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Principle of Superposition and Interference of Diffusion Processes[1]

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1. Superposition Principle in Quantum Mechanics

Let ψ_k be arbitrary wave functions, then

$$\psi = \sum \alpha_k \psi_k \tag{1.1}$$

defines a new wave function. This is the so-called superposition principle of states ψ_k in quantum mechanics. It is nothing but the linearity of the space of wave functions. On the other hand it is claimed in quantum theory that

$$|\psi|^2 = \psi\overline{\psi} \tag{1.2}$$

is a probability distribution density. This gives no problem to probabilists. However, if formula (1.2) is combined with the superposition principle (1.1), then it turns out to be a serious (almost unsolvable) problem in probability theory.

To make things clear let us consider the simplest case of two wave functions

$$\psi_1 + \psi_2, \tag{1.3}$$

where we neglect "normalization" for simplicity. Then

$$|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\mathcal{R}e(\psi_1\overline{\psi}_2). \tag{1.4}$$

The real part $\mathcal{R}e(\psi_1\overline{\psi}_2)$ of cross terms is called "interference" of the wave functions ψ_1 and ψ_2. Probabilists have found no mathematical structure in probability theory providing such, and hence this has been a long standing open problem in probability theory since 1926.

[1] Lecture given at the third European Symposium on Analysis and Probability held at the Henri Poincaré Institut on Jan. 6, 1992

If a probabilist were brave enough, he could have done the following:

(i) He gives nice names to a linear space of real valued functions ϕ, and defines a kind of its "dual space" of functions $\widehat{\phi}$;

(ii) claims

$$\phi\widehat{\phi} \tag{1.5}$$

is a probability distribution density; and

(iii) proposes to use real valued functions ϕ and $\widehat{\phi}$ instead of complex valued functions ψ and $\overline{\psi}$. If do so, formula (1.4) turns out to be

$$(\phi_1 + \phi_2)(\widehat{\phi}_1 + \widehat{\phi}_2) = \phi_1\widehat{\phi}_1 + \phi_2\widehat{\phi}_2 + (\phi_1\widehat{\phi}_2 + \phi_2\widehat{\phi}_1). \tag{1.6}$$

Then, finally:

(iv) He calls the term $\phi_1\widehat{\phi}_2 + \phi_2\widehat{\phi}_1$ in (1.6) "interference" or more probably "correlation" of ϕ_1 and ϕ_2.

Then probabilists could have been liberated from uneasy feeling against quantum theory.

2. A Diffusion Theory

Let us consider a diffusion equation

$$\frac{\partial u}{\partial t}(t,x) + Lu(t,x) = 0, \tag{2.1}$$

on $[a,b]\times\mathbf{R}^d$, $-\infty < a < b < \infty$, where

$$L = \frac{1}{2}\Delta + b(t,x)\cdot\nabla \tag{2.2}$$

$$\Delta = \frac{1}{\sqrt{\sigma_2(t,x)}}\frac{\partial}{\partial x^i}\left(\sqrt{\sigma_2(t,x)}\,\sigma^T\sigma^{ij}(t,x)\frac{\partial}{\partial x^j}\right) \tag{2.3}$$

with $\{(\sigma^T\sigma)^{ij}(t,x)\}$ which is positive definite diffusion coefficient, and $\sigma_2(t,x) = |(\sigma^T\sigma)_{ij}(t,x)|$. $b(t,x)$ is a drift vector satisfying a gauge condition div $b = 0$.

We assume the existence of space-time diffusion processes $\{(t,X_t)\colon \mathrm{P}_{(s,x)}, (s,x) \in [a,b]\times\mathbf{R}^d\}$ determined by the diffusion equation given in (2.1), requiring necessary conditions on the coefficients. The existence of $\mathrm{P}_{(s,x)}$ is not of our interest. Our main concern will be on diffusion processes with additional singular drift, the existence of which is not at all evident.

