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MASATO KIKUCHI

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A NOTE ON THE ENERGY INEQUALITIES FOR INCREASING PROCESSES

Masato KIKUCHI

Department of Mathematics, Toyama University  
Gofuku, Toyama 930, Japan.

1. Introduction.

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space with a filtration  $(\mathcal{F}_t)_{t \geq 0}$  which satisfies the usual conditions. It is well-known that if  $A = (A_t)_{t \geq 0}$  is an adapted right continuous increasing process whose left potential is bounded by a constant  $c > 0$ , then

$$E[A_\infty^n] \leq n! c^{n-1} E[A_\infty] \leq n! c^n$$

holds for every integer  $n \geq 0$ . This is called the "energy inequality".

In this short note, we establish the more general inequality

$$(1) \quad E[\Phi(A_\infty)] \leq \frac{1}{c} \left( \int_0^\infty \Phi(cs) e^{-s} ds \right) E[A_\infty]$$

where  $\Phi$  is a convex function on  $\mathbb{R}_+$  such that  $\lim_{t \downarrow 0} \Phi(t) = \Phi(0) = 0$  (which is not necessarily positive). (1) is used for an alternative proof of the theorem on the equivalence of  $BMO_\Phi$ -norms of martingales given in [1].

Dellacherie and Meyer in [3, p.189] investigated the case where  $\Phi(t) = t^p$ ,  $p > 0$ : their proof of the inequality

$$(2) \quad E[A_\infty^p] \leq \Gamma(p+1) c^p$$

contains an unavoidable error. In fact, (2) is valid only for  $p \geq 1$ , though they asserted otherwise: for  $p < 1$ , the constant process  $A_t = c$  does not satisfy (2). Their proof is based on the following assertion: if  $g$  is a convex function on the interval  $[1, +\infty[$  such that

$$(3) \quad g(n) = \log \Gamma(n) \quad \text{for every integer } n \geq 1;$$

$$(4) \quad g(x+1) - g(x) \leq \log \Gamma(x) \quad \text{for every } x \in [1, +\infty[,$$

then  $g(x) \leq \log \Gamma(x)$  holds for every  $x \geq 1$ . But the following example shows

that this is false. For  $x \geq 1$ , we set

$$g(x) = (\log[x])(x - [x]) + \log \Gamma([x]),$$

where  $[x]$  denotes the integer part of  $x$ . It is clear that  $g$  is convex and satisfies (3) and (4), but unless  $x$  is an integer, we have  $g(x) > \log \Gamma(x)$ . Actually, if  $f$  is a convex function satisfying (3) and (4), then  $\log \Gamma \leq f \leq g$  (cf. [2, Chap.7, §1, no.1, Prop.1]).

## 2. Analytic results.

In this section, we shall give an upper bound of the integral of  $\Phi \circ f$ , where  $f$  is an increasing BMO - function on the interval  $[0, 1[$  and  $\Phi$  is as in (1).

For each right continuous integrable function  $f$  on  $[0, 1[$  (which is not necessarily increasing), we define the function  $f^\#$  by

$$f^\#(t) = \frac{1}{1-t} \int_t^1 (f(s) - f(t)) ds, \quad t \in [0, 1[.$$

Note that  $f$  is uniquely determined by  $f^\#$  and  $\int_0^1 f(t) dt$ . In fact, if  $f_1^\# = f_2^\#$ , then the function

$$F(t) = \int_t^1 (f_1(s) - f_2(s)) ds, \quad t \in [0, 1[$$

is the unique solution of the equation  $F(t) = F(0) - \int_0^t (1-s)^{-1} F(s) ds$  and hence we have  $F(t) = F(0)(1-t)$ . It follows that  $f_1 - f_2 \equiv F(0)$  and therefore  $f_1 = f_2$  if  $\int f_1 dt = \int f_2 dt$ .

Furthermore  $f$  can be expressed by  $f^\#$  as follows :

**Lemma 1.** Let  $f$  be a right continuous integrable function on  $[0, 1[$ . If  $f^\#$  is integrable over  $[0, 1[$ , then

$$(5) \quad f(t) = \int_0^t (1-s)^{-1} f^\#(s) ds + \alpha - f^\#(t), \quad t \in [0, 1[,$$

where  $\alpha = \int_0^1 f(t) dt$ . In particular, if  $\log\left(\frac{1}{1-t}\right) f(t)$  is integrable or  $f^p$  is integrable for some  $p > 1$ , then  $f^\#$  is integrable and (5) holds.

