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QUASI-EVERYWHERE UPPER FUNCTIONS

T.S. Mountford

Abstract: In this paper we find a good approximation for the capacitance of paths with large deviations for the Ornstein-Uhlenbeck process on Wiener space. We use this result to obtain an integral test for a function to be an upper function quasi-everywhere. The criterion differs from the necessary and sufficient condition for a function to be a.s. upper. We believe that this is a qualitatively new result.

Introduction

This paper examines the Ornstein-Uhlenbeck process on one dimensional Wiener space. This is a diffusion $\{O_s(\ ): s \geq 0\}$ on the space of real-valued continuous functions on $[0, \infty)$, with stationary measure equal to Wiener measure and such that the increments are independent one-dimensional Ornstein-Uhlenbeck processes. More concretely

$$O_s(t_1), \quad O_s(t_2) - O_s(t_1), \quad \ldots \ldots O_s(t_n) - O_s(t_{n-1}) \quad t_1 < t_2 < \ldots t_n$$

vary as independent O-U processes with stationary measures equal to the distributions of

$$N(0,t_1), N(0,t_2-t_1), \quad \ldots \ldots N(0,t_n-t_{n-1})$$

respectively.

Since the stationary measure of the O-U process on Wiener space is the Wiener measure, the statement that a path property holds for almost all Brownian paths is equivalent to the property holding on O_s for Lebesgue a.a. times with probability one. If the property holds for all times s, with probability one then the path property is said to hold quasi-everywhere (or q.e.). While the converse is obviously true, it does not follow that if a property holds (Wiener) a.s., then it holds q.e.. A simple and illuminating example is the property that at time 1 the path has value different from 0. This property clearly holds a.s., but since the process $\{O_s(1): s \geq 0\}$ is a standard one dimensional O-U process, there are uncountably many times s at which $O_s(1) = 0$ and so the property does not hold q.e..

However many a.s. properties do indeed hold q.e.. In fact most of the papers on this subject have established results of this type, see e.g. Komatsu and Takashima (1984), Penrose (1989) and Shigekawa (1984). One of the first papers on the subject, Fukushima (1984), showed that as M tends to infinity

$$Cap^1(\{\omega: \sup_{u \in [0,1]} \omega(u) < M\}) \leq KMe^{-M^2/2}. \qquad (*)$$

Since this is of the same exponential order as the Brownian probability of the set of paths in question, many of the a.s. results for Brownian motion could be quickly extended to q.e. results. In particular, Fukushima showed that the L.I.L. held at fixed points q.e.. Another approach to such problems was suggested by Meyer (1980) and Walsh (1984). The process $\{O_s(\ ): s \geq 0\}$ was identified with the process $\{e^{-s/2}W(e^s, \ ): s \geq 0\}$, where W(,) is a Brownian sheet.

Walsh showed that the L.I.L. had to hold q.e., since if it broke down, it had to stay broken for a non-empty random interval of time. This contradicted the necessity of the L.I.L. holding for Lebesgue a.a. times. In this paper we wish to show that the key capacitory inequality (*) of Fukushima is (up to an order of magnitude) an equality.

Theorem One

For the O-U process on Wiener space, the 1-capacitance of the set of paths $\{\omega: \sup_{u\in[0,1]} \omega(u) > M\}$, C(1,M), satisfies

$$(1+o(1))\sqrt{\frac{1}{2\pi}}Me^{-M^2/2} \le C(1,M) \le (1+o(1))\sqrt{\frac{2}{\pi}}Me^{-M^2/2}$$

Our method of derivation is quite different from that of Fukushima. The above result is then used to show

Theorem Two

Let f be a function on $[0, \infty)$ such that eventually $f(t)/t^{1/2}$ is increasing. Then f is a q.e.
upper function if and only if $\sum_n \frac{f(m_n)}{m_n^{1/2}} e^{-f(m_n)^2/2m_n} < \infty$, where $m_n = e^{n/\log(n)}$.

