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KIYOSHI KAWAZU

HIROSHI TANAKA

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# On the maximum of a diffusion process in a drifted Brownian environment

KIYOSHI KAWAZU      AND      HIROSHI TANAKA

## 1. Introduction

In this paper we investigate asymptotic behavior of the tail of the distribution of the maximum of a diffusion process in a drifted Brownian environment. This problem is a diffusion analogue of the Afanas'ev problem([1]). Our result is naturally compatible with that of Afanas'ev[1].

Let  $\{W(x), x \in \mathbf{R}, P\}$  be a Brownian environment, namely, let  $\{W(t), t \geq 0, P\}$  and  $\{W(-t), t \geq 0, P\}$  be independent Brownian motions in one-dimension with  $W(0) = 0$ . We consider a diffusion process  $X(t, W)$  defined formally by

$$X(t, W) = \text{Brownian motion} - \frac{1}{2} \int_0^t \{W'(X(s, W)) + c\} ds,$$

where  $c$  is a positive constant. The precise meaning of  $X(t, W)$  is simply a diffusion process with generator

$$\frac{1}{2} e^{W(x)+cx} \frac{d}{dx} \left( e^{-W(x)-cx} \frac{d}{dx} \right),$$

starting at 0. Such a diffusion process can be constructed from a Brownian motion through changes of scale and time. For a fixed environment  $W = (W(x), x \in \mathbf{R})$  we denote by  $P_W$  the probability law of the process  $\{X(t, W)\}$  and put

$$\mathcal{P} = \int P(dW) P_W.$$

Thus  $\mathcal{P}$  is the full law of  $\{X(t, \cdot)\}$ . We often write  $X(t) = X(t, \cdot)$ . Since  $c > 0$ ,  $\max_{t \geq 0} X(t)$  is finite ( $\mathcal{P}$ -a.s.). The problem is the following : How fast does  $\mathcal{P}\{\max_{t \geq 0} X(t) > x\}$  decay as  $x \rightarrow \infty$  ? Since

$$(1.1) \quad \mathcal{P}\{\max_{t \geq 0} X(t) > x\} = E\{A(A+B)^{-1}\},$$

where

$$(1.2) \quad A = \int_{-\infty}^0 e^{W(t)+ct} dt, \quad B = \int_0^x e^{W(t)+ct} dt,$$

the problem is nothing but to find the asymptotics of  $E\{A(A+B)^{-1}\}$  as  $x \rightarrow \infty$ . The result varies according as  $c > 1$ ,  $c = 1$ ,  $0 < c < 1$ , as will be stated in the following theorem.

THEOREM. (i) If  $c > 1$ , then

$$\mathcal{P}\{\max_{t \geq 0} X(t) > x\} \sim \frac{2c-2}{2c-1} \exp\left\{-\left(c-\frac{1}{2}\right)x\right\}, \quad x \rightarrow \infty.$$

(ii) If  $c = 1$ , then

$$\mathcal{P}\{\max_{t \geq 0} X(t) > x\} \sim (2/\pi)^{1/2} x^{-1/2} \exp\{-x/2\}, \quad x \rightarrow \infty.$$

(iii) If  $0 < c < 1$ , then

$$\mathcal{P}\{\max_{t \geq 0} X(t) > x\} \sim \text{const.} x^{-3/2} \exp\{-c^2 x/2\}, \quad x \rightarrow \infty,$$

where

$$\text{const.} = 2^{5/2-2c} \Gamma(2c)^{-1} \int_0^\infty \int_0^\infty \int_0^\infty z(a+z)^{-1} a^{2c-1} e^{-a/2} y^{2c} e^{-\lambda z} u \sinh u \, da \, dy \, dz \, du,$$

$$\lambda = (1+y^2)/2 + y \cosh u.$$

## 2. Proof of the theorem

Since A and B are independent, the right hand side of (1.1) equals  $E\{Af(A)\}$  where  $f(a) = E\{(a+B)^{-1}\}$ ,  $a \geq 0$ . Fixing  $x > 0$ , we consider the time reversal  $\widehat{W}(t) = W(x-t) - W(x)$ ,  $0 \leq t \leq x$ . Since  $\{\widehat{W}(t), 0 \leq t \leq x\}$  is also a Brownian motion, we have

