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SECOND ORDER LIMIT LAWS FOR THE LOCAL TIMES OF STABLE PROCESSES

by Jay Rosen[1]

1 Introduction

In this paper we obtain second order limit laws for the local times of stable and related Lévy processes. This work generalizes the results obtained for Brownian local time by Papanicolaou, Stroock and Varadhan [1977] and Yor [1983].

For now, let X_t denote the symmetric stable process in $\mathcal{R}^1$ of order $\beta > 1$, which is known to have a jointly continuous local time L_t^x (Boylan [1964]).

Set

$$\langle f, f\rangle_{\beta-1} = -\iint f(x)|x-y|^{\beta-1} f(y)\,dx\,dy. \tag{1.1}$$

Theorem 1.1 *Let f be a bounded Borel function on $\mathcal{R}^1$ with compact support such that*

$$\int f(x)\,dx = 0.$$

If X_s denotes the symmetric stable process of order $\beta > 1$ in $\mathcal{R}^1$, with local time L_t^x then

$$\frac{1}{\lambda^{(1-1/\beta)/2}} \int_0^{\lambda t} f(X_s)\,ds \overset{\mathcal{L}}{\Longrightarrow} \sqrt{2c\langle f, f\rangle_{\beta-1}}\, W_{L_t^0} \tag{1.2}$$

as $\lambda \to \infty$, where $\overset{\mathcal{L}}{\Longrightarrow}$ denotes weak convergence of processes in $C(\mathcal{R}^+)$, W_t denotes a real Brownian motion independent of X and

$$c = \int_0^\infty (p_1(0) - p_1(1/s^{1/\beta}))\frac{ds}{s^{1/\beta}} \tag{1.3}$$

When X is Brownian motion, i.e., $\beta = 2$, this is the result of Papanicolaou, Stroock and Varadhan [1977], which is also presented in Ikeda and Watanabe [1989], p. 147.

We note that by scaling, (1.2) is equivalent to

$$\frac{1}{\epsilon^{(\beta-1)/2}} \int_0^t f_\epsilon(X_s)\,ds \overset{\mathcal{L}}{\Longrightarrow} \sqrt{2c\langle f, f\rangle_{\beta-1}}\, W_{L_t^0} \tag{1.4}$$

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where

$$f_\epsilon(x) = \frac{1}{\epsilon} f(\frac{x}{\epsilon}).$$

To state our next theorem, let

$$\Gamma^{(\gamma)}(x,y) = \frac{1}{2}\left(|x|^\gamma + |y|^\gamma - |x-y|^\gamma\right). \tag{1.5}$$

For any $0 < \gamma < 1$, there exists a continuous mean zero Gaussian process, $B_s^{(\gamma)}(x)$ with covariance

$$E\left(B_s^{(\gamma)}(x)\; B_t^{(\gamma)}(y)\right) = (s \wedge t)\Gamma^{(\gamma)}(x,y), \tag{1.6}$$

see Yor [1988]. In the following, we always take $B_s^{(\gamma)}(x)$ to be independent of X.

Theorem 1.2 *Let L_t^x denote the local time of the symmetric stable process of order $\beta > 1$ in $\mathcal{R}^1$. Then*

$$\frac{1}{\epsilon^{(\beta-1)/2}}\left(L_t^{\epsilon x} - L_t^0\right) \stackrel{\mathcal{L}}{\Longrightarrow} 2\sqrt{c}\; B_{L_t^0}^{(\beta-1)}(x) \tag{1.7}$$

as $\epsilon \to 0$, where $\stackrel{\mathcal{L}}{\Longrightarrow}$ now denotes weak convergence of processes in $C(\mathcal{R}_+ \times R)$, $B^{(\beta-1)}$ is independent of X, and c is given in (1.3).

When X is Brownian motion, i.e. $\beta = 2$, this is the result of Yor [1983].

We next present several variations on Theorem 2, where t is replaced by a random time.

Theorem 1.3 *Let L_t^x denote the local time of the symmetric stable process of order $\beta > 1$ in $\mathcal{R}^1$, and ζ an independent exponential random variable of mean 1, then*

$$\frac{1}{\epsilon^{(\beta-1)/2}}\left(L_\zeta^{\epsilon x} - L_\zeta^0\right) \stackrel{\mathcal{L}}{\Longrightarrow} 2\sqrt{cu^1(0)}\; B_\zeta^{(\beta-1)}(x) \tag{1.8}$$

as $\epsilon \to 0$, where $\stackrel{\mathcal{L}}{\Longrightarrow}$ denotes weak convergence of processes in $C(\mathcal{R})$; $B^{(\beta-1)}$ is independent of ζ, c is given by (1.3) and

$$u^1(0) = \frac{1}{2\pi}\int \frac{dp}{1+|p|^\beta} \tag{1.9}$$

is the 1-potential at 0.

Define

$$\tau_u = \inf\{t \mid L_t^0 > u\} \tag{1.10}$$

We will use $F^{(\beta-1)}(x)$ to denote a fractional Brownian motion of order $\beta - 1$, i.e. the continuous mean zero Gaussian process with covariance

$$E\left(F^{(\beta-1)}(x)\; F^{(\beta-1)}(y)\right) = \Gamma^{(\beta-1)}(x,y). \tag{1.11}$$

Of course, we can take

$$F^{(\beta-1)}(x) \doteq B_1^{(\beta-1)}(x)$$

Theorem 1.4 *Let L^x_t denote the local time of the symmetric stable process of order $\beta > 1$ in $\mathcal{R}^1$.*

Then, for any $u > 0$,

$$\frac{1}{\epsilon^{(\beta-1)/2}}\left(L^{\epsilon x}_{T_u} - u\right) \stackrel{\mathcal{L}}{\Longrightarrow} 2\sqrt{cu}\, F^{(\beta-1)}(x) \tag{1.12}$$

as $\epsilon \to 0$, where $\stackrel{\mathcal{L}}{\Longrightarrow}$ denotes weak convergence of processes in $C(\mathcal{R})$ and c is given by (1.3).

For Brownian motion, i.e. $\beta = 2$ this is contained in Yor [1983].

