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Infinitesimal Behaviour of a Continuous Local Martingale

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a. Summary

We investigate the behaviour of a continuous local martingale $(M_t)_{t \geq 0}$ in the neighbourhood of $t = 0$. We prove that under suitable conditions $\frac{M_t}{\sqrt{t}}$ tends in law to a normal variable as $t \rightarrow 0$. A convergence theorem to Brownian motion as well as an application to continuous Markov processes are also given.

b. The main result

In this paragraph we suppose that $(M_t)_{t \geq 0}$ is a continuous d -dimensional local martingale with $M_0 = 0$. The filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions and \mathcal{F}_0 is degenerate. The process $(W_t)_{t \geq 0}$ is a standard d -dimensional Wiener process defined on some probability space. The \cdot denotes the obvious Euclidean innerproduct between vectors in \mathbb{R}^d . If u is a vector in \mathbb{R}^d then $|u|$ denotes the Euclidean norm of u . We use the results and the notation of [2] and [3].

Theorem:

If the continuous local martingale $(M_t)_{t \geq 0}$, $M_0 = 0$ satisfies

$$\frac{1}{t} \langle M^i, M^j \rangle_t \rightarrow a_{ij} \quad \text{a.e. as } t \rightarrow 0.$$

then for all $0 = s_0 < s_1 \dots < s_n = 1$ we have

$$\left(\frac{M_{s_1 t}}{\sqrt{t}}, \dots, \frac{M_{s_n t}}{\sqrt{t}} \right) \rightarrow \left(A^{1/2} W_{s_1}, \dots, A^{1/2} W_{s_n} \right) \quad \text{in law as } t \rightarrow 0.$$

Here $A^{1/2}$ is the symmetric positive definite square root of the positive definite symmetric matrix $A = (a_{ij})$ and W is a standard d -dimensional Wiener process.

Proof :

We will work with complex valued martingales and we will show that for

$$(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$$

$$E \left[\exp \frac{i}{\sqrt{t}} \left\{ u_1 \cdot \begin{pmatrix} M_{s_1 t} \\ \vdots \\ M_{s_1 t} \end{pmatrix} + u_2 \cdot \begin{pmatrix} M_{s_2 t} - M_{s_1 t} \\ \vdots \\ M_{s_2 t} - M_{s_1 t} \end{pmatrix} + \dots + u_n \cdot \begin{pmatrix} M_{s_n t} - M_{s_{n-1} t} \\ \vdots \\ M_{s_n t} - M_{s_{n-1} t} \end{pmatrix} \right\} \right]$$

$$\rightarrow \exp \left(-\frac{1}{2} \left(s_1 u_1 \cdot A u_1 + (s_2 - s_1) u_2 \cdot A u_2 + \dots + (s_n - s_{n-1}) u_n \cdot A u_n \right) \right)$$

Let $\sigma = \inf \{ t \mid \text{trace} \langle M, M \rangle_t \geq t (\text{trace} (A) + 1) \}$

Since $\frac{1}{t} \langle M, M \rangle_t \rightarrow A$ as $t \rightarrow 0$ we certainly have $\sigma > 0$ a.e.

Stopping M at time σ , the difference between

$$E \left[\exp \frac{i}{\sqrt{t}} \left\{ u_1 \cdot \begin{pmatrix} M_{s_1 t \wedge \sigma} \\ \vdots \\ M_{s_1 t \wedge \sigma} \end{pmatrix} + u_2 \cdot \begin{pmatrix} M_{s_2 t \wedge \sigma} - M_{s_1 t \wedge \sigma} \\ \vdots \\ M_{s_2 t \wedge \sigma} - M_{s_1 t \wedge \sigma} \end{pmatrix} + \dots + u_n \cdot \begin{pmatrix} M_{s_n t \wedge \sigma} - M_{s_{n-1} t \wedge \sigma} \\ \vdots \\ M_{s_n t \wedge \sigma} - M_{s_{n-1} t \wedge \sigma} \end{pmatrix} \right\} \right]$$