Proof. Let  $g(t)$  be the function defined by the right-hand side of (5). Using Fubini's theorem, we easily verify that  $g^\# = f^\#$  and  $\int_0^1 g(t) dt = \int_0^1 f(t) dt$ . Hence we obtain  $f = g$ , as noted above.

If the function  $\log\left(\frac{1}{1-t}\right) f(t)$  is integrable, then

$$\int_0^1 |f^\#(t)| dt \leq \int_0^1 \log\left(\frac{1}{1-t}\right) \cdot |f(t)| dt + \int_0^1 |f(t)| dt < +\infty .$$

If  $f^p$  is integrable for some  $p > 1$ , then  $\log\left(\frac{1}{1-t}\right) f(t)$  is also integrable by Hölder's inequality. Thus the proof is complete.  $\square$

Now let  $\Phi$  be a convex function on  $[0, +\infty[$  such that  $\lim_{t \downarrow 0} \Phi(t) = \Phi(0) = 0$  and let  $\varphi$  be its right-hand derivative. Note that if  $\Phi$  is a positive convex function such that  $\Phi(0) = 0$ , it is necessarily right continuous at 0, that is, the required condition is satisfied. In general, we cannot affirm that  $\varphi(0) > -\infty$ , but in the following lemma, we assume this.

Lemma 2. Let  $\Phi$  and  $\varphi$  be as above and let  $f$  and  $g$  be positive Borel functions on  $[0, 1[$ . If  $\varphi(0) (= \lim_{t \downarrow 0} \varphi(t)) > -\infty$  and  $f, g$  satisfy the conditions

$$\int_0^1 f(t) \varphi \circ f(t) dt \leq \int_0^1 g(t) \varphi \circ f(t) dt, \quad \int_0^1 f(t) \varphi \circ f(t) dt < +\infty$$

then

$$(6) \quad \int_0^1 \Phi \circ f(t) dt \leq \int_0^1 \Phi \circ g(t) dt .$$

This is a well-known lemma for positive convex functions  $\Phi$  ([3, p.180]). The proof is almost the same as that of the case where  $\Phi$  is positive. By the formula for integration by parts, we have for  $u, v \geq 0$ ,

$$v\varphi(u) \leq \int_{]0, u]} t d\varphi(t) + \Phi(v) ;$$

the equality holds if  $u = v$ . From this it follows that

$$\int_0^1 f(t) \varphi \circ f(t) dt = \int_0^1 dt \left( \int_0^\infty s I_{\{f(t) \geq s\}} d\varphi(s) \right) + \int_0^1 \Phi \circ f(t) dt ;$$

$$\int_0^1 g(t) \varphi \circ f(t) dt \leq \int_0^1 dt \left( \int_0^\infty s I_{\{f(t) \geq s\}} d\varphi(s) \right) + \int_0^1 \Phi \circ g(t) dt .$$

To prove (6), we may assume that  $\int_0^1 \Phi \circ f(t) dt > -\infty$  and  $\int_0^1 \Phi \circ g(t) dt < +\infty$ . Then all the integrals in the preceding inequalities are finite, and (6) follows.

Proposition 3. Let  $\Phi$  be as in (1) and  $f$  be a positive right continuous increasing function on  $[0, 1[$ . If  $f^\# \leq 1$ , then

$$(7) \quad \int_0^1 \Phi \circ f(t) dt \leq \left( \int_0^\infty \Phi(t) e^{-t} dt \right) \left( \int_0^1 f(t) dt \right).$$

Note. As  $\Phi$  is negative or bounded to the below on  $[0, 1[$ , integrals in (7) make sense.

Proof. First let  $\psi$  be a bounded increasing function on  $[0, 1[$ . From (5) and Fubini's theorem, it follows that

$$(8) \quad \int_0^1 f(t) \psi(t) dt = \int_0^1 f^\#(t) \psi^\#(t) dt + \left( \int_0^1 f(t) dt \right) \left( \int_0^1 \psi(t) dt \right).$$

Now let  $g$  be an integrable function on  $[0, 1[$  such that  $f^\# \leq g^\#$ ,  $g^\#$  is integrable, and  $\int_0^1 g(t) dt = \int_0^1 f(t) dt$ . Since (8) is also valid for  $g$ , we have

$$\int_0^1 f(t) \psi(t) dt \leq \int_0^1 g(t) \psi(t) dt.$$

If the right-hand derivative  $\varphi$  of  $\Phi$  is bounded, we can set  $\psi = \varphi \circ f$ . Then by Lemma 2 we have

$$(9) \quad \int_0^1 \Phi \circ f(t) dt \leq \int_0^1 \Phi \circ g(t) dt.$$

By the monotone convergence theorem, we obtain (9) for any  $\Phi$  as in (1).