$$(2.1) \quad \begin{aligned} f(a) &= E\left\{\left(a + \int_0^x \exp\{\widehat{W}(t) + ct\} dt\right)^{-1}\right\} \\ &= E\left\{\left(a + e^{-W(x)} \int_0^x \exp\{W(x-t) + ct\} dt\right)^{-1}\right\} \\ &= E\left\{\left(ae^{W(x)-cx} + \int_0^x e^{W(t)-ct} dt\right)^{-1} e^{W(x)-cx}\right\} \\ &= e^{(1/2-c)x} E\left\{\left(ae^{W(x)-cx} + \int_0^x e^{W(t)-ct} dt\right)^{-1} e^{W(x)-x/2}\right\} \\ &= e^{(1/2-c)x} E\left\{\left(ae^{W(x)-(c-1)x} + \int_0^x e^{W(t)-(c-1)t} dt\right)^{-1}\right\} \\ &= e^{(1/2-c)x} E\left\{\left(a + \int_0^x e^{W(t)+(c-1)t} dt\right)^{-1} e^{W(x)+(c-1)x}\right\}. \end{aligned}$$

In deriving the fifth equality in the above we used the formula of Cameron-Martin-Maruyama-Girsanov; the last equality was derived by using  $\widehat{W}(t)$  as in the case of the first equality. From the fifth equality of (2.1) we obtain the following lemma.

LEMMA 1. For any  $c > 0$  and  $x > 0$

$$(2.2) \quad \mathcal{P}\{\max_{t \geq 0} X(t) > x\} = e^{(1/2-c)x} E\{A(Ae^{W(x)-(c-1)x} + \int_0^x e^{W(t)-(c-1)t} dt)^{-1}\},$$

where  $A$  is given by (1.2).

The following lemma due to Yor will also be used.

LEMMA 2 (Yor[2]). For any  $\nu > 0$  we have

$$(2.3) \quad \int_0^\infty \exp(W(t) - \frac{\nu t}{2}) dt \stackrel{d}{=} 2/Z_\nu,$$

where  $\stackrel{d}{=}$  means equality in distribution and  $Z_\nu$  is a gamma variable of index  $\nu$ , that is,

$$P\{Z_\nu \in dt\} = \Gamma(\nu)^{-1} t^{\nu-1} e^{-t} dt \quad (t > 0).$$

### 2.1. Proof of (i)

When  $c > 1$ , Lemma 1 implies

$$\lim_{x \rightarrow \infty} e^{-(1/2-c)x} \mathcal{P}\{\max_{t \geq 0} X(t) > x\} = E\{A(\int_0^\infty e^{W(t)-(c-1)t} dt)^{-1}\}.$$

It is easy to see that the above expectation is finite. To obtain its exact value we use Lemma 2. We thus obtain (i).

### 2.2. Proof of (ii)

For  $x > 0$  we put

$$\varphi(x) = E\{\log \int_0^x e^{W(t)} dt\}, \quad \psi(x) = \frac{d}{dx} \varphi(x).$$

Then it is easy to see that

$$\psi(x) = E\{(\int_0^x e^{W(t)} dt)^{-1} e^{W(x)}\} = E\{(\int_0^x e^{W(t)} dt)^{-1}\};$$

in fact, the second equality is a consequence of the last equality of (2.1) with  $a = 0$  and  $c = 1$ . Thus  $\psi(x)$  is monotone decreasing in  $x$ .

LEMMA 3. When  $c = 1$ , we have

$$(2.4) \quad E\{A(\int_0^x e^{W(t)+t} dt)^{-1}\} \sim \sqrt{2/\pi} x^{-1/2} e^{-x/2} \text{ as } x \rightarrow \infty.$$

Proof. Since  $E\{A\} = 2$  in case  $c = 1$ , the left hand side of (2.4) equals  $2E\{(\int_0^x e^{W(t)+t} dt)^{-1}\}$  which also equals  $2e^{-x/2} E\{(\int_0^x e^{W(t)} dt)^{-1} e^{W(x)}\}$  by virtue of (2.1) with  $a = 0$  and  $c = 1$ .