Let us define, in terms of the parabolic differential operator L,

(2.4) $$c(t,x) = -\frac{L\phi}{\phi}(t,x),$$

on $D = \{(t,x)\colon \phi(t,x) \neq 0\}$ for an arbitrary real valued function[2] $\phi(t,x)$, and call the function $c(t,x)$ the **creation and killing** induced by the function $\phi(t,x)$. The function $\phi(t,x)$ may take negative values, but for simplicity we assume it is non-negative.

As the naming itself indicates already, we consider a diffusion equation

(2.5) $$Lp(t,x) + c(t,x)p(t,x) = 0,$$

with the creation and killing $c(t,x)$, which is singular at the zero set of $\phi(t,x)$, in general, as (2.4) shows. Therefore, the existence of the fundamental solution $p(s,x;t,y)$ of diffusion equation (2.5) is a non-trivial problem. Actually we can solve the existence problem applying a transformation in terms of a multiplicative functional, as will be seen.

Since the function ϕ is p-harmonic, namely, it satisfies

(2.6) $$L\phi(t,x) + c(t,x)\phi(t,x) = 0,$$

(this is trivial, because of definition (2.4)), if we define

(2.7) $$q(s,x;t,y) = \frac{1}{\phi(s,x)} p(s,x;t,y)\phi(t,y),$$

on a subset D of $[a,b]\times\mathbf{R}^{\mathrm{d}}$

(2.8) $$D = \{(s,x)\colon \phi(s,x) \neq 0\},$$

then $q(s,x;t,y)$ is a transition probability density on the subset D.

Taking another arbitrary non-negative function $\hat{\phi}(a,x)$ such that

(2.9) $$\int dx\hat{\phi}(a,x)\phi(a,x) = 1,^{3}$$

we consider a diffusion process $\{(t,X_t), Q\}$ on D with the initial distribution density $\hat{\phi}(a,x)\phi(a,x)$ and the transition probability density $q(s,x;t,y)$.

[2] We consider sufficiently smooth bounded functions

[3] dx denotes the volume element

To construct such diffusion processes we apply the following theorem on a transformation of singular drift. Then, we find that the diffusion process $\{X_t, Q\}$ constructed has an additional drift term

$$a(t,x) = \sigma^T\sigma(t,x)\,\nabla\log\phi(t,x), \tag{2.10}$$

which is singular at the boundary ∂D.

Theorem 2.1. (Nagasawa (1989)) *Require*

$$\mathrm{P}_{(s,x)}[\exp(\int_s^b c(r,X_r)dr)1_{\{b<T_s\}}] < \infty. \tag{2.11}$$

Then, (i) *a multiplicative functional* N_s^t *defined by*

$$N_s^t = \exp(-\int_s^t \frac{L\phi}{\phi}(r,X_r)dr)\,\frac{\phi(t,X_t)}{\phi(s,X_s)}1_{\{t<T_s\}} \tag{2.12}$$

satisfies

$$\mathrm{P}_{(s,x)}[N_s^t] = 1, \quad \textit{for} \ \forall\,(s,x)\in D. \tag{2.13}$$

(ii) *The transformed diffusion process* $\mathrm{Q}_{(s,x)} = N_s^b\mathrm{P}_{(s,x)}$, $\forall\,(s,x)\in D$, *has an additional drift term* $a(t,x) = \sigma^T\sigma\nabla\log\phi(t,x)$.

(iii) *The space-time diffusion process* $\{(t,X_t), \mathrm{Q}_{(s,x)}; (s,x)\in D\}$ *does not hit the zero set N of* ϕ.[4]

Remark. The diffusion process constructed in the above theorem corresponds to the diffusion equation

$$Bq(t,x) = 0, \quad \textit{in } D, \tag{2.14}$$

where B is a time-dependent parabolic differential operator

$$B = B(t) = \frac{\partial}{\partial t} + \frac{1}{2}\Delta + \{b(t,x) + a(t,x)\}\cdot\nabla. \tag{2.15}$$

The drift coefficient $b(t,x)$ is regular, while $a(t,x)$ is so singular that the Novikov or Kazamaki condition cannot be applied. Thus we need a theorem such as Theorem 2.1. For related subject cf. references in Nagasawa (89, 90, Monograph).