The above condition is more elegantly expressed by

Theorem Three

Let f be a function on $[0, \infty)$ such that eventually $f(t)/t^{1/2}$ is increasing. Then f is a q.e. upper function if and only if $\int^{\infty} \frac{f^3(x)}{x^{5/2}} e^{-f(x)^2/2x} dx < \infty$.

The proof of Theorem two relies heavily on arguments found in Erdos (1943), which finds a (different) integral test to determine which functions are upper for a.a. Brownian paths.

Section One

Lemma 1.1

Let $\{O_s^1: s \ge 0\}$ be the O-U process on Wiener space killed at rate 1. Let D_M be the set of continuous paths $\{\omega: \omega(1) \ge M\}$. Let Y be a standard normal random variable. Then for $\omega \in \delta D_M$,

$$V(\omega) = E[\int_0^\infty I_{\{O_s^1 \in D_M\}} ds \mid O_0^1 = \omega] \le \int_0^\infty e^{-s} P[Y > M\left[\frac{1-e^{-s/2}}{1+e^{-s/2}}\right]^{1/2}] ds = \frac{1}{M^2}(1+o(1))$$

Proof

Let ω be a path in δD_M. Then

$$E[\int_0^\infty I_{\{O_s^1 \in D_M\}} ds \mid O_0^1 = \omega] = \int_0^\infty P[O_s^1 \in D_M \mid O_0^1 = \omega] ds$$

$$= \int_0^\infty e^{-s} P[O_s \in D_M \mid O_0^1 = \omega].$$

Now $O_s(1)$ is equal in distribution to $\sqrt{1-e^{-s/2}}B(1) + e^{-s/2}O_0(1)$ where B(.) is a Brownian motion independent of $O_0(.)$. From this distribtional representation we see that $P[O_s \in D_M \mid O_0^1 = \omega]$ is equal to $P[\sqrt{1-e^{-s}}B(1) \geq M(1-e^{-s/2})]$. This latter expression is equal to $P[Y \geq M \left[\frac{1-e^{-s/2}}{1+e^{-s/2}}\right]^{1/2}]$. The result follows.

□

Using a similar argument we can show

Lemma 1.2

For $\omega \in \delta D_{M+x}$,

$$V(\omega) = E[\int_0^\infty I_{\{O_s^1 \in D_M\}} ds \mid O_0^1 = \omega] = o(1)$$

as M tends to infinity, uniformly on x $\in$ [0, 1].

Corollary 1.2

$$E\left[I_{\{O_s^1 \in D_M\}} ds,\ O_0^1 \in D_M\right] = o(\frac{1}{M} e^{-M^2/2})$$ as M tends to infinity.

Using the two lemmas we obtain our first inequality.

Proposition 1.1

The quantity C(1,M) satisfies $C(1,M) \geq Cap^1(D_M) = (1+o(1))\sqrt{\frac{1}{2\pi}} M e^{-M^2/2}$.

Proof

A characterization of the 1-capacity of a set E for a symmetric process with symmetrizing measure m, is

$$Cap^1(E) = \int P^\omega[T_E < \infty] m(d\omega)$$

where T_E is the first hitting time of the set E for the process killed at rate 1. See Fukushima (1980). We know from the Gaussian distribution that for our symmetric process and the set E = D_M, $\int P^\omega[T_E = 0] m(d\omega)$ is of the order $\frac{1}{M} e^{-M^2/2}$. Corollary 1.2 states that $E[\int_0^\infty I_{\{O_s^1 \in D_M,\ T_{D_M}=0\}} ds] = o(\frac{1}{M} e^{-M^2/2})$ as M tends to infinity. From Fubini's Theorem

$$\frac{2}{\sqrt{2\pi}} \frac{1}{M} e^{-M^2/2}(1 + o(1)) = \int_0^\infty P[O_s^1(\ ) \in D_M] ds =$$

$$E[\int_0^\infty I_{\{O_s^1 \in D_M,\ T_{D_M}=0\}} ds] + E[\int_0^\infty I_{\{O_s^1 \in D_M,\ 0<T_{D_M}<\infty\}} ds]$$

Given our observation on the first term we conclude that

$$\frac{2}{\sqrt{2\pi}}\frac{1}{M}e^{-M^2/2}(1+o(1)) = E[I_{\{0<T_{D_M}<\infty\}}\int_{T_{D_M}}^{\infty} I_{\{O_s^1 \in D_M\}}ds]$$

Using the Strong Markov property, we can rewrite the right hand side as $E\left[I_{\{0<T_{D_M}<\infty\}}V(O^1_{T_{D_M}})\right]$. By Lemma 1.1, this quantity is less than (1 + o(1))P[0 < $T_{D_M} < \infty]/M^2$ and the result follows. □

We complete the proof of Theorem One with the following.

