-e^k)$. We will use these notations throughout this paper.

Our main aim in this section is to prove the following Proposition:

Proposition 1. *For any* $(\varepsilon_k, k \ge 0) \in \Sigma$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} X_{\varepsilon_k,\delta_k} = \frac{1}{2\pi} \quad a.s.$$

In fact, our proof also implies that $\lim_{\varepsilon\to 0+} |\log\varepsilon|^{-1} X_{\varepsilon,\delta(\varepsilon)} = (2\pi)^{-1}$ in Probability for $\delta(\varepsilon) = \exp(-(\log|\log\varepsilon|)^2)$, but Proposition 1 will be more useful in Section 2.

1.2 Preliminary results

We now recall some known results, which will be useful in this proof: If $T_\delta(z) = \inf\{t \geq 0,\ |Z_t - z| < \delta\}$ is the hitting time of $\mathcal{D}(z,\delta)$, one has

$$P(T_\delta(z) < t) \leq \frac{\phi_t(z)}{|\log \delta|} \tag{2}$$

for all $\delta < 1$, where $\phi_t \in L^p(\mathbb{R}^2)$ for all $p \geq 1$ (see for instance [LG], chapter 6).

We also restate Lemma 8 from $[W_2]$, which is an easy consequence of estimations on Bessel functions: For all $\varepsilon < 1/2$,

$$E((n_z^\varepsilon)^2) \leq \psi(z)\ |\log \varepsilon|, \tag{3}$$

where $\psi(z) = A + B\log|z| + C|z|^{-1/2}$ for some constants A, B, C. This implies readily that for all $\varepsilon < \delta < 1/2$,

$$E((m_z^{\varepsilon,\delta})^2) \leq 4\psi(z)\ |\log \varepsilon|. \tag{4}$$

1.3 The moments of $X_{\varepsilon,\delta}$

We now estimate the first two moments of $X_{\varepsilon,\delta}$.

Lemma 1. *For all* $\varepsilon < \delta$,

$$(i) \qquad E(X_{\varepsilon,\delta}) = \frac{1}{2\pi}\log\frac{\delta}{\varepsilon}$$

$$(ii) \qquad E((X_{\varepsilon,\delta})^2) \leq \frac{8}{\pi^2}(\log\frac{\delta}{\varepsilon})^2.$$

Proof: (i) is a straightforward consequence of (1) and of Fubini's Theorem: For all $z \neq 0$, (1) implies that

$$E((m_z^{\varepsilon,\delta})^2) = \frac{1}{4\pi^2}\int_0^1 ds\ E\left(\frac{1_{|Z_s - z| \in]\varepsilon,\delta]}}{|Z_s - z|^2}\right).$$

So,

$$\begin{aligned} E(X_{\varepsilon,\delta}) &= \int_{\mathbb{R}^2} dz\ E((m_z^{\varepsilon,\delta})^2) \\ &= \frac{1}{4\pi^2}\int_0^1 ds\ E(\int_{\mathbb{R}^2} \frac{1_{|Z_s - z| \in]\varepsilon,\delta]}}{|Z_s - z|^2} dz) \\ &= \frac{1}{4\pi^2}\int_0^1 ds\ 2\pi\log\frac{\delta}{\varepsilon} \\ &= \frac{1}{2\pi}\log\frac{\delta}{\varepsilon}. \end{aligned}$$

(ii) Let us denote for all $t \geq 0$, $f_t = f(X_t)$, $g_t = g(X_t)$, $F_t = \int_0^t f_s dX_s$ and $G_t = \int_0^t g_s dX_s$, where X is a linear Brownian motion and f and g two measurable bounded functions. Itô's formula yields

$$F_t G_t = \int_0^t (F_s g_s + G_s f_s) dX_s + \int_0^t f_s g_s ds$$

and

$$E(F_t^2 G_t^2) \leq 4E\left(\int_0^t (F_s^2 g_s^2 + G_s^2 f_s^2 + t f_s^2 g_s^2) ds\right) \tag{5}$$

for all $t \geq 0$.