Finally, let

$$T_a = \inf\{t > 0 \mid X_t = a\} \tag{1.13}$$

Theorem 1.5 *Let L^x_t denote the local time of the symmetric stable process of order $\beta > 1$ in $\mathcal{R}^1$. Then for any a,*

$$\frac{1}{\epsilon^{(\beta-1)/2}}\left(L^{\epsilon x}_{T_a} - L^0_{T_a}\right) \stackrel{\mathcal{L}}{\Longrightarrow} \sqrt{2c}\, B^{(\beta-1)}_{L^0_{T_a}}(x) \tag{1.14}$$

as $\epsilon \to 0$, where $\stackrel{\mathcal{L}}{\Longrightarrow}$ denotes weak convergence of processes in $C(\mathcal{R})$, and c is given by (1.3).

Our basic approach is the method of moments, aided by a simple identity (Lemma 1) concerning the moments of differences of local times of the form that appear in Theorem 2.

Remark: Let now X_t denote a symmetric Lévy process, with characteristic exponent $\psi(p)$ defined by

$$E\left(e^{ipX_t}\right) = e^{-t\psi(p)}.$$

If $\psi(p)$ is regularly varying at ∞ of order $\beta > 1$, then the methods of this paper can be used to show that theorems 2–5, as well as (1.4), will hold if we replace the factor $\frac{1}{\epsilon^{(\beta-1)/2}}$ by

$$\sqrt{\epsilon\psi(\frac{1}{\epsilon})}$$

For some other work on second order limit theorems, see Weinryb- Yor [1988], Biane [1989] and Adler-Rosen [1990].

It is a pleasure to thank Marc Yor for several helpful comments.

2 A Simple Identity

L_t^x continues to denote the local time of X, the symmetric stable process in $\mathcal{R}^1$ of order $\beta > 1$. We use $p_t(x)$ to denote the transition density of X.

Lemma 1

$$E_0\left((L_t^x - L_t^y)^n\right) \tag{2.1}$$
$$= n! \int_{0\le t_1 \cdots \le t_n \le t} \cdots \int \left(p_{t_1}(x) - (-1)^{n-1} p_{t_1}(y)\right) \prod_{i=2}^n \left(p_{\Delta t_i}(0) - (-1)^{n-i} p_{\Delta t_i}(x-y)\right) dt_1, \ldots dt_n$$

Remark: We are most interested in the case of $y = 0$, for which we obtain

$$E_0\left(\left(L_t^x - L_t^0\right)^n\right)$$
$$= n! \int_{0\le t_1 \cdots \le t_n \le t} \cdots \int \prod_{i=1}^n \left(p_{\Delta t_i}(0) - (-1)^{n-i} p_{\Delta t_i}(x)\right) dt_i \tag{2.2}$$

Pf: We first rewrite (2.1) as

$$E_0\left((L_t^x - L_t^y)^n\right)$$
$$= n!\, E_0\left(\int_{0\le t_1 \cdots \le t_n \le t} \cdots \int \prod_{i=1}^n \left(dL_{t_i}^x - dL_{t_i}^y\right)\right)$$
$$= n!\, E_0\left(\int_{0\le t_1 \cdots \le t_{n-2} \le t} \cdots \int \left\{\int_{t_{n-2}}^t \left(\int_{t_{n-1}}^t dL_{t_n}^x - dL_{t_n}^y\right)\left(dL_{t_{n-1}}^x - dL_{t_{n-1}}^y\right)\right\} \prod_{i=1}^{n-2} \left(dL_{t_i}^x - dL_{t_i}^y\right)\right)$$
$$= n!\, E_0\left(\int_{0\le t_1 \cdots \le t_{n-2} \le t} \cdots \int \left(\int_{t_{n-2}}^t A_{t_{n-1}} dL_{t_{n-1}}^x - B_{t_{n-1}} dL_{t_{n-2}}^y\right) \prod_{i=1}^{n-2} \left(dL_{t_i}^x - dL_{t_i}^y\right)\right) \tag{2.3}$$

where

$$A_{t_{n-1}} = E_x\left(\int_0^{t-t_{n-1}} dL_{t_n}^x - dL_{t_n}^y\right)$$
$$= \int_{t_{n-1}}^t (p_{\Delta t_n}(0) - p_{\Delta t_n}(x-y))dt_n$$

and

$$B_{t_{n-1}} = E_y\left(\int_0^{t-t_{n-1}} dL_{t_n}^x - dL_{t_n}^y\right)$$
$$= \int_{t_{n-1}}^t p_{\Delta t_n}(x-y) - p_{\Delta t_n}(0)dt_n$$
$$= -A_{t_{n-1}}$$

Hence (2.3) can be written as

$$
\begin{aligned}
&E_0\left((L_t^x - L_t^y)^n\right) \\
&= n!\, E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-2}\le t} \left(\int_{t_{n-2}}^t A_{t_{n-1}}\left(dL^x_{t_{n-1}} + dL^y_{t_{n-1}}\right)\right)\prod_{i=1}^{n-2}\left(dL^x_{t_i} - dL^y_{t_i}\right)\right) \\
&= n!\, E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-3}\le t} \left\{\int_{t_{n-3}}^t \left(\int_{t_{n-2}}^t A_{t_{n-1}}\left(dL^x_{t_{n-1}} + dL^y_{t_{n-1}}\right)\right)\left(dL^x_{t_{n-2}} - dL^y_{t_{n-2}}\right)\right\}\right. \\
&\qquad \left.\prod_{i=1}^{n-3}\left(dL^x_{t_i} - dL^y_{t_i}\right)\right) \\
&= n!\, E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-3}\le t} \left(\int_{t_{n-3}}^t C_{t_{n-2}}dL^x_{t_{n-2}} - D_{t_{n-2}}dL^y_{t_{n-2}}\right)\prod_{i=1}^{n-3}\left(dL^x_{t_i} - dL^y_{t_i}\right)\right) \qquad (2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{t_{n-2}} &= \int_{t_{n-2}}^t A_{t_{n-1}}\left(p_{\Delta t_{n-1}}(0) + p_{\Delta t_{n-1}}(x-y)\right)dt_{n-1} \\
&= \int_{t_{n-2}}^t \left(\int_{t_{n-1}}^t p_{\Delta t_n}(0) - p_{\Delta t_n}(x-y)dt_n\right)\left(p_{\Delta t_{n-1}}(0) + p_{\Delta t_{n-1}}(x-y)\right)dt_{n-1}
\end{aligned}
$$

and

$$
\begin{aligned}
D_{t_{n-2}} &= \int_{t_{n-2}}^t A_{t_{n-1}}\left(p_{\Delta t_{n-1}}(x-y) + p_{\Delta t_{n-1}}(0)\right)dt_{n-1} \\
&= C_{t_{n-2}}
\end{aligned}
$$

hence

$$
\begin{aligned}
&E_0\left((L_t^x - L_t^y)^n\right) \\
&= n!\, E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-2}\le t} C_{t_{n-2}}\prod_{i=1}^{n-2}\left(dL^x_{t_i} - dL^y_{t_i}\right)\right) \qquad (2.5)
\end{aligned}
$$

and it is now clear that Lemma 1 follows on iterating this procedure.