Thus we have

$$(2.5) \quad E\{A(\int_0^x e^{W(t)+t} dt)^{-1}\} = 2e^{-x/2} \psi(x).$$

On the other hand, using the scaling property  $\{W(t)\} \stackrel{d}{=} \{\sqrt{x}W(t/x)\}$  we have

$$\varphi(x) = E\{\log \int_0^1 e^{\sqrt{x}W(t)} dt\} + \log x,$$

and hence

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{-1/2} \varphi(x) &= \lim_{x \rightarrow \infty} E\{\frac{1}{\sqrt{x}} \log \int_0^1 e^{\sqrt{x}W(t)} dt\} \\ &= E\{\max_{0 \leq t \leq 1} W(t)\} = \sqrt{2/\pi}, \end{aligned}$$

which combined with the monotonicity of  $\psi(x) = \varphi'(x)$  implies

$$(2.6) \quad \psi(x) \sim (2\pi x)^{-1/2} \quad \text{as } x \rightarrow \infty.$$

This together with (2.5) proves the lemma.

LEMMA 4. For  $x > 0$  we have

$$(2.7) \quad E\{(\int_0^x e^{W(t)} dt)^{-2} e^{W(x)}\} \leq \psi(x/2)^2.$$

Proof. The left hand side of (2.7) is dominated by

$$\begin{aligned} &E\{(\int_0^{x/2} e^{W(t)} dt)^{-1} (\int_{x/2}^x e^{W(t)} dt)^{-1} e^{W(x)}\} \\ &= E\{(\int_0^{x/2} e^{W(t)} dt)^{-1} (\int_{x/2}^x e^{W(t)-W(x/2)} dt)^{-1} e^{W(x)-W(x/2)}\} \\ &= E\{(\int_0^{x/2} e^{W(t)} dt)^{-1}\} E\{(\int_0^{x/2} e^{W(t)} dt)^{-1} e^{W(x/2)}\} \\ &= \psi(x/2)^2; \end{aligned}$$

in deriving the second equality in the above we used the fact that  $\{W(t + \frac{x}{2}) - W(\frac{x}{2}), t \geq 0\}$  is a Brownian motion independent of  $\{W(t), 0 \leq t \leq x/2\}$ .

The proof of (ii) is now given as follows. By (1.1) we have

$$\begin{aligned} (2.8) \quad 0 &\leq E\{A(\int_0^x e^{W(t)+t} dt)^{-1}\} - \mathcal{P}\{\max_{t \geq 0} X(t) > x\} \\ &= E\{AB^{-1} - A(A+B)^{-1}\} \\ &\leq E\{2^{-1}A^{3/2}B^{-3/2}\} = 2^{-1}E\{A^{3/2}\}E\{B^{-3/2}\}. \end{aligned}$$

We prove

$$(2.9) \quad E\{A^{3/2}\} < \infty,$$

$$(2.10) \quad E\{B^{-3/2}\} < \text{const. } x^{-3/4} e^{-x/2}.$$

(2.9) follows immediately from Lemma 2 ; a direct proof can also be given as follows. Using Hölder's inequality we have

$$\begin{aligned} E\{A^{3/2}\} &= E\left\{\left(\int_0^\infty e^{W(t)-4t/5} e^{-t/5} dt\right)^{3/2}\right\} \\ &\leq (5/3)^{1/2} E\left\{\int_0^\infty \exp\left\{\frac{3}{2}\left(W(t) - \frac{4t}{5}\right)\right\} dt\right\} = (5/3)^{1/2} \cdot (40/3). \end{aligned}$$

(2.10) can be proved by making use of the CMMG formula, the Schwarz inequality, Lemma 4 and then (2.6) ; in fact, putting  $B_0 = \int_0^x e^{W(t)} dt$  we have

$$\begin{aligned} E\{B^{-3/2}\} &= E\{B_0^{-3/2} e^{W(x)-x/2}\} \\ &\leq e^{-x/2} E\{B_0^{-1} e^{W(x)}\}^{1/2} E\{B_0^{-2} e^{W(x)}\}^{1/2} \\ &\leq e^{-x/2} \psi(x)^{1/2} \psi(x/2) \\ &\leq \text{const. } e^{-x/2} x^{-1/4} \cdot x^{-1/2}. \end{aligned}$$

The assertion (ii) of our theorem follows from Lemma 3, (2.8), (2.9) and (2.10).

### 2.3. Proof of (iii)

The proof of (iii) relies essentially on the following Yor's formula.