Similarly, if $f_t = f(X_t, Y_t)$, $g_t = g(X_t, Y_t)$, $F_t = \int_0^t f_s dX_s$ and $G_t = \int_0^t g_s dY_s$, where X and Y are two independent Brownian motions, Itô's formula yields

$$F_t G_t = \int_0^t F_s g_s dY_s + \int_0^t G_s f_s dX_s$$

and

$$E(F_t^2 G_t^2) \leq \ 2E\left(\int_0^t ((F_s g_s)^2 + (G_s f_s)^2) ds\right). \tag{6}$$

Now,

$$E((X_{\varepsilon,\delta})^2) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} E((m_z^{\varepsilon,\delta})^2 (m_{z'}^{\varepsilon,\delta})^2) \; dz \; dz'$$

and we deduce from (1) that

$$\begin{aligned}
&(m_z^{\varepsilon,\delta} m_{z'}^{\varepsilon,\delta})^2 \\
&\leq \frac{1}{16\pi^4}\left(\Big(\int_0^1 \frac{(Y_s - y)}{|Z_s - z|^2} 1_{|Z_s - z| \in]\varepsilon,\delta]} dX_s\Big)^2 + \Big(\int_0^1 \frac{(X_s - x)}{|Z_s - z|^2} 1_{|Z_s - z| \in]\varepsilon,\delta]} dY_s\Big)^2\right) \\
&\times \left(\Big(\int_0^1 \frac{(Y_s - y')}{|Z_s - z'|^2} 1_{|Z_s - z'| \in]\varepsilon,\delta]} dX_s\Big)^2 + \Big(\int_0^1 \frac{(X_s - x')}{|Z_s - z'|^2} 1_{|Z_s - z'| \in]\varepsilon,\delta]} dY_s\Big)^2\right).
\end{aligned}$$

Hence, (5) and (6) lead easily (using also a symmetry argument) to

$$\begin{aligned}
E((X_{\varepsilon,\delta})^2) \leq & 16 \int_{\mathbb{R}^2 \times \mathbb{R}^2} E\left(\int_0^1 ds \frac{1_{|Z_s - z| \in]\varepsilon,\delta]}}{4\pi^2 |Z_s - z|^2} (m_{z'}^{\varepsilon,\delta}(s))^2\right) dz dz' \\
&+ 16 \int_{\mathbb{R}^2 \times \mathbb{R}^2} E\left(\int_0^1 ds \frac{1_{|Z_s - z| \in]\varepsilon,\delta]} 1_{|Z_s - z'| \in]\varepsilon,\delta]}}{16\pi^4 |Z_s - z|^2 |Z_s - z'|^2}\right) dz dz'.
\end{aligned}$$

Finally, Fubini's Theorem and (i) imply that

$$E((X_{\varepsilon,\delta})^2) \leq \frac{8}{\pi^2} \log \frac{\delta}{\varepsilon},$$

which completes the proof of Lemma 1.

1.4 Cutting the Brownian path.

We will now use the independence of the increments of Brownian motion to prove that $X_{\varepsilon_k,\delta_k} \sim E(X_{\varepsilon_k,\delta_k})$ as $k \to \infty$ (where $(\varepsilon_k, k \geq 0) \in \Sigma$). Let us define, for all time-intervals $[a,b]$, $m_z^{\varepsilon,\delta}([a,b]) = m_z^{\varepsilon,\delta}(b) - m_z^{\varepsilon,\delta}(a)$ and denote

$$m_z^{\varepsilon,\delta,1} = m_z^{\varepsilon,\delta}([0,1/2]),\ m_z^{\varepsilon,\delta,2} = m_z^{\varepsilon,\delta}([1/2,1]).$$