and

$$E \left[\exp \frac{i}{\sqrt{t}} \left\{ u_1 \cdot \begin{pmatrix} M_{s_1 t} \\ \vdots \\ M_{s_1 t} \end{pmatrix} + u_2 \cdot \begin{pmatrix} M_{s_2 t} - M_{s_1 t} \\ \vdots \\ M_{s_2 t} - M_{s_1 t} \end{pmatrix} + \dots + u_n \cdot \begin{pmatrix} M_{s_n t} - M_{s_{n-1} t} \\ \vdots \\ M_{s_n t} - M_{s_{n-1} t} \end{pmatrix} \right\} \right]$$

is clearly bounded by $2 P [\sigma < t]$,

and hence we only have to prove the theorem for M^σ instead of M . From now on we therefore suppose w.l.o.g. that $\langle M, M \rangle_t \leq t (\text{trace} (A) + 1)$.

$$\text{Let } h = u_1 \mathbf{1}_{[0, s_1 t]} + u_2 \mathbf{1}_{[s_1 t, s_2 t]} + \dots + u_n \mathbf{1}_{[s_{n-1} t, s_n t]} .$$

h is a deterministic process and we have to calculate

$$E \left[\exp \left(i \int_0^t h(s) \cdot dM_s \right) \right]$$

Since the martingale M is continuous we know from Itô's lemma that the process (indexed by v):

$$\exp \left(i \left(\int_0^v h(s) \cdot dM_s \right) + \frac{1}{2} \int_0^v h(s) \cdot d \langle M, M \rangle_s h(s) \right)$$

is a martingale.

Therefore for all $(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$

$$E \left[\exp \left\{ i \left(u_1 \cdot \begin{pmatrix} M_{s_1 t} \\ \vdots \\ M_{s_1 t} \end{pmatrix} + u_2 \cdot \begin{pmatrix} M_{s_2 t} - M_{s_1 t} \\ \vdots \\ M_{s_2 t} - M_{s_1 t} \end{pmatrix} + \dots + u_n \cdot \begin{pmatrix} M_{s_n t} - M_{s_{n-1} t} \\ \vdots \\ M_{s_n t} - M_{s_{n-1} t} \end{pmatrix} \right) \right\} \right]$$

$$+ \frac{1}{2} u_1 \cdot \langle M, M \rangle_{s_1 t} u_1 + \dots + \frac{1}{2} u_n \cdot \left(\langle M, M \rangle_{s_n t} - \langle M, M \rangle_{s_{n-1} t} \right) u_n \Big\} = 1.$$

Replacing u_i by $\frac{u}{\sqrt{t}}$ gives

$$E \left[\exp i \left\{ \left(u_1 \cdot \frac{M}{\sqrt{t}} + u_2 \cdot \left(\frac{M}{\sqrt{t}} - \frac{M}{\sqrt{t}} \right) + \dots + u_n \cdot \left(\frac{M}{\sqrt{t}} - \frac{M}{\sqrt{t}} \right) \right) \right. \right. \\ \left. \left. + \frac{1}{2t} \left(u_1 \cdot \langle M, M \rangle_{s_1 t} u_1 + \dots + u_n \left(\langle M, M \rangle_{s_n t} - \langle M, M \rangle_{s_{n-1} t} \right) u_n \right) \right\} \right] = 1.$$

We will use this equality to prove the theorem. Let K denote the quantity

$$K = \exp \left(\frac{1}{2} \left(s_1 u_1 \cdot A u_1 + (s_2 - s_1) u_2 \cdot A u_2 + \dots + (s_n - s_{n-1}) u_n \cdot A u_n \right) \right) \\ E \left[\exp i \sum_{m=1}^n \frac{u}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right) \right] \\ = E \left[\exp \left(i \sum_{m=1}^n \frac{u}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right) + \frac{1}{2t} \sum_{m=1}^n u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right) \right] \\ \exp \left\{ \frac{1}{2t} \sum_{m=1}^n (s_m - s_{m-1}) u_m \cdot A u_m - \frac{1}{2t} \sum_{m=1}^n u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right\}$$