Proposition 1.2

The quantity C(1,M) satisfies C(1,M) ≤ $(1+o(1))\sqrt{\frac{2}{\pi}}Me^{-M^2/2}$.

Proof

Fix $\lambda \in (0, 1)$. Define the sets

$$A_n(M) = \{\omega: \sup_{u\in[\lambda^n,\lambda^{n-1}]} \omega(u) > M\} \quad n = 1, 2, ..$$

Then $A_M = \bigcup_{n=1}^{\infty} A_n(M)$. Therefore C(1,M) = $Cap^1(A_M) \le \sum_{n=1}^{\infty} Cap^1(A_n(M))$. Also by scaling, $Cap^1(A_n(M)) = Cap^1(A_1(M/\lambda^{\frac{n-1}{2}}))$. It will suffice to estimate quantities $Cap^1(A_1(M))$. By reasoning similar to that employed with Lemma 1.1 we find that

$$V'(\omega) = E[\int_0^{\infty} I_{\{O_s^1\in A_1(M)\}}ds \mid O_0^1=\omega]$$

satisfies $\inf_{\omega\in A_1(M)} V'(\omega) \ge (1+o(1))\frac{\lambda^{1/2}}{M^2}$. Therefore, reasoning as in Proposition 1.1, we find

$$(1+o(1))\sqrt{\frac{2}{\pi}}\frac{1}{M}e^{-M^2/2} = E[I_{\{0<T_{A_1(M)}<\infty\}}V'(O^1_{T_{A_1(M)}})] \ge P[0<T_{A_1(M)}<\infty]\frac{\lambda^{1/2}}{M^2}.$$

Therefore $Cap^1(A_1(M)) \le (1+o(1))\sqrt{\frac{2}{\pi}}\frac{M}{\lambda^{1/2}}e^{-M^2/2}$. Consequently $Cap^1(A_M) \le (1+o(1))\sqrt{\frac{2}{\pi}}\frac{M}{\lambda^{1/2}}e^{-M^2/2}$. Since λ is arbitrarily close to one, our proof is complete. □

We will need the following corollaries.

Corollary 1.3

$$P[T_{D_M} < \infty \mid O_0^1(\omega,1) = 0] \le 2(1+o(1))\sqrt{\frac{2}{\pi}}Me^{-M^2/2}.$$

Corollary 1.4

For a fixed, finite interval I,

$$1 \le \underline{\lim}_{M\to\infty} \frac{P[t\in I:\, O_t(\omega)\in A_M]}{\frac{1}{\sqrt{2\pi}}|I|Me^{-M^2/2}} \le \overline{\lim}_{M\to\infty} \frac{P[t\in I:\, O_t(\omega)\in A_M]}{\frac{1}{\sqrt{2\pi}}|I|Me^{-M^2/2}} \le 2$$

This limiting behaviour is uniform on $|I| \le 1$. In the above inequalities A_M may be replaced by D_M.

The proposition below follows directly from Karatzas and Shreve (1988), page 339-342.

Proposition 1.3

Let $S_y = \inf\{t > 0 : O_t(1) = y\}$. Then for $0 < x < M$, $P[T_M < T_0 \mid O_0(1) = x] = \frac{\psi(x)}{\psi(M)}$, where $\psi(y) = \int_0^y e^{v^2/2}dv$.