Let also denote S_δ^1 and S_δ^2 the Wiener sausages of radius δ on the time-intervals $[0,1/2]$ and $[1/2,1]$. Obviously,

$$X_{\varepsilon,\delta} = \int_{\mathbb{R}^2} (m^{\varepsilon,\delta,1})^2 dz + \int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta,2})^2 dz + Y_{\varepsilon,\delta} \tag{7}$$

where

$$Y_{\varepsilon,\delta} = 2\int_{\mathbb{R}^2} m_z^{\varepsilon,\delta,1} m_z^{\varepsilon,\delta,2} dz.$$

The independence of Brownian increments before and after time 1/2 and the fact that $m_z^{\varepsilon,\delta,1} = 0$ as soon as $z \notin S_\delta^1$ yield

$$\begin{aligned} E(|Y_{\varepsilon,\delta}|) &\leq 2\, E\left(\int_{\mathbb{R}^2} (1_{z\in S_\delta^1}\ |m_z^{\varepsilon,\delta,2}|^2 + 1_{z\in S_\delta^2} |m_z^{\varepsilon,\delta,1}|^2) dz\right) \\ &\leq 4\int_{\mathbb{R}^2} P(T_\delta(z') < 1/2) E((m_{z'}^{\varepsilon,\delta,1})^2)\ dz' \end{aligned}$$

(where $z' = z - Z_{1/2}$). (2) and (4) now imply immediately that for all $\varepsilon < \delta < 1/2$,

$$E(|Y_{\varepsilon,\delta}|) \leq C\frac{|\log \varepsilon|}{|\log \delta|} \tag{8}$$

for some constant $C > 0$.

On the other hand, the Markov property and a scaling argument show that $X^1_{\varepsilon\sqrt{2},\delta\sqrt{2}} = 2\int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta,1})^2 dz$ and $X^2_{\varepsilon\sqrt{2},\delta\sqrt{2}} = 2\int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta,2})^2 dz$ are two independent copies of $X_{\varepsilon\sqrt{2},\delta\sqrt{2}}$.

Now, repeating p times (7) gives:

$$X_{\varepsilon,\delta} = \frac{1}{2^p}\left(X^1_{\varepsilon 2^{p/2},\delta 2^{p/2}} + \ldots + X^{2^p}_{\varepsilon 2^{p/2},\delta 2^{p/2}}\right) + Y^p_{\varepsilon,\delta}, \tag{9}$$

where $(X^1_{\varepsilon 2^{p/2},\delta 2^{p/2}}, \ldots, X^{2^p}_{\varepsilon 2^{p/2},\delta 2^{p/2}})$ are 2^p independent copies of $X_{\varepsilon 2^{p/2},\delta 2^{p/2}}$ and where

$$E(|Y^p_{\varepsilon,\delta}|) \leq p\, C\frac{|\log \varepsilon|}{|\log(\delta 2^{p/2})|} \tag{10}$$

if $\delta 2^{p/2} < 1/2$.

Now, let us fix $(\varepsilon_k, k \geq 0) \in \Sigma$ and let p_k be the integer part of $(2\log k)/\log 2$, so that $k^{3/2} \leq 2^{p_k} \leq k^2$ for all sufficiently large k. (10) can be rewritten as:

$$E(|Y^{p_k}_{\varepsilon_k,\delta_k}|) \leq C' \frac{\log k}{k^2} |\log \varepsilon_k|$$

for some new constant C'. Chebyshev's inequality and Borel-Cantelli's Lemma now imply that

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} Y^{p_k}_{\varepsilon_k,\delta_k} = 0 \quad \text{a.s.} \tag{11}$$

Similarly, Lemma 1 and Chebyshev's inequality yield, for all $\varepsilon < \delta < 1/2$,

$$\begin{aligned} P\left(\left| \frac{X^1_{\varepsilon 2^{p/2},\delta 2^{p/2}} + \ldots + X^{2^p}_{\varepsilon 2^{p/2},\delta 2^{p/2}}}{2^p E(X_{\varepsilon 2^{p/2},\delta 2^{p/2}})} - 1 \right| > \frac{1}{\log k} \right) \\ \leq \frac{E((X_{\varepsilon 2^{p/2},\delta 2^{p/2}})^2)}{2^p (\log k)^{-2} E(X_{\varepsilon 2^{p/2},\delta 2^{p/2}})^2} \\ \leq \frac{32}{2^p} (\log k)^2. \end{aligned}$$