The integrand is bounded since $\frac{1}{v} \langle M, M \rangle_v \leq (\text{trace } (A) + 1)$ for all v . We therefore can apply Lebesgue's theorem

$$|K - 1| \leq$$

$$\left| E \left[\exp \left\{ i \sum_{m=1}^n \frac{u}{\sqrt{t}} \cdot \left(M_{s_m t} - M_{s_{m-1} t} \right) + \frac{1}{2t} \sum_{m=1}^n u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right\} \right] \right|$$

$$\begin{aligned} & \exp \left\{ \frac{1}{2t} \sum_{m=1}^n (s_m - s_{m-1}) u_m \cdot A u_m - \frac{1}{2t} \sum_{m=1}^n u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right\}^{-1} \Big\| \\ & \leq E \left[\exp \left\{ \sum_{m=1}^n \frac{1}{2t} \sum_{m=1}^n u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right\} \right. \\ & \left. \left| \exp \left\{ \frac{1}{2t} \sum_{m=1}^n \left((s_m - s_{m-1}) u_m \cdot A u_m - u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \right) \right\}^{-1} \right| \right]. \end{aligned}$$

Since $(s_m - s_{m-1}) u_m \cdot A u_m - \frac{1}{t} u_m \left(\langle M, M \rangle_{s_m t} - \langle M, M \rangle_{s_{m-1} t} \right) u_m \rightarrow 0$ as $t \rightarrow 0$
 we obtain $K \rightarrow 1$ as $t \rightarrow 0$. qed.

Remarks

1. If the 1-dimensional martingale $(M_t)_{t \geq 0}$ is such that $\left(\frac{M_t}{\sqrt{t}}\right)_{t \geq 0}$ is bounded in L^p and $\frac{1}{t} \langle M, M \rangle_t \rightarrow c$ then we find for all $r < p$

$$E \left[\left| \frac{M_t}{\sqrt{t}} \right|^r \right] \rightarrow \gamma(r) c^{r/2}.$$

Indeed $\frac{M_t}{\sqrt{t}}$ tends to a normal variable with mean zero and variance c . The theorem now

follows with $\gamma(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^r e^{-x^2/2} dx$.

Theorem.

Let $(M_t)_{t \geq 0}$ be a d -dimensional continuous local martingale as in the previous theorem.

Let A be as in the previous theorem.

Put $B_s^t = \frac{M_{st}}{\sqrt{t}}$ for $s \in [0, 1]$.

Then $B \xrightarrow{t \text{ law}} A^{1/2} W$ as $t \rightarrow 0$,

where W is a standard d -dimensional Wiener process.

Proof.



Since the finite dimensional distributions converge we only need to prove that the image laws on $C[0,1]$, the space of d -dimensional continuous functions on $[0,1]$, form a tight family as $t \rightarrow 0$. We will use the Aldous' criterion [1].

As in the previous theorem we may suppose that the martingale is bounded and satisfies

$$\left| \frac{1}{t} \langle M^i, M^j \rangle_t \right| \leq c$$

for a fixed constant c .

(a) We first verify the uniform boundedness

$$\begin{aligned} P \left[\sup_{s \in [0,1]} \left| B_s^t \right| > k \right] &= P \left[\sup_{s \in [0,1]} \left| \frac{M_{st}}{\sqrt{t}} \right| > k \right] \\ &= P \left[\sup_{s \in [0,1]} \left| M_{st} \right| > k \sqrt{t} \right] \leq (tk)^{2-1} E \left[\text{trace} \langle M, M \rangle_t \right] \leq \frac{dc}{k} \end{aligned}$$

this quantity tends to zero uniformly in t .