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[4] Cf. also Nagasawa (1990), Aebi-Nagasawa (1992) for another methods based on variational principle and large deviation

In terms of the diffusion process $\{(t,X_t), Q_{(s,x)}; (s,x)\in D\}$ given in Theorem 2.1 we put

$$Q[\cdot] = \int dx\widehat{\phi}(a,x)\phi(a,x)Q_{(a,x)}[\cdot] \tag{2.16}$$

The finite dimensional distributions of the diffusion process $\{X_t, Q\}$ is given by

(2.17) $Q[f(X_{t_0},X_{t_1},\dots,X_{t_n})]$

$$= \int dx_0\widehat{\phi}_a(x_0)\phi_a(x_0)\frac{1}{\phi_a(x_0)}p(a,x_0;t_1,x_1)\phi_{t_1}(x_1)dx_1\frac{1}{\phi_{t_1}(x_1)}p(t_1,x_1;t_2,x_2)\times$$

$$\times\phi_{t_2}(x_2)dx_2\cdots\frac{1}{\phi_{t_{n-1}}(x_{n-1})}p(t_{n-1},x_{n-1};b,x_n)\phi_b(x_n)dx_n f(x_0,x_1,\dots,x_n),$$

where we denote $\widehat{\phi}_a(x) = \widehat{\phi}(a,x)$ and $\phi_t(x) = \phi(t,x)$. Then, it is clear that formula (2.17) turns out to be

(2.18) $Q[f(X_{t_0},X_{t_1},\dots,X_{t_n})]$

$$= \int dx_0\widehat{\phi}(a,x_0)p(a,x_0;t_1,x_1)dx_1p(t_1,x_1;t_2,x_2)dx_2\cdots$$

$$\cdots p(t_{n-1},x_{n-1};b,x_n)\phi(b,x_n)dx_n f(x_0,\dots,x_n),$$

where $a < t_1 < \cdots < t_{n-1} < b$.

Adopting formula (2.18) instead of (2.17) was one of genius ideas of Schrödinger (1931).

We will call the probability measure given on the right-hand side of formula (1.18) **Schrödinger's representation** (or **p-representation**) with an **entrance-exit law** $\{\widehat{\phi}_a, \phi_b\}$ of the diffusion process $\{X_t, Q\}$, and denote it symbolically as

$$Q = [\,\widehat{\phi}_a p \gg\ll p\,\phi_b\,], \tag{2.19}$$

while we denote the probability measure given on the right-hand side of formula (2.17) and its time-reversal as $[\,\widehat{\phi}_a\phi_a q\gg$ and $\ll\widehat{q}\,\widehat{\phi}_b\phi_b\,]$, respectively, and call them **Kolmogorov's representation** (**q-representation**) and **time reversed $\widehat{q}$-representation.**

Therefore, we have

Theorem 2.2.[5] *A diffusion process Q has three representations*

$$Q = [\hat{\phi}_a \phi_a q >> \tag{2.20}$$

$$= << \hat{q} \hat{\phi}_b \phi_b]$$

$$= [\hat{\phi}_a p >> << p \, \phi_b],$$

where

$$\hat{q}(s,x;t,y) = \hat{\phi}(s,x) p(s,x;t,y) \frac{1}{\hat{\phi}(t,y)}, \tag{2.21}$$

with

$$\hat{\phi}(t,x) = \int \hat{\phi}(a,z) dz p(a,z;t,x). \tag{2.22}$$

Moreover,

$$Q[f(X_t)] = \int dx \hat{\phi}(t,x) \phi(t,x) f(x). \tag{2.23}$$

Namely, the distribution density $\mu_t(x)$ of $\{X_t, Q\}$ is given by

$$\mu_t(x) = \hat{\phi}(t,x) \phi(t,x). \tag{2.24}$$

Formula (2.24) is exactly (1.5)! We will call (2.24) **Schrödinger's factorization** of the distribution density of the diffusion process $\{X_t, Q\}$.