3 The Basic Limit Theorem

In this section we illustrate the basic idea of this paper by using Lemma 1 to prove weak convergence of the following marginal distributions. In sections 4 and 5 we will elaborate on this idea to complete the proof of Theorem 2.

Proposition 1 *For fixed x, t*

$$
\frac{1}{\epsilon^{(\beta-1)/2}}\left(L_t^{\epsilon x} - L_t^0\right) \stackrel{\mathcal{L}}{\Longrightarrow} 2\sqrt{c}B^{(\beta-1)}_{L_t^0}(x) \qquad (3.1)
$$

where

$$
c = \int_0^\infty \left(p_1(0) - p_1\left(\frac{1}{s^{1/\beta}}\right)\right)\frac{ds}{s^{1/\beta}} \qquad (3.2)
$$

Pf: We first prove

Lemma 2

$$\int_0^t p_s(0) - p_s(x)\, ds - c|x|^{\beta-1} = O\left(\frac{x^2}{t^{3/\beta-1}}\right) \tag{3.3}$$

where $0 < c < \infty$ *is given by (3.2).*

Pf of Lemma 2 We recall the scaling

$$p_s(x) = \frac{1}{s^{1/\beta}} p_1\left(\frac{x}{s^{1/\beta}}\right) \tag{3.4}$$

so that

$$\begin{aligned}
&\int_0^t p_s(0) - p_s(x) ds \\
&= \int_0^t \left(p_1(0) - p_1\left(\frac{x}{s^{1/\beta}}\right)\right) \frac{ds}{s^{1/\beta}} \\
&= |x|^{\beta-1} \int_0^{t/x^\beta} \left(p_1(0) - p_1\left(\frac{1}{s^{1/\beta}}\right)\right) \frac{ds}{s^{1/\beta}} \\
&= c|x|^{\beta-1} - |x|^{\beta-1} \int_{t/x^\beta}^{\infty} \left(p_1(0) - p_1\left(\frac{1}{s^{1/\beta}}\right)\right) \frac{ds}{s^{1/\beta}}
\end{aligned} \tag{3.5}$$

and our lemma now follows from the fact that $p_1(x)$ is C^∞ and symmetric with bounded derivatives so that:

$$\left|p_1(0) - p_1\left(\frac{1}{s^{1/\beta}}\right)\right| \le c\frac{1}{s^{2/\beta}}.$$

Pf of Proposition 1: We calculate moments using (2.2):

$$\begin{aligned}
&E_0\left(\left(L_t^{\epsilon x} - L_t^0\right)^n\right) \\
&= n! \int_{0\le t_1 \cdots \le t_n \le t} \cdots \int \prod_{i=1}^n \left(p_{\Delta t_i}(0) - (-1)^{n-i} p_{\Delta t_i}(\epsilon x)\right) dt_i
\end{aligned} \tag{3.6}$$

Consider first the case $n = 2K + 1$. In (3.6) there will be $K + 1$ factors

$$p_{\Delta t_i}(0) - p_{\Delta t_i}(\epsilon x), \qquad i.e. \quad i = 1, 3, 5, \ldots, 2K + 1 \tag{3.7}$$

Since (3.7) is positive, we can bound (3.6) by

$$\begin{aligned}
&\int_{[0,t]^n} \cdots \int \prod_{i=1}^n \left(p_{s_i}(0) - (-1)^{n-i} p_{s_i}(\epsilon x)\right) ds_i \\
&\le \left(\int_0^t 2p_s(0) ds\right)^K \left(\int_0^t p_s(0) - p_s(\epsilon x) ds\right)^{K+1} \\
&= O\left(\epsilon^{(\beta-1)(K+1)}\right)
\end{aligned} \tag{3.8}$$

by Lemma 2.

Hence

$$E_0\left[\left(\frac{L_t^{\epsilon x}-L_t^0}{\epsilon^{(\beta-1)/2}}\right)^{2K+1}\right]\longrightarrow 0 \quad \text{as } \epsilon\to 0. \tag{3.9}$$

Similarly, for $n=2K$, we can replace each factor

$$p_{\Delta t_i}(0)+p_{\Delta t_i}(\epsilon x) \tag{3.10}$$

by $2p_{\Delta t_i}(0)$, since the error term introduced simply adds another factor of the form (3.7), giving at least $K+1$ such factors, which can be bounded as in (3.8).

Hence

$$\begin{aligned}&\lim_{\epsilon\to 0}E_0\left[\left(\frac{L_t^{\epsilon x}-L_t^0}{\epsilon^{(\beta-1)/2}}\right)^{2K}\right] \qquad (3.11)\\ &= \lim_{\epsilon\to 0}\frac{(2K)!\,2^K}{\epsilon^{(\beta-1)K}}\int\cdots\int_{0\le t_1\cdots\le t_{2K}\le t}\prod_{i=1}^{K}p_{\Delta t_{2i-1}}(0)\left(p_{\Delta t_{2i}}(0)-p_{\Delta t_{2i}}(\epsilon x)\right)dt_{2i-1}dt_{2i}\end{aligned}$$

Let

$$h(t) = p_t(0)\ ,\quad t\ge 0 \tag{3.12}$$

$$f_\epsilon(t) = p_t(0)-p_t(\epsilon x),\quad t\ge 0 \tag{3.13}$$

Using the commutativity of convolution we can write the integral in (3.11) as

$$\begin{aligned}&\int_0^t h*f_\epsilon*h*f_\epsilon*\ldots*h*f_\epsilon(s)ds\\ &= \int_0^t H*F_\epsilon(s)ds \qquad (3.14)\end{aligned}$$

where

$$H(r)=h^{*K}(r) \tag{3.15}$$

the K-fold convolution of h, and

$$F_\epsilon(r)=f_\epsilon^{*K}(r) \tag{3.16}$$

the K-fold convolution of f_ϵ.