So, Borel-Cantelli's Lemma and Lemma 1-(i) imply that

$$\lim_{k\to\infty} \left(\frac{X^1_{\varepsilon_k 2^{p_k/2},\delta_k 2^{p_k/2}} + \ldots + X^{2^{p_k}}_{\varepsilon_k 2^{p_k/2},\delta_k 2^{p_k/2}}}{2^{p_k} |\log \varepsilon_k|} \right) = \frac{1}{2\pi} \quad \text{a.s.}$$

(11) and (9) finally show that,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} X_{\varepsilon_k,\delta_k} = \frac{1}{2\pi} \quad \text{a.s.}$$

and the proof of Proposition 1 is completed.

1.5 Localization

Finally, we estimate the difference between $(n^\varepsilon_z)^2$ and $(m^{\varepsilon,\delta}_z)^2$:

Lemma 2. *For any $(\varepsilon_k, k \geq 0) \in \Sigma$ and for any compact set $A \subset \mathbb{R}^2$,*

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \left(\int_A (n^{\varepsilon_k}_z)^2 dz - \int_A (m^{\varepsilon_k,\delta_k}_z)^2 dz \right) = 0 \quad \text{a.s.}$$

Proof: Notice that

$$\begin{aligned} E(|(n^\varepsilon_z)^2 - (m^{\varepsilon,\delta}_z)^2|) &\leq E(|n^\varepsilon_z - m^{\varepsilon,\delta}_z|^2)^{1/2} E(|n^\varepsilon_z + m^{\varepsilon,\delta}_z|^2)^{1/2} \\ &\leq 4E((n^\delta_z)^2)^{1/2}\ E(|n^\varepsilon_z|^2 + |n^\delta_z|^2)^{1/2}. \end{aligned}$$

Using (3) shows that for all $k \geq 1$,

$$\frac{1}{|\log \varepsilon_k|} \left| \int_A (n^{\varepsilon_k}_z)^2 dz - \int_A (m^{\varepsilon_k,\delta_k}_z)^2 dz \right| \leq C'' \frac{|\log \delta_k|^{1/2}}{|\log \varepsilon_k|^{1/2}}$$

for some constant C''. A Borel-Cantelli argument ends the proof.

In the sequel, we will use Lemma 2 in the following form:

Corollary 1. *Let Γ be any event such that $P(\Gamma) > 0$. Then, conditional on Γ, one has*

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A |(n_z^{\varepsilon_k})^2 - (m_z^{\varepsilon_k,\delta_k})^2| dz = 0 \quad \text{a.s.}$$

for any $(\varepsilon_k, k \geq 0) \in \Sigma$ and any compact $A \subset \mathbb{R}^2$.

2. The Brownian loop

2.1 Preliminaries

If $\gamma = (\gamma_t, 0 \leq t \leq 1)$ denotes a Brownian loop with $\gamma_0 = \gamma_1 = 0$, let us recall that for $\lambda < 1$, the law of $(\gamma_t, t \leq \lambda)$ has the same negligible sets as the law of $(Z_t, t \leq \lambda)$. Hence, an almost sure result depending only on $(Z_t, t \leq \lambda)$ is also true for γ. As in [W_2], that is the basic idea we will use to obtain results on the Brownian loop.