The q-representation belongs to the real world, since it describes the real observable evolution of the distribution density of a diffusion process. It should be remarked that the p-representation, in contrast, describes an evolution in an "fictitious" world, because p does not concern the probability distribution of the diffusion process directly, but it describes the evolution of an entrance-exit law $\{\hat{\phi}, \phi\}$. This is exactly the real-valued counterpart of the relation between wave functions $\{\psi, \overline{\psi}\}$ and the product $\psi\overline{\psi}$ in quantum theory.

We can identify the two distribution densities

$$\psi(t,x) \overline{\psi}(t,x) = \hat{\phi}(t,x) \phi(t,x). \tag{2.25}$$

This identification of the products is enough to construct a diffusion process (see the following corollaries), but in addition we need to establish the equivalence of the Schrödinger equation and a pair of diffusion equations. This solves Schrödinger's conjecture, which will be explained in the next section.

We formulate simple corollaries of Theorem 2.1.

[5] Cf. Nagasawa (1961, 1964, monograph) for time reversal and the duality of diffusion processes

Corollary 2.1. *Let $\mu_t, t \in [a,b]$, be a flow of non-negative distribution densities and set*

$$\phi_t(x) = \sqrt{\mu_t(x)}\, e^{S(t,x)},$$
(2.26)
$$\hat{\phi}_t(x) = \sqrt{\mu_t(x)}\, e^{-S(t,x)}.$$

Let $\phi_t \in C^{1,2}(D)$ (or $H^{1,2}(D)$), where $D = \{(t,x): \phi(t,x) \neq 0\}$ and assume the integrability condition (2.11) for the creation and killing $c = -L\phi_t/\phi_t$ induced by ϕ_t. Then, there exist a space-time diffusion process $\{(t,X_t), Q_{(s,x)}; (s,x) \in D\}$ with an additional drift coefficient $\boldsymbol{a}(t,x) = \sigma^T\sigma\nabla\log\phi_t(x)$.

The same arguments applied to $\hat{\phi}_t$ with $\hat{c} = -\hat{L}\hat{\phi}_t/\hat{\phi}_t$ implies the existence of a space-time diffusion process (in reversed time) $\{(s,X_s), \hat{Q}_{(t,x)}; (t,x) \in \hat{D}\}$ with an additional drift term $\hat{\boldsymbol{a}}(t,x) = \sigma^T\sigma\nabla\log\hat{\phi}_t(x)$.

As is seen in Corollary 2.1, a flow $\mu_t, t \in [a,b]$ does not determine a diffusion process uniquely, since we can choose a function $S(t,x)$ freely. To specify a distribution of the process $\{X_t, Q_{\mu_a}\}$ (resp. $\{X_t, \hat{Q}_{\mu_b}\}$) we need an additional requirement, as will be explained in the following corollaries.

Corollary 2.2. *Keep the assumptions of Corollary 2.1. Then:*
(i) *The distribution density of the process $\{(t,X_t), Q_{\mu_a}\}$, where*

$$Q_{\mu_a}[\cdot] = \int \mu_a(x)dx\, Q_{(a,x)}[\cdot], \tag{2.27}$$

coincide with the given flow $\mu_t, t \in [a,b]$, if and only if, with

$$\frac{\partial R}{\partial t} + \frac{1}{2}\Delta S + (\sigma\nabla S)\cdot(\sigma\nabla R) + \boldsymbol{b}\cdot\nabla R = 0, \tag{2.28}$$

where $R = \frac{1}{2}\log\mu_t$.