Section Two

Note that by scaling Theorem One states that

$$\sqrt{\frac{1}{2\pi}}\frac{1}{M}e^{-M^2/2}(1+o(1)) \le Cap^1(\{\omega:\ \sup_{u\in[s,t]} \omega(u) - \omega(s) > M\sqrt{t-s}\ \}) \le \sqrt{\frac{2}{\pi}}\frac{1}{M}e^{-M^2/2}(1+o(1)).$$

Recall that a function f is an upper function for a path ω if for all t sufficiently large, $\omega(t) < f(t)$. It is well known that $f(t) = \sqrt{(2+\varepsilon)t\, loglog(t)}$ is an upper function for a.a. Brownian paths and indeed for q.e. Brownian path, if ε is stricly positive but not if ε is strictly negative. Erdos (1943) proves that if $f(t)/t^{1/2}$ is eventually increasing, then f is upper for a.a. Brownian paths (or a.s. upper), if and only if

$$\int^{\infty} \psi(t)e^{\psi^2/2}\frac{dt}{t} < \infty$$

where $\psi(t) = f(t)/t^{1/2}$.

We will adapt Erdos's proof to the capacity case.

Define the sequence m_n by $m_n = e^{n/log(n)}$. The following inequalities will be made use of in this section:

There exist finite strictly positive constants c and c' such that

A) $c(\frac{m_n}{loglog(m_n)})^{1/2} < (m_{n+1}loglog(m_{n+1}))^{1/2} - (m_n\, loglog(m_n))^{1/2} < c'(\frac{m_n}{loglog(m_n)})^{1/2}$.

B) $c\frac{m_n}{loglog(m_n)} < m_{n+1} - m_n < c'\frac{m_n}{loglog(m_n)}$.

Lemma 2.1

If f is an increasing function and $\sum_{n\geq 1} \frac{f(m_n)}{m_n^{1/2}} e^{-f(m_n)^2/2m_n} < \infty$, then f is a quasi-everywhere upper function.

Proof

Given Fukushima's LIL we can and will assume that for all t, f(t) < $\sqrt{3t\, loglog(t)}$. For notational simplicity we denote the n'th term of the above series by a_n. Consider

$$\frac{f(m_n)}{m_{n+1}^{1/2}} e^{-f(m_n)^2/2m_{n+1}} = \frac{f(m_n)}{m_n^{1/2}} e^{-f(m_n)^2/2m_n} \left[\frac{m_n}{m_{n+1}}\right]^{1/2} e^{\frac{f(m_n)^2}{2}\frac{m_{n+1}-m_n}{m_{n+1}m_n}}$$

$$= a_n \left[\frac{m_n}{m_{n+1}}\right]^{1/2} e^{-\frac{f(m_n)^2}{2}\frac{m_{n+1}-m_n}{m_{n+1}m_n}}$$

Clearly $\left[\frac{m_n}{m_{n+1}}\right]^{1/2}$ is bounded. Also inequality B) enables us to bound $e^{\frac{f(m_n)^2}{2}\frac{m_{n+1}-m_n}{m_{n+1}m_n}}$ by $e^{f(m_n)^2/2\frac{c}{m_n\, loglog(m_n)}}$. We assumed that f(t) < $\sqrt{3t\, loglog(t)}$ so this latter expression is bounded. Consequently

$$\sum_n \frac{f(m_n)}{m_{n+1}^{1/2}} e^{-f(m_n)^2/2m_{n+1}} \leq K\sum_n a_n < \infty$$

Therefore, by Theorem One, $\sum_n Cap^1(\{\omega : \sup_{u\in[0,m_{n+1}]} \omega(u) > f(m_n)\}) < \infty$.

So by the "Borel-Cantelli Lemma " for capacitances (see Fukushima (1984)), $Cap^1(\{\omega : \sup_{u\in[0,m_{n+1}]} \omega(u) > f(m_n)$ for infinitely many n $\}) = 0$.

This establishes the lemma since f is assumed to be increasing.

□

The proof of Theorem Two is all but completed with the following proposition.

Proposition 2.2

Let f be a function such that $\phi(t) = f(t)/t^{1/2}$ increases with t, $\lim_{t\to\infty} t\phi'(t) = 0$, and $\sum_{n\geq 1} \frac{f(m_n)}{m_n^{1/2}} e^{-f(m_n)^2/2m_n} = \infty$, then for each non-trivial interval I and each n, there exists s ∈ I and t > m_n so that $O_s(t) > f(t)$ with probability one.