To prove (7), it is sufficient to show that (9) applies to the function  $g(t) = \log^+ \{ \alpha(1-t)^{-1} \}$ , where  $\alpha = \int_0^1 f(t) dt$ . So we must verify that  $\int_0^1 g(t) dt = \alpha$  and  $f^\#(t) \leq g^\#(t) = \alpha(1-t)^{-1} \wedge 1$ . But the first condition is obvious and the other follows from the definition of  $f^\#$  and the inequality  $f^\# \leq 1$ . Thus the proof is complete.  $\square$

### 3. Application.

The following theorem is an easy consequence of Proposition 3, and as remarked later, they are equivalent.

Theorem 4. Let  $A = (A_t)_{t \geq 0}$  be an adapted, right continuous, increasing process (resp. predictable, right continuous, increasing process which is zero at  $t=0$ ), and let  $\Phi$  be a convex function on  $[0, +\infty[$  such that  $\lim_{t \downarrow 0} \Phi(t) = \Phi(0) = 0$ . If

$$(10) \quad E[A_\infty - A_{T-} | F_T] \leq c \quad \text{a.s.} \quad (\text{resp. } E[A_\infty - A_T | F_T] \leq c \quad \text{a.s.})$$

holds for every stopping time (resp. predictable stopping time)  $T$ , then

$$(1) \quad E[\Phi(A_\infty)] \leq \frac{1}{c} \left( \int_0^\infty \Phi(ct) e^{-t} dt \right) E[A_\infty] .$$

Corollary. Let  $A = (A_t)$  be as in Theorem 4. Then

$$E[A_\infty^p] \leq c^{p-1} \Gamma(p+1) E[A_\infty] \leq c^p \Gamma(p+1) \quad (1 \leq p < +\infty);$$

$$E[A_\infty^p] \geq c^{p-1} \Gamma(p+1) E[A_\infty] \quad (0 < p < 1);$$

$$E[\exp(\alpha A_\infty)] \leq \frac{\alpha}{1 - c\alpha} E[A_\infty] + 1 \leq \frac{1}{1 - c\alpha} \quad (0 \leq \alpha < 1/c).$$

Previous to proving Theorem 4, we note that for every  $A = (A_t)$  as in the statement, we have

$$(11) \quad \int_{\{A_\infty > \lambda\}} (A_\infty - \lambda) dP \leq c P\{A_\infty > \lambda\} \quad \text{for all } \lambda > 0 ,$$

(for the proof see [4, p.346]). We use only this inequality to prove (1), so (1) is true for every random variable  $A_\infty$  satisfying (11).

Proof of Theorem 4. Without loss of generality, we can assume that  $c=1$ . Let  $f$  be the (unique) right continuous increasing function on  $[0, 1[$  with the same distribution as  $A_\infty$ , with respect to the Lebesgue measure. We set, for each  $t \in [0, 1[$ ,

$$\tau(t) = \inf\{s \in [0, 1[; f(s) > f(t)\} \wedge 1 .$$

It is obvious that  $]\tau(t), 1[ \subset \{s : f(s) > f(t)\} \subset ]\tau(t), 1[$  and hence that

$$P\{A_\infty > f(t)\} = 1 - \tau(t) \quad \text{and} \quad \int_{\{A_\infty > f(t)\}} A_\infty dP = \int_{\tau(t)}^1 f(s) ds .$$

Since the function  $t \rightarrow \frac{1}{1-t} \int_t^1 f(s) ds$  is increasing and  $\tau(t) \geq t$ , we have

$$\begin{aligned} f^\#(t) &\leq \frac{1}{1-\tau(t)} \int_{\tau(t)}^1 (f(s) - f(t)) ds \\ &= \frac{1}{P\{A_\infty > f(t)\}} \int_{\{A_\infty > f(t)\}} (A_\infty - f(t)) dP \leq 1, \end{aligned}$$

where the last inequality follows from (11) with  $c = 1$ . In order to obtain (1), it only remains to apply Proposition 3 to this  $f$ .  $\square$