We now rewrite (3.14) as

$$\begin{aligned}&\int_0^t H*F_\epsilon(s)ds\\ &= \int_0^t\left(\int_0^s H(r)F_\epsilon(s-r)dr\right)ds\\ &= \int_0^t H(r)\left(\int_0^{t-r}F_\epsilon(s)ds\right)dr \qquad (3.17)\end{aligned}$$

From Lemma 2, we see that for each fixed $t > 0$,

$$\int_0^t \frac{f_\epsilon(s)}{\epsilon^{\beta-1}} ds \longrightarrow c|x|^{\beta-1} \tag{3.18}$$

Hence for each fixed $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} \frac{f_\epsilon(t)}{\epsilon^{\beta-1}} d \longrightarrow c|x|^{\beta-1} \tag{3.19}$$

Hence

$$\int_0^\infty e^{-\lambda t} \frac{F_\epsilon(t)}{\epsilon^{(\beta-1)K}} dt \longrightarrow (c|x|^{\beta-1})^K \tag{3.20}$$

This implies in turn that for each fixed $r > 0$

$$\int_0^r \frac{F_\epsilon(s)}{\epsilon^{(\beta-1)K}} ds \longrightarrow (c\,|x|^{\beta-1})^K. \tag{3.21}$$

Using monotonicity, it is clear that the convergence in (3.21) is uniform in $r,\ \ \delta \le r \le t$.

On the other hand, for any r

$$\begin{aligned} \int_0^r F_\epsilon(s)ds \ &\le\ \int_0^\infty F_\epsilon(s)ds \\ \le (\int_0^\infty f_\epsilon(s)ds)^K \ &=\ (c(\epsilon|x|)^{\beta-1})^K. \end{aligned} \tag{3.22}$$

This shows that

$$\begin{aligned} &\int_0^t H(r)\left(\int_0^{t-r} \frac{F_\epsilon(s)}{\epsilon^{(\beta-1)K}} ds\right) dr \\ &\qquad \longrightarrow (c|x|^{\beta-1})^K \int_0^t H(r)dr \\ &= \ (c|x|^{\beta-1})^K \ \int\!\!\cdots\!\!\int_{0\le t_1 \cdots \le t_K \le t} \prod_{i=1}^K p_{\Delta t_i}(0) dt_i \\ &= \ (c|x|^{\beta-1})^K \frac{1}{K!} E_0\left(\left(L_t^0\right)^K\right) \end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned} &\lim_{\epsilon\to 0} E_0\left[\left(\frac{L_t^{\epsilon x} - L_t^0}{\epsilon^{(\beta-1)/2}}\right)^{2K}\right] \\ &\quad = \ \frac{(2K)!}{2^K K!}(4c|x|^{\beta-1})^K E_0\left(\left(L_t^0\right)^K\right) \\ &\quad = \ (4c)^K E_0\left[\left(B^{(\beta-1)}_{L_t^0}(x)\right)^{2K}\right] \end{aligned} \tag{3.24}$$

which proves proposition 1.

4 Convergence of Finite Dimensional Distributions

Here is the next step in our proof of Theorem 2.

Proposition 2

$$\frac{1}{\epsilon^{(\beta-1)\frac{1}{2}}}\left(L_t^{\epsilon x}-L_t^0\right)\overset{\mathcal{L}}{\Longrightarrow}2\sqrt{c}B^{(\beta-1)}_{L_t^0}(x)$$

in the sense of finite dimensional distributions (c is defined in (3.2))

Proof of Proposition 2: When we consider joint moments, we no longer have a formula as simple as that of Lemma 1. However, we will prove the following lemma, where we use the notation

$$\begin{aligned}\Delta p_r(x) &= p_r(0)-p_r(x) &(4.1)\\ \gamma_r(x,y) &= \Delta p_r(x)+\Delta p_r(y)-\Delta p_r(x-y) &(4.2)\end{aligned}$$

Lemma 3

$$\begin{aligned}&E_0\left(\prod_{i=1}^n\left(L_t^{\epsilon x_i}-L_t^0\right)\right)\\ &=\sum_\pi\int\cdots\int_{0\le t_1\cdots\le t_n\le t}\prod_{j=1}^K p_{\Delta t_{2j-1}}(0)\gamma_{\Delta t_{2j}}(\epsilon x_{\pi_{2j-1}},\epsilon x_{\pi_{2j}})dt_{2j-1}dt_{2j}\\ &\quad+O\left(\epsilon^{(\beta-1)(K+1)}\right),\quad \text{if } n=2K \qquad (4.3)\end{aligned}$$

and

$$=O\left(\epsilon^{(\beta-1)(K+1)}\right),\quad \text{if } n=2K+1$$

These estimates are uniform in $0\le t\le T$ for any fixed $T<\infty$. The sum in (4.3) is over all permutations π of $\{1,\dots,n\}$.

Remark: Our proof will show that the r.h.s (4.3) is unchanged if we start our process at ϵz instead of 0.