(ii) *The diffusion processes with the additional drift coefficients $\boldsymbol{a} = \sigma^T\sigma\nabla\log\phi_t$ and $\hat{\boldsymbol{a}} = \sigma^T\sigma\nabla\log\hat{\phi}_t$, respectively, are time reversal of each other (in duality with respect to the given μ_t), namely, Q_{μ_a} is equal to*

$$[\mu_a q >> = << \hat{q}\mu_b],$$

in the q-representation, which can be represented also in the p-representation as

$$[\hat{\phi}_a p >><< p\,\phi_b].$$

Proof. The two diffusion processes are in duality with respect to $\mu_t(x)$, if and only if Fokker-Planck's equations

$$B^{\circ}\mu = -\partial\mu/\partial t + \frac{1}{2}\Delta\mu - \frac{1}{\sqrt{\sigma_2}}\nabla(\sqrt{\sigma_2}\{b(t,x) + a(t,x))\mu\} = 0,$$

$$(\widehat{B})^{\circ}\mu = \partial\mu/\partial t + \frac{1}{2}\Delta\mu - \frac{1}{\sqrt{\sigma_2}}\nabla\{\sqrt{\sigma_2}(-b(t,x) + \widehat{a}(t,x))\mu\} = 0,$$

hold, and hence after subtracting one from another we get (2.28).

Corollary 2.3. *Keep the assumptions of Corollary* 2.1. *If the given flow* $\mu_t, t \in [a,b]$, *satisfies the Fokker-Planck equation* $B^{\circ}\mu = 0$, *then the distribution density of the process* $\{(t,X_t), Q_{\mu_a}\}$, *where* Q_{μ_a} *is defined in* (2.27), *coincides with the given flow.*

Proof. $\phi_t(x)$ can be determined through $a(t,x) = \sigma^T\sigma\nabla\log\phi_t(x)$ (and hence $S(t,x)$ by the first equation of (2.26)), where we assume $a(t,x)$ is given.

Remark. A typical example of a flow is given by $\mu_t = \overline{\psi}_t\psi_t$, where ψ_t is a normalized wave function.

3. Schrödinger's conjecture

Based on formula (2.25) Schrödinger (1931, 32) was convinced that diffusion theory must provide better understanding of quantum mechanics.

Let us call this Schrödinger's conjecture, more precisely,

Schrödinger's conjecture: *Quantum mechanics is a diffusion theory.*

His conjecture was shown to be correct, because we can prove the following (cf. Nagasawa (1989, 91, monograph)):

We assume a gauge condition

(3.1)
$$\text{div}\, b = \frac{1}{\sqrt{\sigma_2}}\nabla(\sqrt{\sigma_2}\, b) = 0.$$

Theorem 3.1. *Let* $\phi(t,x)$ *and* $\widehat{\phi}(t,x)$ *be solutions of a diffusion equation and its adjoint equation*

(3.2)
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\Delta\phi + b(t,x).\nabla\phi + c(t,x)\phi = 0,$$

$$-\frac{\partial\widehat{\phi}}{\partial t} + \frac{1}{2}\Delta\widehat{\phi} - b(t,x)\cdot\nabla\widehat{\phi} + c(t,x)\widehat{\phi} = 0,$$

respectively, and on $\{(t,x)\colon \hat\phi(t,x)\phi(t,x) > 0\}$ *set*

(3.3) $$R = \tfrac{1}{2}\log\hat\phi\phi \quad and \quad S = \tfrac{1}{2}\log\frac{\phi}{\hat\phi}.$$

Define a complex valued function $\psi(t,x)$ *by*

(3.4) $$\psi(t,x) = e^{R(t,x) + iS(t,x)}.$$

Then, the function ψ *is a solution of the Schrödinger equation*

(3.5) $$i\frac{\partial\psi}{\partial t} + \tfrac{1}{2}\Delta\psi + ib(t,x)\cdot\nabla\psi - V(t,x)\psi = 0,$$

with a potential function

(3.6) $$V(t,x) = -c(t,x) - 2\partial S/\partial t(t,x) - (\sigma\nabla S)^2(t,x) - 2b\cdot\nabla S(t,x).$$

Conversely, let ψ *be a solution of the Schrödinger equation* (3.5) *of the form*