We make some preliminary remarks.

Remarks

1) Our assumptions on f guarantee that $f(m_{n+1}) - f(m_n) > c\left[\frac{m_n}{loglog(m_n)}\right]^{1/2}$ for some strictly positive c.

2) Let I_n be the indicator of the event { s ∈ I such that $O_s(m_n) > f(m_n)$}. The σ-field of these events is trivial, so to establish the lemma, it will be sufficent to show

that there exists c > 0 so that for each n, $P[\sum_{m\geq n} I_m > 0] > c$.

3) We may assume that I is an interval of small length and that $f(m_n)/m_n^{1/2}$ increases to infinity.

4) As with Lemma 2.1, we may assume that for n large $f(m_n) < \sqrt{3m_n \log\log(m_n)}$.

5) Given Corollary 1.4, we may assume that for n large enough, $P[I_n = 1] \sim |I| \dfrac{f(m_n)}{m_n^{1/2}} e^{-f(m_n)^2/2m_n}$. Therefore for n arbitrarily large we can find m > n so that $E[\sum_{j=n}^{m} I_j] \in (1/3, 1/2)$. As Remark 2 makes plain, Proposition 2.2 will be proven if we can show that for such an (n,m) pair with n sufficiently large we have $E[(\sum_{j=n}^{m} I_j)^2]$ uniformly bounded. Therefore it will be enough to find K so that for n large enough and every v ∈ [n, m], $E[I_v \sum_{j>0}^{m-v} I_{v+j}] < KE[I_v]$.

6) Given I = [a, b], define the stopping times $T_{k,a}$ by

$$T_{k,a} = \inf_{s\geq a} \{O_s(m_k) > f(m_k)\}, \quad T_k^{k+j} = \inf_{s\geq T_{k,a}} \{O_s(m_{k+j}) > f(m_{k+j})\}.$$

By the reversibility of the process

$$E[I_v I_{v+j}] \leq 2P[T_{v+j}^v \in I] \leq 2E[I_v P[T_{v+j}^v < T_v + |I| \mid O_{T_v}]]$$

Remarks 2, 5 and 6 show that Proposition 2.2 is implied by the following proposition.

Proposition 2.3

Suppose that f satisfies the assumptions of Proposition 2.3 and the assumption made in Remark 4. Then there exist K and $B_{v,j}$ so that for v large enough and every j

$$P[T_{v+j}^v < |I| + T_v] < KE[I_{v+j}] + B_{v,j}$$

and $\sup_v \sum_j B_{v,j}$ is bounded.

A good first step towards proving Proposition 2.3 and therefore Proposition 2.2 is the lemma below.

Lemma 2.2

There exists a finite K so that for v sufficiently large and all j,

$$P[T_{v+j}^v < T_v + |I|] < K \left\{ E[I_{v+j}] + \frac{(m_{j+v} - m_v)^{1/2}}{f(m_v) - f(m_{v+j})\dfrac{m_v}{m_{v+j}}} e^{-\frac{(f(m_{v+j}) - f(m_v))^2}{2(m_{v+j} - m_v)}} \right\}.$$

Proof

The processes $O_s(m_v)$ and $O_s(m_{v+j}) - O_s(m_v)$ are independent. Since T_v is a stopping time with respect to the filtration of the first process, it follows that $O_{T_v}(m_{v+j}) - O_{T_v}(m_v)$ has a $N(0, m_{v+j} - m_v)$ distribution. Let the stopping time S_x =

$\inf\{ s > 0 : O_s(m_{v+j}) = x \}$. From the strong Markov property

$$P[T^v_{v+j} < T_v + |I|] < \int_R P[O_{T_v}(m_{v+j}) - O_{T_v}(m_v) \in dy] P[S_{f(m_{v+j})} < S_0 \mid O_0(m_{v+j}) = f(m_v)+y]$$
$$+ \int_R P[O_{T_v}(m_{v+j}) - O_{T_v}(m_v) \in dy] P[S_{f(m_{v+j})} < |I| \mid O_0(m_{v+j}) = 0]$$
$$= P[S_{f(m_{v+j})} < |I| \mid O_0(m_{v+j}) = 0] + \int_{-\infty}^{\infty} \frac{dy}{\sqrt{m_{v+j}-m_v}} e^{-\frac{y^2}{2(m_{v+j}-m_v)}} P[S_{f(m_{v+j})} < S_0 \mid O_0(m_{v+j}) = f(m_v)+y]$$

Corollary 1.3 tells us that the first of the two terms above is already of the desired form so it will suffice to deal with the last term.