Yor's formula([3: the formula(6.e)]). *For any bounded Borel functions  $f$  and  $g$  we have*

$$\begin{aligned} &E\left\{f\left(\int_0^t e^{2W(s)} ds\right)g(e^{W(t)})\right\} \\ &= c_t \int_0^\infty dy \int_0^\infty dz g(y)f(1/z) \exp\{-z(1+y^2)/2\} \psi_{yz}(t), \end{aligned}$$

where

$$\begin{aligned} c_t &= (2\pi^2 t)^{-1/2} \exp\{\pi^2/2t\}, \\ \psi_r(t) &= \int_0^\infty \exp\{-u^2/2t\} e^{-r(\cosh u)} (\sinh u) \sin(\pi u/t) du. \end{aligned}$$

To proceed to the proof of (iii) we put

$$f(a, z) = a(a + 4z)^{-1}, \quad g(y) = y^{2c},$$

$$B^{(\nu)}(t) = \int_0^t e^{2(W(s)+\nu s)} ds.$$

Using first the CMMG formula and then Yor's formula we have

$$\begin{aligned} &E\left\{a\left(a + \int_0^x e^{W(t)+ct} dt\right)^{-1}\right\} = E\left\{a\left(a + 4B^{(2c)}(x/4)\right)^{-1}\right\} \\ &= E\left\{a\left(a + 4B^{(0)}(x/4)\right)^{-1} \exp\left(2cW(x/4) - \frac{c^2 x}{2}\right)\right\} \\ &= \exp(-c^2 x/2) E\left\{f\left(a, B^{(0)}(x/4)\right)g\left(e^{W(x/4)}\right)\right\} \\ &= \exp(-c^2 x/2) c_{x/4} \int_0^\infty dy \int_0^\infty dz g(y)f(a, 1/z) \exp\{-z(1+y^2)/2\} \psi_{yz}(x/4). \end{aligned}$$

Since Lemma 2 implies

$$P\{A \in da\} = 2^{2c}\Gamma(2c)^{-1}a^{-2c-1}e^{-2/a}da \quad (a > 0),$$

we have

$$(2.11) \quad \begin{aligned} & \mathcal{P}\{\max_{t \geq 0} X(t) > x\} \\ &= 2^{2c+1/2}\Gamma(2c)^{-1}\pi^{-1} \exp(2\pi^2/x)x^{-1/2} \exp(-c^2x/2) \\ & \quad \times \int_0^\infty dy \int_0^\infty dz \int_0^\infty du y^{2c}h(z)e^{-\lambda z} \exp(-2u^2/x)(\sinh u) \sin(4\pi u/x), \end{aligned}$$

where

$$\begin{aligned} h(z) &= \int_0^\infty az(az+4)^{-1}a^{-2c-1}e^{-2/a}da, \\ \lambda &= (1+y^2)/2 + y \cosh u. \end{aligned}$$

LEMMA 5. Let  $0 < c < 1$  and put

$$F(y, z, u) = y^{2c}h(z)e^{-\lambda z}u \sinh u.$$

Then we have

$$M = \int_0^\infty \int_0^\infty \int_0^\infty F(y, z, u) dy dz du < \infty,$$

Proof. By a change of variable  $\cosh u = v$ , we have

$$M = \int_0^\infty dy \int_0^\infty dz \int_1^\infty dv y^{2c}h(z)e^{-\lambda z} \log(v + \sqrt{v^2 - 1}),$$

where  $\lambda = (1 + y^2)/2 + yv$ . Since

$$h(z) = 2^{-2c-1}z \int_0^\infty u^{2c-1}e^{-u}(u + \frac{z}{2})^{-1}du,$$

it is easy to see that

$$(2.12) \quad h(z) \longrightarrow 2^{-2c}\Gamma(2c) \quad \text{as } z \rightarrow \infty,$$

$$(2.13) \quad h(z) \sim_{\text{as } z \downarrow 0} \begin{cases} 2^{-2c-1}\Gamma(2c-1)z & \text{if } c > 1/2, \\ 2^{-2}z \log 1/z & \text{if } c = 1/2, \\ 2^{-4c} \int_0^\infty a^{2c-1}(a+1)^{-1}da \cdot z^{2c} & \text{if } 0 < c < 1/2. \end{cases}$$