(3.7) $$\psi(t,x) = e^{R(t,x) + iS(t,x)}.$$

Define a pair $\{\phi(t,x), \hat\phi(t,x)\}$ *of real valued functions by*

(3.8) $$\phi(t,x) = e^{R(t,x) + S(t,x)},$$

(3.9) $$\hat\phi(t,x) = e^{R(t,x) - S(t,x)}.$$

Then, the functions ϕ and $\hat\phi$ *are solutions of diffusion equations in* (3.2), *where the creation and killing* $c(t,x)$ *is given by*

(3.10) $$c(t,x) = -V(t,x) - 2\partial S/\partial t(t,x) - (\sigma\nabla S)^2(t,x) - 2b\cdot\nabla S(t,x).$$

Therefore, diffusion and Schrödinger equations are equivalent.[6]

We must aware of the fact that if the Schrödinger equation is linear, the corresponding diffusion equation is non-linear, and vice versa.[7] Since we know now Schrödinger's conjecture is correct, we we can also adopt a pair $\{\hat\phi, \phi\}$ of real valued functions instead of a pair $\{\psi, \overline{\psi}\}$ of complex valued functions .

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[6] Notice that no "mechanics" is necessary in any form for the equivalence. In other words we need no "quantization" for Schrödinger equation. Cf. Nagasawa (1989, 91, monograph)

[7] Cf. Nagasawa (monograph) for the non-linearity appearing here and in (3.5) and (3.9)

4. Principle of Superposition of Diffusion Processes

Let $\widehat{\phi}_t^{(k)}\phi_t^{(k)}$ be Schrödinger's factorization of diffusion processes $Q^{(k)}$, k = 1, 2, ... For α_k, $\beta_k \geq 0$ we set

(4.1) $$\widehat{\phi}_t = \sum_k \alpha_k \widehat{\phi}_t^{(k)} \quad and \quad \phi_t = \sum_k \beta_k \phi_t^{(k)},$$

where

(4.2) $$1 = \sum_{k,j} \alpha_k \beta_j$$

so that

$$\int dx \widehat{\phi}_t(x)\phi_t(x) = 1.$$

If $\widehat{\phi}_t\phi_t$ is Schrödinger's factorization of a diffusion process Q, then we call the process Q the **superposition of the diffusion processes** $\{Q^{(k)}: k = 1, 2, \cdots\}$. It is clear that the claim in Section 1 on "interference" turns out to be correct for the superposition of diffusion processes defined above.

As an example let us consider an inverse problem; namely, we decompose a diffusion process Q into, say, two diffusion processes. We ignore, for simplicity, normalization. We decompose entrance and exit laws $\widehat{\phi}_a$ and ϕ_b as

(4.3) $$\widehat{\phi}_a = \widehat{\phi}_a^{(1)} + \widehat{\phi}_a^{(2)}$$
$$\phi_b = \phi_b^{(1)} + \phi_b^{(2)},$$

and consider diffusion processes $Q^{(1)}$ and $Q^{(2)}$ with the $p^{(i)}$-representation

(4.4) $$Q^{(i)} = [\, \widehat{\phi}_a^{(i)} p^{(i)} >< p^{(i)} \phi_b^{(i)} \,], \quad i = 1,2,$$

where $p^{(i)}$ denotes a transition density. If Q is the superposition of $Q^{(1)}$ and $Q^{(2)}$, then, for $\forall t \in [a, b]$,

(4.5) $$\widehat{\phi}_a^{(1)} p^{(1)}(a,t) + \widehat{\phi}_a^{(2)} p^{(2)}(a,t) = \widehat{\phi}_a p(a,t),$$
$$p^{(1)}(t,b)\phi_b^{(1)} + p^{(2)}(t,b)\phi_b^{(2)} = p(t,b)\phi_b,$$

where notations are self-explanatory. This is the requirement on $p^{(i)}$, $i = 1,2$, and p for Q to be the superposition of $Q^{(1)}$ and $Q^{(2)}$.