Remark. Although Theorem 4 is probabilistic, it is equivalent to Proposition 3, which is purely analytic. To see this, let  $\Omega = [0, 1[$ ,  $dP$  be the Lebesgue measure on  $\Omega$ , and  $F_t$  be the augmentation of the  $\sigma$ -field generated by the set  $]t \wedge 1, 1[$  and the Borel subsets of  $[0, t \wedge 1]$ . If  $f$  is a function as in Proposition 3, the increasing process  $A_t(\omega) = f(t \wedge \omega)$  satisfies (10), and hence (7) follows from (1). Furthermore, if we set  $f(t) = c \log^+(\alpha(1-t)^{-1})$  for  $\alpha \in ]0, 1]$  and  $c > 0$  and if we define  $A_t$  as above, then  $(A_t)$  satisfies (10) and the equality holds in (1). Therefore (1) cannot be improved any more. There is a more interesting such example in [3].

We now give an application of Theorem 4. Using general Young functions, Bassily and Mogyoródi in [1] introduced the  $BMO_\Phi$ -norm corresponding to  $\Phi$ , and proved that it is equivalent to the usual  $BMO_1$ -norm, if  $\Phi$  has a finite power. Their proof is elementary, but somewhat complicated. We give a more straightforward proof of it.

Let  $\Phi$  be an increasing convex function on  $[0, +\infty[$  such that  $\Phi(0) = 0$ , and let  $M = (M_t)_{t \geq 0}$  be a right continuous, uniformly integrable martingale. We set

$$\|M\|_{BMO_\Phi} = \inf\{\lambda > 0 : \sup_T \|E[\Phi(\frac{1}{\lambda} |M_\infty - M_{T-}|) | F_T]\|_\infty \leq 1\},$$

where the supremum is taken over all stopping times. The original definition of the  $BMO_\Phi$ -norm is seemingly different from this definition, but they are identical; see the proof of Theorem 6 in [1].

Theorem 5 (Bassily and Mogyoródi). Let  $\Phi$  be an increasing convex function on  $[0, +\infty[$  such that  $\Phi(0) = 0$ . If  $\int_0^\infty \Phi(ct) e^{-t} dt < +\infty$  for some constant

$c > 0$ , then, for every right continuous uniformly integrable martingale  $M = (M_t)$  we have

$$c_\Phi \|M\|_{\text{BMO}_1} \leq \|M\|_{\text{BMO}_\Phi} \leq C_\Phi \|M\|_{\text{BMO}_1},$$

where  $c_\Phi > 0$  and  $C_\Phi > 0$  depend only on  $\Phi$ , and  $\|\cdot\|_{\text{BMO}_1}$  denotes the norm corresponding to the function  $\Psi(t) = t$ .

Proof. We prove the right-hand inequality only: the left-hand inequality is an easy consequence of Jensen's inequality. We set  $M_t^* = \sup_{s \leq t} |M_s|$ ,  $t \geq 0$ . It is well-known (e.g. [3, p.193]) that

$$E[M_\infty^* - M_{T-}^* | \mathcal{F}_T] \leq 4 \|M\|_{\text{BMO}_1}.$$

Let  $C_\Phi^{-1} = \inf\{c > 0 : \int_0^\infty \Phi(ct) e^{-t} dt > 1\}$ . It then follows from the hypotheses that  $0 < C_\Phi < +\infty$ . By Theorem 4, setting  $\beta = 4 \|M\|_{\text{BMO}_1}$ , we have

$$E[\Phi(\frac{1}{\beta C_\Phi} M_\infty^*)] \leq \frac{1}{\beta} \left( \int_0^\infty \Phi(\frac{t}{C_\Phi}) e^{-t} dt \right) E[M_\infty^*] \leq 1.$$

We put this inequality in conditional form in the usual manner (cf. [3, p.190]). Then we have

$$E[\Phi(\frac{1}{\beta C_\Phi} \sup_t |M_{T+t} - M_{T-}|) | \mathcal{F}_T] \leq 1 \quad \text{a.s.}$$

and hence  $\|M\|_{\text{BMO}_\Phi} \leq \beta C_\Phi = 4 C_\Phi \|M\|_{\text{BMO}_1}$ . This completes the proof.  $\square$

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