Therefore for any  $\epsilon > 0$  and  $\alpha > 0$  we have

$$\begin{aligned} M_1 &= \int_0^\infty dy \int_1^\infty dz \int_1^\infty dv y^{2c}h(z)e^{-\lambda z} \log(v + \sqrt{v^2 - 1}) \\ &\leq \text{const.} \int_0^\infty \int_1^\infty y^{2c}v^\epsilon \lambda^{-1}e^{-\lambda} dy dv \\ &\leq \text{const.} \int_0^\infty \int_1^\infty y^{2c}v^\epsilon \lambda^{-\alpha} dy dv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-\epsilon-1}(1+y^2)^{-\alpha+1+\epsilon} z^\epsilon (1+z)^{-\alpha} dy dz \end{aligned}$$

(by putting  $v = (2y)^{-1}(1 + y^2)z$  with  $y$  fixed ),

which is finite if  $\varepsilon > 0$  is sufficiently small and  $\alpha > 0$  sufficiently large. Note that const. in the above may vary from place to place and depend on  $\varepsilon$  and  $\alpha$  . Next we prove that

$$(2.14) \quad M_2 = \int_0^\infty dy \int_0^1 dz \int_1^\infty dv y^{2c} h(z) e^{-\lambda z} \log(v + \sqrt{v^2 - 1}) < \infty .$$

Assume  $1/2 < c < 1$  . Then by (2.13)

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv y^{2c} z e^{-\lambda z} v^\varepsilon \\ &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-2} y^{2c} v^\varepsilon dy dv \quad (\text{we used } \int_0^1 z e^{-\lambda z} dz \leq \lambda^{-2} ) \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-1-\varepsilon} (1 + y^2)^{-1+\varepsilon} z^\varepsilon (1 + z)^{-2} dy dz \end{aligned}$$

(by putting  $v = (2y)^{-1}(1 + y^2)z$  with  $y$  fixed )

which is finite for sufficiently small  $\varepsilon > 0$  by virtue of  $1/2 < c < 1$  . When  $c = 1/2$  , (2.13) implies

$$M_2 \leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv y z^{1-\varepsilon} e^{-\lambda z} v^\varepsilon$$

for  $0 < \varepsilon < 1$  . Since  $\int_0^1 z^{1-\varepsilon} e^{-\lambda z} dz \leq \text{const.} \lambda^{-2+\varepsilon}$  , we have

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-2+\varepsilon} y v^\varepsilon dy dv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{-\varepsilon} (1 + y^2)^{-1+2\varepsilon} z^\varepsilon (1 + z)^{-2+\varepsilon} dy dz < \infty \end{aligned}$$

provided that  $\varepsilon > 0$  is small enough. Finally assume  $0 < c < 1/2$  . Then by (2.13)

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv y^{2c} z^{2c} e^{-\lambda z} v^\varepsilon \\ &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-1-2c} y^{2c} v^\varepsilon dy dv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-\varepsilon-1} (1 + y^2)^{-2c+\varepsilon} z^\varepsilon (1 + z)^{-1-2c} dy dz < \infty \end{aligned}$$

provided that  $\varepsilon > 0$  is small enough. Thus (2.14) is proved.

We can now complete the proof of (iii) as follows. From (2.11) we have

$$(2.15) \quad \mathcal{P}\{\max_{t \geq 0} X(t) > x\} = 2^{2c+5/2} \Gamma(2c)^{-1} \exp(2\pi^2/x) x^{-3/2} \exp(-c^2 x/2) M(x) ,$$

where

$$M(x) = \int_0^\infty \int_0^\infty \int_0^\infty F(y, z, u) \sin(4\pi u/x) / (4\pi u/x) \exp(-2u^2/x) dy dz du .$$



By Lemma 5 we have  $\lim_{x \rightarrow \infty} M(x) = M$  which equals

$$2^{-4c} \int_0^\infty \int_0^\infty \int_0^\infty z(a+z)^{-1} a^{2c-1} e^{-a/2} y^{2c} e^{-\lambda z} u \sinh u \, da \, dy \, dz \, du .$$

Thus the assertion (iii) follows from (2.15).

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Kiyoshi Kawazu  
 Department of Mathematics  
 Faculty of Education  
 Yamaguchi University  
 Yosida, Yamaguchi 753  
 Japan

Hiroshi Tanaka  
 Department of Mathematics  
 Faculty of Science and Technology  
 Keio University  
 Hiyoshi, Yokohama 223  
 Japan