We will use the same notations for the functionals of γ and of Z. To avoid confusion, we will specify each time which case we consider: We will refer to the Brownian loop (respectively motion) as the 'L-case' (resp. 'M-case'). Let us define, as for Z,

$$m_z^{\varepsilon,\delta}(\mathrm{I}) = \frac{1}{2\pi} \Im \left(\int_I \frac{d\gamma_s}{\gamma_s - z} 1_{|\gamma_s - z| \in]\varepsilon,\delta]} \right)$$

for all $z \neq 0$, $\varepsilon < \delta$ and all intervals $I \subset [0,1]$. We put, $m_z^{\varepsilon,\delta} = m_z^{\varepsilon,\delta}([0,1])$. Similarly we define $n_z^\varepsilon(I)$ and n_z^ε.

We cut $(\gamma_t, 0 \leq t \leq 1)$ (and $(Z_t, t \leq 1)$) in three parts corresponding to the time-intervals $I_1 = [0,1/3], I_2 = [1/3,2/3]$ and $I_3 = [2/3,1]$. We denote, for $i \in \{1,2,3\}$,

$$m_z^{\varepsilon,\delta,i} = m_z^{\varepsilon,\delta}(I_i) \text{ and } n_z^{\varepsilon,\delta,i} = n_z^{\varepsilon,\delta}(I_i)$$

in both M- and L-cases.

2.2 The analogue of Proposition 1

We will now derive the analogue of Proposition 1 in the L-case. Obviously, for all $\varepsilon < \delta$,

$$\int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta})^2 dz = \sum_{i=1}^{3} \int_{\mathbb{R}^2} (m_z^{\varepsilon,\delta,i})^2 dz + 2 \sum_{1 \leq i < j \leq 3} \int_{\mathbb{R}^2} m_z^{\varepsilon,\delta,i} m_z^{\varepsilon,\delta,j} dz \tag{12}$$

in both M- and L-cases.

Let us fix $(\varepsilon_k, k \geq 0) \in \Sigma$. A scaling argument and Proposition 1 show that in the M-case,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} (m_z^{\varepsilon_k,\delta_k,1})^2 dz = \frac{1}{6\pi} \quad \text{a.s.}$$

Since this depends only on $(Z_s, s \in I_1)$, it also holds in the L-case. Now, by symmetry, for $i \in \{1,2,3\}$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} (m_z^{\varepsilon_k,\delta_k,i})^2 dz = \frac{1}{6\pi} \quad \text{a.s.} \tag{13}$$

in the L-case.

On the other hand, (8) readily gives (using scaling, Chebyshev's inequality and Borel-Cantelli's Lemma) that,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} m_z^{\varepsilon_k,\delta_k,1} m_z^{\varepsilon_k,\delta_k,2} dz = 0 \quad \text{a.s.} \tag{14}$$

Since this depends only on $(Z_s, s \le 2/3)$, it holds also in the L-case. By symmetry, for any $i \neq j$ in $\{1,2,3\}$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} m_z^{\varepsilon_k,\delta_k,i} m_z^{\varepsilon_k,\delta_k,j} dz = 0 \quad \text{a.s.} \tag{15}$$

in the L-case. Finally (12), (13) and (15) show that

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} (m_z^{\varepsilon_k,\delta_k})^2 dz = \frac{1}{2\pi} \quad \text{a.s.} \tag{16}$$

in the L-case.

2.3 Localization

We now want to derive Theorem 1. Let us first put down some notations: Define $\mathrm{diam}(Z,I) = \sup_{(s,t)\in I^2} |Z_s - Z_t|$ and $\mathrm{diam}(\gamma,I) = \sup_{(s,t)\in I^2} |\gamma_s - \gamma_t|$. For $N > 1$, $\mathcal{E}_N^I$ (respectively $\mathcal{H}_N^I$) will denote the event $\{\mathrm{diam}(Z,I) < N\}$ (resp. $\{\mathrm{diam}(\gamma,I) < N\}$) We also keep the notations introduced in section 2.1.