If we assume $p = p^{(1)} = p^{(2)}$, then the system of equations in (4.5) reduces to the system of equations in (4.3), and hence we can formulate

Proposition 4.1. *Let $p = p^{(1)} = p^{(2)}$. A diffusion process Q can be decomposed into two diffusion processes $Q^{(1)}$ and $Q^{(2)}$ with $\widehat{\phi}_a$ and ϕ_b given in* (4.3), *if $\mu_a^{(i)} = \widehat{\phi}_a^{(i)}\phi_a^{(i)}$ and $\mu_b^{(i)} = \widehat{\phi}_b^{(i)}\phi_b^{(i)}$, $i = 1, 2$, are admissible to p.*[8] *The two processes $Q^{(1)}$ and $Q^{(2)}$ interfere of each other with the correlation $\widehat{\phi}_t^{(1)}\phi_t^{(2)} + \widehat{\phi}_t^{(2)}\phi_t^{(1)}$* (cf. (1.6)).

This corresponds to the so-called "two slits problem" in quantum mechanics.

More generally we have :

Theorem 4.1. (Superposition Principle) *Let $\{X_t, Q^{(k)}\}$, $k = 1, 2, \ldots$, $Q^{(k)} = [\,\widehat{\phi}_a^{(k)} p^{(k)} \gg\ll p^{(k)} \phi_b^{(k)}\,]$, be diffusion processes with Schrödinger's representation.*

Define

$$\widehat{\phi}_t = \sum_k \alpha_k \widehat{\phi}_t^{(k)} \quad \text{and} \quad \phi_t = \sum_k \beta_k \phi_t^{(k)}. \tag{4.6}$$

with $\alpha_k, \beta_k \geq 0$ such that

$$1 = \sum_{k,j} \alpha_k \beta_j. \tag{4.7}$$

Consider a flow $\mu_t = \widehat{\phi}_t \phi_t$ of probability distribution densities and determine $\tilde{b}(t,x)$ with

$$\frac{\partial R}{\partial t} + \frac{1}{2}\Delta S + (\sigma\nabla S)\cdot(\sigma\nabla R) + \tilde{b}\cdot\nabla R = 0, \tag{4.8}$$

where $R = \frac{1}{2}\log \mu_t$ and $S = \frac{1}{2}\log \widehat{\phi}_t \phi_t$. Let $\tilde{L}$ be a parabolic differential operator defined in (2.2) *with this $\tilde{b}(t,x)$ in place of $b(t,x)$, and assume the integrability condition* (2.11) *with $\tilde{c}(t,x)$ defined in* (2.4) *with $\tilde{L}$ in place of L.*[9] *Then there exists a diffusion process Q with the p-representation:*

$$Q = [\sum_k \alpha_k \widehat{\phi}_a^{(k)} p \gg\ll p \sum_k \beta_k \phi_b^{(k)}], \tag{4.9}$$

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[8] For the admissibility see Aebi-Nagasawa (1992), Nagasawa (Monograph)

[9] We can assume the drift coefficient $\tilde{b}$ is good enough so that we can have a diffusion process corresponding to the parabolic operator $\tilde{L}$

which is the superposition of $Q^{(k)}$; *namely, it holds that for a bounded measurable* f, *and* $\forall\, t \in [a,b]$

$$Q[f(X_t)] = \sum_{k,j} \int dy \alpha_k \widehat{\phi}_t^{(k)}(y)\, \beta_j \phi_t^{(j)}(y) f(y), \tag{4.10}$$

where

$$\widehat{\phi}_t^{(k)}(y) = \int dx \widehat{\phi}_a^{(k)}(x) p(a,x;t,y),$$

(4.11)

$$\phi_t^{(j)}(y) = \int p(t,y;b,z) dz \phi_b^{(j)}(z).$$

The cross terms in (4.10) *represent "interference" of diffusion processes* $\{X_t, Q^{(k)}\}$.