(b) For fixed stopping times $S \leq T \leq S+\theta$, (with respect to the filtration $(\mathcal{F}_{st})_{0 \leq s \leq t}$) we have

$$\begin{aligned} &P \left[\left| B_S^t - B_T^t \right| \geq \varepsilon \right] \\ &= P \left[\left| M_{St} - M_{Tt} \right| \geq \varepsilon \sqrt{t} \right] \\ &\leq (t\varepsilon)^{2-1} E \left[\left| M_{St} - M_{Tt} \right|^2 \right] \\ &\leq (t\varepsilon)^{2-1} E \left[\text{trace} \left(\langle M, M \rangle_{Tt} - \langle M, M \rangle_{St} \right) \right] \\ &\leq (t\varepsilon)^{2-1} E \left[\text{trace} \langle M, M \rangle_{(S+\theta)t} - \langle M, M \rangle_{St} \right] \\ &\rightarrow \frac{\theta}{\varepsilon} \text{trace}(A) \end{aligned}$$

as $t \rightarrow 0$.

Aldous' criterion is therefore satisfied and the theorem is proved. qed.

c. Application to continuous Markov processes

Let E be a locally compact space on which a strongly continuous Feller semi group $(P_t)_{t \geq 0}$ is given. We suppose that the domain \mathcal{D}_A of the infinitesimal generator A is an algebra and we denote by $\Gamma(f, g) = A(fg) - fAg - gAf$ the carré du champ operator. The semi-group $(P_t)_{t \geq 0}$ is supposed to generate a continuous Markov process with values in E . For

$x \in E$ we denote by E_x the corresponding expectation operator. Clearly $X_0 = x$ P_x a.e.

Theorem

Let $f_1 \dots f_n \in \mathcal{D}_A$ and let α is the $n \times n$ matrix consisting of the elements $\alpha_{ij} = \Gamma(f_i, f_j)(x)$.

Let $(B_s^{t1} \dots B_s^{tn})_{0 \leq s \leq 1}$ denote the n-dimensional process

$$B_s^{ti} = \frac{f_i(X_{ts}) - f_i(x) - \int_0^{st} Af_i(X_u) du}{\sqrt{t}} \quad \text{viewed under } P_x.$$

Then $B^t \rightarrow \alpha^{1/2} W$ in distribution, where W is a standard n-dimensional Wiener process.

Proof

$$\text{Let } M_t^i = f_i(X_t) - f_i(x) - \int_0^t Af_i(X_u) du$$

$$\text{This is a martingale and } \langle M^i, M^j \rangle_t = \int_0^t \Gamma(f_i, f_j)(X_u) du.$$

$$\text{Clearly } \frac{1}{t} \langle M^i, M^j \rangle_t = \frac{1}{t} \int_0^t \Gamma(f_i, f_j)(X_u) du \rightarrow \Gamma(f_i, f_j)(x).$$

We now can apply the main theorem.

Corollary.

If $f \in \mathcal{D}_A$ then

$$E_x \left[\left| \frac{f(X_t) - f(x)}{\sqrt{t}} \right|^p \right] \rightarrow \gamma(p) \Gamma(f, f)(x)^{p/2} \quad \text{as } t \rightarrow 0$$

$$\text{where } \gamma(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^p e^{-x^2/2} dx.$$

Proof.

$$\text{Putting } M_t = f(X_t) - f(x) - \int_0^t Af(X_u) du,$$

we have $\langle M, M \rangle_t = \int_0^t \Gamma(f, f)(X_u) du$

Clearly $\left\| \int_0^t Af(X_u) du \frac{1}{\sqrt{t}} \right\|_p \rightarrow 0$ as $t \rightarrow 0$,

so that $E_x \left[\left| \frac{f(X_t) - f(x)}{\sqrt{t}} \right|^p \right]$ and $E_x \left[\left| \frac{M_t}{\sqrt{t}} \right|^p \right]$ have the same limit

Since $\left\| \frac{M_t}{\sqrt{t}} \right\|_p \leq c \left\| \left(\frac{1}{t} \int_0^t \Gamma(f, f)(X_u) du \right)^{1/2} \right\|_p$ by the Burkholder inequality, we

obtain $\left\| \frac{M_t}{\sqrt{t}} \right\|_p \leq c \left(\max_{y \in E} \Gamma(f, f)(y) \right)^{1/2} t^{1/2}$.

We can therefore apply a previous remark for all p between 0 and ∞ . \quad qed.

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