Pf of Lemma 3:

$$\begin{aligned}&E_0\left(\prod_{i=1}^n\left(L_t^{\epsilon x_i}-L_t^0\right)\right)\\ &=\sum_\pi E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_n\le t}\prod_{i=1}^n\left(dL_{t_i}^{\epsilon x_{\pi_i}}-dL_{t_i}^0\right)\right) \qquad (4.4)\end{aligned}$$

It suffices to deal with the case $\pi_i = i$. We write this term as

$$E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-2}\le t}\left[\int_{t_{n-2}}^t\left(\int_{t_{n-1}}^t dL_{t_n}^{\epsilon x_n}-dL_{t_n}^0\right)\left(dL_{t_{n-1}}^{\epsilon x_{n-1}}-dL_{t_{n-1}}^0\right)\right]\prod_{i=1}^{n-2}\left(dL_{t_i}^{\epsilon x_i}-dL_{t_i}^0\right)\right)$$

$$= E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-2}\le t} C_{t_{n-2}}\prod_{i=1}^{n-2}\left(dL_{t_i}^{\epsilon x_i}-dL_{t_i}^0\right)\right) \tag{4.5}$$

where, as in the proof of Lemma 1,

$$C_{t_{n-2}} = \int_{t_{n-2}}^t f_{t_{n-1}}\, dL_{t_{n-1}}^{\epsilon x_{n-1}} + \int_{t_{n-2}}^t g_{t_{n-1}}\, dL_{t_{n-1}}^0 \tag{4.6}$$

and

$$\begin{aligned} f_{t_{n-1}} &= \int_{t_{n-1}}^t p_{\Delta t_n}(\epsilon x_n-\epsilon x_{n-1})-p_{\Delta t_n}(\epsilon x_{n-1})dt_n \\ &= \int_{t_{n-1}}^t \Delta p_{\Delta t_n}(\epsilon x_{n-1})-\Delta p_{\Delta t_n}(\epsilon x_n-\epsilon x_{n-1})dt_n \qquad (4.7)\\ g_{t_{n-1}} &= \int_{t_{n-1}}^t p_{\Delta t_n}(0)-p_{\Delta t_n}(\epsilon x_n)dt_n \\ &= \int_{t_{n-1}}^t \Delta p_{\Delta t_n}(\epsilon x_n)dt_n \qquad (4.8)\end{aligned}$$

We now rewrite (4.5) as

$$E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-3}\le t}\left[\int_{t_{n-3}}^t C_{t_{n-2}}(dL_{t_{n-2}}^{\epsilon x_{n-2}}-dL_{t_{n-2}}^0)\right]\prod_{i=1}^{n-3}\left(dL_{t_i}^{\epsilon x_i}-dL_{t_i}^0\right)\right)$$

$$= E_0\left(\int\cdots\int_{0\le t_1\cdots\le t_{n-3}\le t} D_{t_{n-3}}\prod_{i=1}^{n-3}\left(dL_{t_i}^{\epsilon x_i}-dL_{t_i}^0\right)\right) \tag{4.9}$$

where

$$\begin{aligned} D_{t_{n-3}} &= \int_{t_{n-3}}^t\left(\int_{t_{n-2}}^t f_{t_{n-1}}p_{\Delta t_{n-1}}(\epsilon x_{n-1}-\epsilon x_{n-2})+g_{t_{n-1}}p_{\Delta t_{n-1}}(\epsilon x_{n-2})\, dt_{n-1}\right)dL_{t_{n-2}}^{\epsilon x_{n-2}} \\ &\quad -\int_{t_{n-3}}^t\left(\int_{t_{n-2}}^t f_{t_{n-1}}p_{\Delta t_{n-1}}(\epsilon x_{n-1})+g_{t_{n-1}}p_{\Delta t_{n-1}}(0)dt_{n-1}\right)dL_{t_{n-2}}^0 \\ &= \int_{t_{n-3}}^t\left(\int_{t_{n-2}}^t\left(f_{t_{n-1}}+g_{t_{n-1}}\right)p_{\Delta t_{n-1}}(0)dt_{n-1}\right)\left(dL_{t_{n-2}}^{\epsilon x_{n-2}}-dL_{t_{n-2}}^0\right) \\ &\quad +\int_{t_{n-3}}^t \bar f_{t_{n-2}}dL_{t_{n-2}}^{x_{n-2}}+\int_{t_{n-3}}^t \bar g_{t_{n-2}}dL_{t_{n-2}}^0 \qquad (4.10)\end{aligned}$$

where

$$\bar{f}_{t_{n-2}} = -\int_{t_{n-2}}^{t} f_{t_{n-1}} \Delta p_{\Delta t_{n-1}}(\epsilon x_{n-1} - \epsilon x_{n-2}) + g_{t_{n-1}} \Delta p_{\Delta t_{n-1}}(\epsilon x_{n-2})\, dt_{n-1} \tag{4.11}$$

$$\bar{g}_{t_{n-2}} = \int_{t_{n-2}}^{t} f_{t_{n-1}} \Delta p_{\Delta t_{n-1}}(\epsilon x_{n-1})\, dt_{n-1} \tag{4.12}$$

If we now replace $D_{t_{n-3}}$ in (4.9) by its expression in (4.10) we obtain three terms. The first term can be rewritten as

$$E_0 \left(\int \cdots \int_{0 \le t_1 \cdots \le t_{n-2} \le t} h_{t_{n-2}} \prod_{i=1}^{n-2} \left(dL_{t_i}^{\epsilon x_i} - dL_{t_i}^{0} \right) \right) \tag{4.13}$$

where

$$\begin{aligned} h_{t_{n-2}} &= \int_{t_{n-2}}^{t} \left(f_{t_{n-1}} + g_{t_{n-1}} \right) p_{\Delta t_{n-1}}(0) dt_{n-1} \\ &= \int_{t_{n-2}}^{t} \int_{t_{n-1}}^{t} p_{\Delta t_{n-1}}(0) \gamma_{\Delta t_n}(\epsilon x_n,\, \epsilon x_{n-1})\, dt_n\, dt_{n-1} \end{aligned} \tag{4.14}$$

It is easy to see how the main term in Lemma 3 will be generated on iterating this argument.

As for the other terms in (4.10), and their analogues which are obtained in iteration, we note that $\bar{f}$ and $\bar{g}$ [(4.11), (4.12)] each contain two factors of the form Δp, and in general we can check that all terms generated by our iteration aside from the terms in (4.3) will contain at least $K+1$ factors of the form Δp, if $n = 2K$ or $2K+1$. We already know that such factors give rise to the uniform estimate

$$O\left(\epsilon^{(\beta-1)(K+1)} \right)$$

This completes the proof of Lemma 3.