Using corollary 1, one has, conditional on $\mathcal{E}_N^{[0,2/3]}$, for any compact set $A \subset \mathbb{R}^2$, for any $(\varepsilon_k, k \ge 0) \in \Sigma$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A ((n_z^{\varepsilon_k,1})^2 - (m_z^{\varepsilon_k,\delta_k,1})^2) dz = 0 \quad \text{a.s.} \tag{17}$$

and

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A \left((n_z^{\varepsilon_k,1} + n_z^{\varepsilon_k,2})^2 - (m_z^{\varepsilon_k,\delta_k,1} + m_z^{\varepsilon_k,\delta_k,2})^2\right) dz = 0 \quad \text{a.s.} \tag{18}$$

in the M-case. (17) and (18) imply (using a symmetry argument) that

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A (n_z^{\varepsilon_k,1} n_z^{\varepsilon_k,2} - m_z^{\varepsilon_k,\delta_k,1} m_z^{\varepsilon_k,\delta_k,2}) dz = 0 \quad \text{a.s.}, \tag{19}$$

in the M-case, conditional on $\mathcal{E}_N^{[0,2/3]}$.

Since (17), (18) and (19) depend only on $(Z_s, s \leq 2/3)$, they also hold in the L-case (conditional on $\mathcal{H}_N^{[0,2/3]}$). As $\mathcal{H}_N^{[0,1]} \subset \mathcal{H}_N^{[0,2/3]}$ and $P(\mathcal{H}_N^{[0,1]}) > 0$, (17), (18) and (19) also hold conditional on $\mathcal{H}_N^{[0,1]}$ (in the L-case).

By symmetry, this implies that, conditional on $\mathcal{H}_N^{[0,1]}$, for any $(\varepsilon_k, k \geq 0) \in \Sigma$, for any compact set $A \subset \mathbb{R}^2$, and for all $i \neq j$ in $\{1,2,3\}$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A ((n_z^{\varepsilon_k,i})^2 - (m_z^{\varepsilon_k,\delta_k,i})^2) dz = 0 \quad \text{a.s.} \tag{20}$$

and

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A (n_z^{\varepsilon_k,i} n_z^{\varepsilon_k,j} - m_z^{\varepsilon_k,\delta_k,i} m_z^{\varepsilon_k,\delta_k,j}) dz = 0 \quad \text{a.s.} \tag{21}$$

in the L-case. Finally, as $n_z^\varepsilon = n_z^{\varepsilon,1} + n_z^{\varepsilon,2} + n_z^{\varepsilon,3}$, (20) and (21) show immediately that, conditional on $\mathcal{H}_N^{[0,1]}$, for any $(\varepsilon_k, k \geq 0) \in \Sigma$ and for any compact set $A \subset \mathbb{R}^2$

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_A ((n_z^{\varepsilon_k})^2 - (m_z^{\varepsilon_k,\delta_k})^2) dz = 0 \quad \text{a.s.} \tag{22}$$

in the L-case. For $A = \mathcal{D}(0, N+1)$, it is obvious that conditional on $\mathcal{H}_N^{[0.1]}$, $n_z^\varepsilon = m_z^{\varepsilon,\delta} = 0$ for all $\varepsilon < \delta < 1$, as soon as $z \notin A$. Hence, one can replace A by $\mathbb{R}^2$ in (22).

Finally, (22) and (16) show that, conditional on $\mathcal{H}_N^{[0,1]}$, for any $(\varepsilon_k, k \geq 0) \in \Sigma$,

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \int_{\mathbb{R}^2} (n_z^{\varepsilon_k})^2 dz = \frac{1}{2\pi} \quad \text{a.s.}$$

in the L-case. This implies that, conditional on $\mathcal{H}_N^{[0,1]}$,

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^2} (n_z^\varepsilon)^2 dz = \frac{1}{2\pi} \quad \text{in Probability}$$

(since for every sequence $\varepsilon_k \to 0$, there exists a subsequence in Σ for which almost sure convergence holds). Finally, as $\lim_{N\to\infty} P(\mathcal{H}_N^{[0,1]}) = 1$, Theorem 1 follows.

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