Given that the probability $P[S_{f(m_{v+j})} < S_0 \mid O_0(m_{v+j}) = f(m_v)+y]$ is an increasing function of y, we may majorize the second term by

$$2\int_0^{\infty} \frac{dy}{\sqrt{m_{v+j}-m_v}} e^{-\frac{y^2}{2(m_{v+j}-m_v)}} \left[\frac{\psi(\frac{y+f(m_v)}{\sqrt{m_{v+j}}})}{\psi(\frac{f(m_{v+j})}{\sqrt{m_{v+j}}})} \wedge 1 \right].$$

It is easily seen that for large x, $\psi(x)$ is approximately equal to $e^{x^2/2}/x$ and so the above expression is bounded by

$$K\int_0^{\infty} \frac{dy}{\sqrt{m_{v+j}-m_v}} e^{-\frac{y^2}{2(m_{v+j}-m_v)}} \left[\frac{e^{(y+f(m_v))^2/2m_{v+j}} f(m_{v+j})}{e^{f(m_{v+j})^2/2m_{v+j}} (y+f(m_v))} \wedge 1 \right]$$
$$\le K \left[\int_0^{f(m_{v+j})-f(m_v)} \frac{dy}{\sqrt{m_{v+j}-m_v}} e^{-\frac{y^2}{2(m_{v+j}-m_v)}} \frac{e^{(y+f(m_v))^2/2m_{v+j}} f(m_{v+j})}{e^{f(m_{v+j})^2/2m_{v+j}} (y+f(m_v))} + \frac{\sqrt{m_{v+j}-m_v}}{f(m_{v+j})-f(m_v)} e^{-\frac{(f(m_{v+j})-f(m_v))^2}{2(m_{v+j}-m_v)}} \right]$$

The second term satisfies the desired inequality so it remains to examine

$$e^{-f(m_{v+j})^2/2m_{v+j}} \frac{f(m_{v+j})}{\sqrt{m_{v+j}-m_v}} \int \frac{dy}{y+f(m_v)} e^{-\frac{y^2}{2(m_{v+j}-m_v)} + \frac{(y+f(m_v))^2}{2(m_{v+j})}}$$

The exponent inside the above integral, $-\frac{y^2}{2(m_{v+j}-m_v)} + \frac{(y+f(m_v))^2}{2(m_{v+j})}$, is maximized at $y = \frac{f(m_v)}{m_v}(m_{v+j}-m_v)$. Given our assumptions on f, this is greater than $f(m_{v+j})-f(m_v)$. Accordingly, the exponent $-\frac{y^2}{2(m_{v+j}-m_v)} + \frac{(y+f(m_v))^2}{2(m_{v+j})}$, is increasing on $[0, f(m_{v+j})-f(m_v)]$. Furthermore, the slope of the exponent decreases to the value at the right endpoint of the interval,

$$\frac{m_{v+j}f(m_v) - m_v f(m_{v+j})}{m_{v+j}(m_{v+j} - m_v)}$$

Given these facts the conclusion follows easily.

□

The above lemma will provide a good bound for $E[I_v I_{v+j}]$ if m_{v+j}/m_v is not too large. It will not be sufficient when this ratio is large. We require the following lemma.

Lemma 2.3

Let c be a strictly positive constant. There exists a positive constant K so that for v large enough and any j with

i $\dfrac{m_{v+j}}{m_v} \geq (log(m_v))^c$,

ii $f(m_{v+j}) < \sqrt{3m_{v+j} loglog(m_{v+j})}$, $f(m_{v+j}) < \sqrt{3m_{v+j} loglog(m_{v+j})}$,

$E[I_v I_{v+j}] < KE[I_v]E[I_{v+j}]$.