We now continue with the proof of Proposition 2. Consider first a fixed t: by Lemma 3 we have

$$E_0 \left[\prod_{i=1}^{2K+1} \left(\frac{L_t^{\epsilon x_i} - L_t^0}{\epsilon^{(\beta-1)(\frac{1}{2})}} \right) \right] \longrightarrow 0 \tag{4.15}$$

while

$$\begin{aligned} &\lim_{\epsilon \to 0} E_0 \left[\prod_{i=1}^{2K} \left(\frac{L^{\epsilon x_i} - L_t^0}{\epsilon^{(\beta-1)(\frac{1}{2})}} \right) \right] \\ &= \lim_{\epsilon \to 0} \sum_{\pi} \int \cdots \int_{0 \le t_1 \cdots \le t_{2K} \le t} \prod_{j=1}^{K} p_{\Delta t_{2j-1}}(0) \left[\frac{\gamma_{\Delta t_{2j}}(\epsilon x_{\pi_{2j-1}}, \epsilon x_{\pi_{2j}})}{\epsilon^{\beta-1}} \right] dt_i \end{aligned} \tag{4.16}$$

From Lemma 1 we see that

$$\int_0^t \frac{\gamma_s(\epsilon x, \epsilon y)}{\epsilon^{(\beta-1)}} ds \longrightarrow 2c\Gamma^{(\beta-1)}(x, y) \tag{4.17}$$

Hence, as in Section 3, we find that

$$\lim_{\epsilon\to 0} E_0\left[\prod_{i=1}^{2K}\left(\frac{L_t^{x_i}-L_t^0}{\epsilon^{(\beta-1)\frac{1}{2}}}\right)\right]$$
$$= \sum_{\pi}(c2)^K\prod_{j=1}^{K}\Gamma^{(\beta-1)}\left(x_{\pi_{2j-1}},x_{\pi_{2j}}\right)\frac{1}{K!}E_0\left(\left(L_t^0\right)^K\right)$$
$$= \sum_{\text{pairings}}(c4)^K\prod_{j=1}^{K}\Gamma^{(\beta-1)}\left(x_{1_i},x_{2_i}\right)E_0\left(\left(L_t^0\right)^K\right) \tag{4.18}$$

where the sum is over all pairings of the integers $\{1,\ldots,2K\}$ into K pairs $(1_i,2_i)$, $i=1,\ldots,K$.

However, this says exactly that

$$\lim_{\epsilon\to 0} E_0\left[\prod_{i=1}^{2K}\left(\frac{L_t^{x_i}-L_t^0}{\epsilon^{(\beta-1)\frac{1}{2}}}\right)\right]$$
$$= (4c)^K E_0\left(\prod_{i=1}^{2K}B_{L_t^0}^{(\beta-1)}(x_i)\right) \tag{4.19}$$

This shows Theorem 5 for fixed t. When we allow t to depend on i in the l.h.s. of (4.19), it is best to study increments of $L_t^x - L_t^0$ in t—and use the additivity

$$L_{t+s}^x - L_s^x = L_t^x\circ\theta_s. \tag{4.20}$$

To illustrate, let us compute the $\epsilon\to 0$ limit of

$$E_0\left(\prod_{i=1}^{n}\left(L_s^{\epsilon x_i}-L_s^0\right)\prod_{j=1}^{m}\left(L_t^{\epsilon y_j}-L_t^0\right)\circ\theta_s\right)$$
$$= \sum_{\pi}E_0\left(\int_{0\le s_1\cdots\le s_n\le s}\!\!\cdots\int\ \prod_{j=1}^{m}\left(L_t^{\epsilon y_j}-L_t^0\right)\circ\theta_s\prod_{i=1}^{m}\left(dL_{s_i}^{\epsilon x_{\pi_i}}-dL_{s_i}^0\right)\right)$$
$$= \sum_{\pi}E_0\left(\int_{0\le s_1\cdots\le s_n\le s}\!\!\cdots\int\ \left[E_{\epsilon x_{\pi_n}}\left(A_{s_n}\right)dL_{s_n}^{\epsilon x_{\pi_n}}-E_0\left(A_{s_n}\right)dL_{s_n}^0\right]\right.$$
$$\left.\prod_{i=1}^{n-1}\left(dL_{s_i}^{\epsilon x_{\pi_i}}-dL_{s_i}^0\right)\right) \tag{4.21}$$

where

$$A_{s_n} = \prod_{j=1}^{m}\left(L_t^{\epsilon y_j}-L_t^0\right)\circ\theta_{s-s_n} \tag{4.22}$$

Exactly as in the proof of Lemma 3 we find that

$$E_0(A_r) = \sum_{\tilde\pi}E_0\left(\int_{0\le t_1\cdots\le t_m\le t}\!\!\cdots\int\ \prod_{j=1}^{m}\left(dL_{t_j}^{\epsilon y_{\tilde\pi_j}}-dL_{t_j}^0\right)\circ\theta_{s-r}\right)$$

$$= \sum_{\tilde\pi} E_0 \left(\int \cdots \int_{s-r \le t_1 \le \cdots \le t_m \le t+s-r} \prod_{j=1}^{m} \left(dL_{t_j}^{\epsilon y_{\tilde\pi_j}} - dL_{t_j}^{0} \right) \right)$$

$$= \sum_{\tilde\pi} \int \cdots \int_{s-r \le t_1 \cdots \le t_m \le s+t-r} \prod_{j=1}^{\ell} p_{\Delta t_{2j-1}}(0) \gamma_{\Delta t_{2j}}(\epsilon y_{\tilde\pi_{2j-1}}, \epsilon y_{\tilde\pi_{2j}}) dt_{2j-1} dt_{2j}$$
$$+ O\left(\epsilon^{(\beta-1)(\ell+1)}\right) \quad \text{if } m = 2\ell, \ (t_0 \equiv 0) \tag{4.23}$$

and

$$= O\left(\epsilon^{(\beta-1)(\ell+1)}\right) \quad \text{if } m = 2\ell + 1.$$

Changing variables, the sum in the r.h.s. of (4.23) can be written as

$$\sum_{\tilde\pi} \int \cdots \int_{0 \le t_1 \cdots \le t_m \le t} \prod_{j=1}^{\ell} p_{\Delta t_{2j-1}}(0) \gamma_{\Delta t_{2j}}(\epsilon y_{\tilde\pi_{2j-1}}, \epsilon y_{\tilde\pi_{2j}}) dt \tag{4.24}$$