Proof. We consider the flow $\mu_t = \widehat{\phi}_t \phi_t$ and the parabolic differential operator $\widetilde{L}$ with $\widetilde{b}(t,x)$, which is determined by (4.8), and require the integrability condition (2.11) with this $\widetilde{c}(t,x)$. Then, we can apply corollary 2.2, which claims the existence of a space-time diffusion process $\{(t, X_t), Q_{\mu_a}\}$ whose distribution density coincides with the $\mu_t = \widehat{\phi}_t \phi_t$, and moreover the process has the p-representation (4.9).

5. Complex or Real Superposition

Because of the non-linear dependence

$$c(t,x) + V(t,x) = -2\partial S/\partial t(t,x) - (\sigma \nabla S)^2(t,x) - 2b \cdot \nabla S(t,x), \tag{5.1}$$

if we assume the Schrödinger equation is linear with a given potential $V(t,x)$, then the corresponding diffusion equation turns out to be non-linear. Therefore, it is not reasonable to apply the real-valued superposition in terms of ϕ_i, and hence we should apply the complex-valued superposition: namely, defining wave functions ψ_i in terms of R_i and S_i of ϕ_i (cf. (3.4)), we apply superposition of the wave functions.

In the simplest case of (1.3), we get the interference

$$2\mathcal{Re}(\psi_1 \overline{\psi}_2) = 2e^{R_1 + R_2} \cos(S_1 - S_2). \tag{5.2}$$

On the other hand, if a rate of creation and killing $c(t,x)$ is given, then the corresponding Schrödinger equation turns out to be non-linear. Therefore, it is reasonable to apply real-valued superposition, namely, defining ϕ_i in terms of R_i and S_i (cf. (3.8)), we apply superposition of ϕ_i. In the simplest case of (1.6) we get the interference

$$\phi_1\widehat{\phi}_2 + \phi_2\widehat{\phi}_1 = 2e^{R_1 + R_2}\cosh(S_1 - S_2). \tag{5.3}$$

Remark. We have assumed $\phi(t,x)$ and $\widehat{\phi}(t,x)$ are non-negative for simplicity. However, since the space-time diffusion process $\{(t,X_t), Q\}$ does not cross over the zero set of the probability distribution density $\mu(t,x) = \phi(t,x)\widehat{\phi}(t,x)$ (ergodic decomposition occurs), even though functions $\phi(t,x)$ and $\widehat{\phi}(t,x)$ take negative values, the distribution density $\mu(t,x)$ is always non-negative. Therefore, $\phi(t,x)$ and $\widehat{\phi}(t,x)$ can be real-valued in general. This consideration applies also to equation (5.3). We have assumed ϕ_i and $\widehat{\phi}_i$, $i = 1,2$, are non-negative in (5.3). However, if they take negative values too, the interference in (5.3) turns out to be

$$(-1)^{K(t,x)}2e^{R_1 + R_2}\cosh(S_1 - S_2), \tag{5.4}$$

where $K(t,x) = 0$ *or* 1, which may vary only on the zero set of the function $2e^{R_1 + R_2}\cosh(S_1 - S_2)$, because of the ergodic decomposition of the space-time state space by the zero set, cf. Nagasawa (1989, 91, Monograph).

The non-linear dependence appeared in (5.1) and inaccessibility of the diffusion process $\{X_t, Q\}$ to the zero set of its distribution density indicate that it is necessary to find out a statistical mechanical structure behind the Schrödinger equation. For this see Nagasawa (1980, 90, monograph), Aebi-Nagasawa (1992).

Based on the superposition principle in quantum theory it has been claimed often that quantum theory is the third way of describing natural laws besides deterministic and stochastic ways, since deterministic theory and probability theory do not provide such a mathematical structure. We have shown that this claim was false, and moreover that quantum theory is an application of diffusion theory which provide the Schrödinger equation naturally without "quantization". Therefore, we can now fully rely on the theory of diffusion processes, even when we consider quantum theory. As many probabilists actually felt, there are only two mathematical ways describing natural laws: Deterministic mathematics and stochastic mathematics; more precisely, classical mathematics and the theory of probability and stochastic processes. There is no "third mathematics" at the moment.

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