Proof

Remark 4 makes plain that the second constraint does not really represent a loss of generality, since we are only interested in I_v, I_{v+j} where this is the case.

If $y \leq \sqrt{4m_{v+j} loglog(m_{v+j})}$ then

$$\frac{(y-f(m_v))^2}{2(m_{v+j}-m_v)} - \frac{y^2}{2m_{v+j}} \geq \frac{-yf(m_v)}{m_{v+j} - m_v} \geq h$$

for some constant h. It follows that for $y \leq \sqrt{4m_{v+j} loglog(m_{v+j})}$,

$$\frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}}$$

$$\leq \frac{H}{\sqrt{2\pi m_{v+j}}} e^{\frac{-y^2}{2m_{v+j}}} \text{ for some H.}$$

So

$$P[T^v_{v+j} < |I|] = \int P[T_{v+j} < |I| \,|\, O_0(m_{v+j})=y] \frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}} dy$$

$$\leq \int_{-\infty}^{4\sqrt{m_{v+j} loglog(m_{v+j})}} P[T_{v+j} < |I| \,|\, O_0(m_{v+j})=y] \frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}} dy$$

$$+ \int_{4\sqrt{m_{v+j} loglog(m_{v+j})}}^{\infty} \frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}} dy$$

$$\leq H \int_{-\infty}^{4\sqrt{m_{v+j} loglog(m_{v+j})}} P[T_{v+j} < |I| \,|\, O_0(m_{v+j})=y] \frac{1}{\sqrt{2\pi m_{v+j}}} e^{\frac{-y^2}{2m_{v+j}}} dy$$

$$+ \int_{4\sqrt{m_{v+j} loglog(m_{v+j})}}^{\infty} \frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}} dy$$

$$\leq H \int_{-\infty}^{4\sqrt{m_{v+j}loglog(m_{v+j})}} P[T_{v+j} < |I| \mid O_0(m_{v+j})=y] \frac{1}{\sqrt{2\pi m_{v+j}}} e^{\frac{-y^2}{2m_{v+j}}} dy$$

$$+ \int_{4\sqrt{m_{v+j}loglog(m_{v+j})}}^{\infty} \frac{1}{\sqrt{2\pi(m_{v+j}-m_v)}} e^{\frac{-(y-f(m_v))^2}{2(m_{v+j}-m_v)}} dy$$

This last expression is equal to HE[I_{v+j}] (1 + o(1)), where the o( ) is used as v+j tends to infinity. □.

Proof of Proposition 2.3

There are three cases to examine:

1) $m_{v+j} \leq 2m_v$,

2) $m_{v+j} \in (2m_v, m_v(log(m_v))^c]$,

3) $m_{v+j} \in (m_v(log(m_v))^c, m_m]$.

We will choose c later, while dealing with Case 2. Note that however small c is required to be, Lemma 2.3 deals with Case 3, so we need not comment further on v+j ∈ Case 3..

Case 1

We must show that there exists K large enough and $B_{v,j}$ so that for v large and v+j in Case 1,

$$P[T^{v}_{v+j} < T_v + |I|] < K[I_{v+j}] + B_{v,v+j}$$

and $\sum_{B_{v,j}} < M$ for M not depending on v. Given Lemma 2.2, it is sufficient to show that

$$\sum_{v+j \in Case\,1} \frac{(m_{j+v} - m_v)^{1/2}}{f(m_v) - f(m_{v+j})\frac{m_v}{m_{v+j}}} e^{-\frac{(f(m_{v+j})-f(m_v))^2}{2(m_{v+j}-m_v)}}$$

is bounded for large v. Our conditions on f ensure that for v+j in Case 1, $\dfrac{(m_{j+v} - m_v)^{1/2}}{f(m_v) - f(m_{v+j})\frac{m_v}{m_{v+j}}}$ is bounded. Also, the assumptions on the function f ensure

$$f(m_{v+j}) - f(m_v) \geq c_1 j \left[\frac{m_v}{loglog(m_v)}\right]^{1/2}.$$

By inequality B), $m_{v+j} - m_v \leq c_2 j \frac{m_v}{loglog(m_v)}$. Therefore for v+j in Case 1, the above inequalities guarantee that $\dfrac{(m_{j+v} - m_v)^{1/2}}{f(m_v) - f(m_{v+j})\frac{m_v}{m_{v+j}}} \leq |I|Ce^{-c_3 j}$ for suitable C, c_3 not depending on v,j. So $B_{v,j} = |I|Ce^{-c_3 j}$, clearly satisfy our requirements.