where now $t_0 \equiv -(s-r)$ so that

$$\Delta t_1 = s - r + t_1 \tag{4.25}$$

If we temporarily replace $E_{\epsilon x_{\pi_n}}(\cdot)$ in (4.21) by $E_0(\cdot)$ we can use (4.23) and the analogue of Lemma 3 to obtain a main contribution to (4.21):

$$\sum_{\pi} \int \cdots \int_{0 \le s_1 \cdots \le s_n \le s} \prod_{j=1}^{K} p_{\Delta s_{2j-1}}(0) \gamma_{\Delta s_{2j}}(\epsilon x_{\pi_{2j-1}}, \epsilon x_{\pi_{2j}})$$
$$\left(\sum_{\tilde\pi} \int \cdots \int_{0 \le t_1 \cdots \le t_n \le t} \prod_{j=1}^{\ell} p_{\Delta t_{2j-1}}(0) \gamma_{\Delta t_{2i}}(\epsilon y_{\tilde\pi_{2i-1}}, \epsilon y_{\tilde\pi_{2i}}) dt \right) ds$$
$$\text{if } \ n = 2K, \ m = 2\ell, \ (t_0 = -(s - s_n)) \tag{4.26}$$

As before, if we divide by $\epsilon^{(\beta-1)(n+m)}$ and take the limit we obtain

$$\int \cdots \int_{\substack{0 \le s_1 \le \cdots \le s_n \le s \\ 0 \le t_1 \le \cdots \le t_m \le t}} p_{s_1}(0) p_{\Delta s_2}(0) \dots p_{\Delta s_n}(0) p_{t_1+s-s_n}(0) p_{\Delta t_2}(0) \dots p_{\Delta t_m}(0) ds dt \tag{4.27}$$

$$\sum_{\pi,\tilde\pi} (2c)^{m+n} \prod_{j=1}^{K} \Gamma^{(\beta-1)}(x_{\pi_{2j-1}}, x_{2j}) \prod_{i=1}^{\ell} \Gamma^{(\beta-1)}(y_{\tilde\pi_{2i-1}}, y_{\tilde\pi_{2i}}).$$

We can check that all other terms, including those which arise by replacing $E_{\epsilon x_{\pi_n}}(\cdot)$ by $E_0(\cdot)$ go to zero in the limit. (See the remark following Lemma 3.) Finally, we check that

$$E_0\left(\left(L_s^0\right)^n \left(L_t^0\right)^m \circ \theta_s \right)$$
$$= n!m! \int \cdots \int_{\substack{0 \le s_1 \le \cdots \le s_n \le s \\ 0 \le t_1 \le \cdots \le t_m \le t}} p_{s_1}(0)\ p_{\Delta s_2}(0) \cdots p_{\Delta s_n}(0)\ p_{t_1+s-s_n}(0)$$
$$p_{\Delta t_2}(0) \cdots p_{\Delta t_m}(0) ds dt \tag{4.28}$$

which implies that (4.27) equals

$$(4c)^{m+n}E_0\left(\left(\prod_{i=1}^{n}B_{L_s^0}^{(\beta-1)}(x_i)\right)\left(\prod_{j=1}^{m}B_{L_{s+t}^0}^{(\beta-1)}(y_j)-B_{L_s^0}^{(\beta-1)}(y_j)\right)\right) \tag{4.29}$$

which proves our proposition for moments such as (4.21). The general case is analogous, completing the proof of proposition 2.

5 Tightness; Proof of Theorem 1 and 2

We have for $s<t$

$$\begin{aligned} &E_0\left[\left(L_t^{\epsilon x}-L_t^0\right)-\left(L_s^{\epsilon y}-L_s^0\right)\right]^{2n} \\ \leq\ & c\,E_0\left(L_t^{\epsilon x}-L_t^{\epsilon y}\right)^{2n} \\ +\ & c\,E_0\left[\left(L_t^{\epsilon y}-L_t^0\right)-\left(L_s^{\epsilon y}-L_s^0\right)\right]^{2n} \end{aligned} \tag{5.1}$$

By Lemmas 1 and 2, we have

$$\begin{aligned} &E_0\left(L_t^{\epsilon x}-L_t^{\epsilon y}\right)^{2n} \\ \leq\ & c\left(\int_0^t p_s(0)\,ds\right)^n\left(\int_0^\infty p_s(0)-p_s(\epsilon\,(x-y))ds\right)^n \\ \leq\ & c\,\epsilon^{(\beta-1)n}|x-y|^{(\beta-1)n} \end{aligned} \tag{5.2}$$

while

$$\begin{aligned} &E_0\left[\left(L_t^{\epsilon y}-L_t^0\right)-\left(L_s^{\epsilon y}-L_s^0\right)\right]^{2n} \\ =\ & E_0\left[\left(L_{t-s}^{\epsilon y}-L_{t-s}^0\right)\circ\theta_s\right]^{2n} \\ =\ & E_0\left(E_{X_s}\left[\left(L_{t-s}^{\epsilon y}-L_{t-s}^0\right)^{2n}\right]\right) \end{aligned} \tag{5.3}$$

and for any z, using Lemmas 1 and 2 again

$$\begin{aligned} &E_z\left(L_{t-s}^{\epsilon y}-L_{t-s}^0\right)^{2n} \\ =\ & E_0\left(L_{t-s}^{\epsilon y-z}-L_{t-s}^{-z}\right)^{2n} \\ =\ & \int_{0\leq r_1\cdots\leq r_{2n}\leq t-s}\cdots\int (p_{r_1}(\epsilon y-z)+p_{r_1}(-z))\prod_{i=2}^{2n}(p_{\Delta r_i}(0)-(-1)^{n-i}p_{\Delta r_i}(\epsilon y))dr_i \\ \leq\ & c\left(\int_0^\infty p_r(0)-p_r(\epsilon y)\right)^n\left(\int_0^{t-s}p_r(0)dr\right)^n \\ \leq\ & c\,\epsilon^{(\beta-1)n}|t-s|^{n(1-1/\beta)} \end{aligned} \tag{5.4}$$

Putting this all together, we have

$$\begin{aligned} &E_0\left[\frac{(L_t^{\epsilon x}-L_t^0)-(L_s^{\epsilon y}-L_s^0)}{\epsilon^{(\beta-1)/2}}\right]^{2n} \\ \leq\ & c\,\left(|x-y|^{(\beta-1)n}+|t-s|^{(1-1/\beta)n}\right) \end{aligned} \tag{5.5}$$

which gives tightness.