Case 2

In this case, as in the above case we can use Lemma 2.2 to reduce our task to bounding

$$\sum_{Case\,2} \frac{(m_{v+j}-m_v)^{1/2}}{f(m_v)-f(m_{v+j})\frac{m_v}{m_{v+j}}} e^{-\frac{(f(m_{v+j})-f(m_v))^2}{2(m_{v+j}-m_v)}}$$

Note

$$f(m_{v+j}) - f(m_v) \geq c_4 \sum_{r=0}^{j-1} \left[\frac{m_{v+r}}{loglog(m_{v+r})}\right]^{1/2} > c_5(m_{v+j} loglog(m_{v+j}))^{1/2}.$$

It follows that for v large enough the above sum is less than

$$4 \sum_{Case\,2} \frac{m_{v+j}^{1/2}}{f(m_v)} e^{-\frac{(f(m_{v+j})-f(m_v))^2}{2m_{v+j}}}$$

$$\leq 4 \sum_{Case\,2} \left[\frac{m_{v+j}}{m_v}\right]^{1/2} e^{-c_5 loglog(m_v)/2}$$

But as Erdos notes, there are only of order $(loglog(m_v))^2$ j's with $m_{v+j} \in (2m_v, m_v(log(m_v))^c]$. So when v is large,

$$\sum_{Case\,2} \frac{(m_{v+j}-m_v)^{1/2}}{f(m_v)-f(m_{v+j})\frac{m_v}{m_{v+j}}} e^{-\frac{(f(m_{v+j})-f(m_v))^2}{2(m_{v+j}-m_v)}} < K(loglog(m_v))^2 (log(m_v))^{c/2} e^{-c_5 loglog(m_v)/2}$$

s is equal to $K(loglog(m_v)^2(log(m_v))^{(c-c_5)/2}$. So to bound this term we simply take c to be less than c_5. □

Proof of Theorem Two

Given Lemma 2.1 and Proposition 2.2 it is sufficient to show that a function f satisfying the hypotheses of Proposition 2.2 is not quasi-everywhere upper. Proposition 2.2 was proven for a fixed interval I. But this means that the conclusion must hold for any countable collection of intervals, such as the intervals with rational endpints, and therefore for all intervals. So given f satisfying the conditions of Proposition 2.2 and an interval I, we can find (random) s_1 in I so that $O_{s_1}(m_{n_1}) > f(m_{n_1})$. By path continuity, we can find an interval (again random) I_1 s.t. $s_1 \in I_1 \subset I$ so that for all t $\in I_1$, $O_t(m_{n_1}) > f(m_{n_1})$. Invoking Proposition 2.2 again we find $s_2 \in I_1$, n_2, I_2 so that $s_2 \in I_2 \subset I_1$, $n_2 > n_1$ and $O_t(m_{n_2}) > f(m_{n_2})$ for t $\in I_2$. Continuing in this manner we obtain a nested sequence of intervals I_j. If t is a cluster point for the intervals then clearly f is not an upper function for $O_t(\,)$. This completes the proof. □

Remark: Theorem Three which is easily seen to be equivalent to Theorem Two tells us that functions of the form f(t) = $\sqrt{t}(\sqrt{2loglog(t)} + \frac{x}{2}\frac{logloglog(t)}{\sqrt{2loglog(t)}})$ are quasi-everywhere upper if x is greater than 5/2, are a.s upper but not q.e upper if x is in the interval (3/2, 5/2] and a.s. non-upper if x is less than or equal to 3/2.

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