Theorem 2 now follows from proposition 2 and tightness.

We now obtain Theorem 1, in the form (1.4), from Theorem 2 and the continuous mapping theorem, since $w(x,t) \longrightarrow \int f(x)w(x,t)dx$ is a continuous mapping from $C(R_+ \times R) \longrightarrow C(R_+)$, and

$$\begin{aligned} \int L_t^{\epsilon x} f(x)dx &= \int L_t^x f_\epsilon(x)dx \\ &= \int_0^t f_\epsilon(X_s)ds \end{aligned} \tag{5.6}$$

while the process

$$\int f(x) B_{L_t^0}^{(\beta-1)}(x)dx$$

has the same law as

$$\left(\iint \Gamma^{(\beta-1)}(x,y) f(x) f(y) dx dy \right)^{\frac{1}{2}} W_{L_t^0}$$

6 Proof of Theorem 3

We have

$$E_0\left(\prod_{i=1}^n \left(L_\zeta^{\epsilon x_i} - L_\zeta^0\right)\right) = \int_0^\infty e^{-s} E_0\left(\prod_{i=1}^n \left(L_s^{\epsilon x_i} - L_s^0\right)\right) ds. \tag{6.1}$$

If we return to the calculations of Lemma 4, we obtain a main term if $n = 2K$:

$$\begin{aligned} &\sum_\pi \int_0^\infty e^{-s} \underset{0 \le t_1 \cdots \le t_n \le s}{\int \cdots \int} \prod_{j=1}^K p_{\Delta t_{2j-1}}(0) \gamma_{\Delta t_{2j}}(\epsilon x_{\pi_{2j-1}}, \epsilon x_{\pi_{2j}})\, dt ds \\ &= \sum_\pi \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \cdots \int_{t_n}^\infty \prod_{j=1}^K e^{-\Delta t_{2j-1}} p_{\Delta t_{2j-1}}(0) e^{-\Delta t_{2j}} \gamma_{\Delta t_{2j}}(\epsilon x_{\pi_{2j-1}}, \epsilon x_{\pi_{2j}}) \\ &= \sum_\pi (u^1(0))^K \prod_{j=1}^K \int_0^\infty e^{-s} \gamma_s(\epsilon x_{\pi_{2j-1}}, \epsilon x_{\pi_{2j}}) ds \end{aligned} \tag{6.2}$$

as in (3.9)

$$\frac{1}{\epsilon^{\beta-1}} \int_0^\infty e^{-s} \gamma_s(\epsilon x, \epsilon y) ds \longrightarrow 2c\, \Gamma^{(\beta-1)}(x,y) \tag{6.3}$$

while

$$\int_0^\infty e^{-s} \gamma_s(\epsilon x, \epsilon y) ds \le \bar{c} \epsilon^{\beta-1}. \tag{6.4}$$

The other terms from Lemma 3 thus contribute

$$O\left(\epsilon^{(\beta-1)(K+1)}\right), \qquad n = 2k,\ 2k+1.$$

Hence we see that

$$E_0\left[\prod_{i=1}^{n}\left(\frac{L_\zeta^{\epsilon x_i}-L_\zeta^0}{\epsilon^{(\beta-1)\frac{1}{2}}}\right)\right]$$
$$\longrightarrow 0 \quad \text{if} \quad n=2K+1$$

and

$$\longrightarrow \sum_{\substack{\text{pairings}\\(1_j,2_j)}}(u^1(0))^K(4c)^K K!\prod_{j=1}^{K}\Gamma^{(\beta-1)}(x_{1_j},x_{2_j}) \quad \text{if} \quad n=2K. \tag{6.5}$$

while on the other hand

$$\begin{aligned}
&E\left(\prod_{i=1}^{n}B_\zeta^{(\beta-1)}(x_i)\right)\\
&= \int_0^\infty e^{-s}E\left(\prod_{i=1}^{n}B_s^{(\beta-1)}(x_i)\right)\\
&= 0 \quad \text{if} \quad n=2K+1
\end{aligned}$$

and

$$\begin{aligned}
&= \int_0^\infty s^K e^{-s}ds\sum_{\substack{\text{pairings}\\(1_j,2_j)}}\prod_{j=1}^{K}\Gamma^{(\beta-1)}(x_{1_j},x_{2_j})\\
&= K!\sum_{\substack{\text{pairing}\\(1_j,2_j)}}\prod_{j=1}^{K}\Gamma^{(\beta-1)}(x_{1_j},x_{2_j}) \quad \text{if} \quad n=2K
\end{aligned} \tag{6.6}$$

Tightness follows as before.

7 Proof of Theorems 4 and 5

We will need the following joint convergence:

Proposition 3 *If L_t^x denotes the local time for the symmetric stable process in $\mathcal{R}^1$ of order $\beta > 1$, then*

$$\begin{aligned}
&\left(L_t^0;\ \frac{1}{\epsilon^{(\beta-1)\frac{1}{2}}}\left(L_t^{\epsilon x}-L_t^0\right)\right)\\
&\stackrel{\mathcal{L}}{\Longrightarrow}\left(L_t^0;\ 2\sqrt{c}B_{L_t^0}^{(\beta-1)}(x)\right)
\end{aligned} \tag{7.1}$$

as $\epsilon \to 0$, in the sense of weak convergence of processes in

$$C\left(\mathcal{R}_+\times\mathcal{R},\mathcal{R}^2\right).$$

Proof of Theorem 4: This follows as in the proof of Yor's result for Brownian motion [1983] from the continuous mapping theorem for weak convergence and the fact that for each fixed u we have $\mathcal{P}(\tau_{u^-}=\tau_u)=1$.

Proof of Proposition 3: We consider the analogue of Lemma 3 for

$$E_0\left(\left(L_t^0\right)^\ell \prod_{i=1}^n \left(L_t^{\epsilon x_i} - L_t^0\right)\right) \tag{7.2}$$

Running through the proof of that lemma, we at first find a sum over permutations $\tilde{\pi}$ of $\{1,2,\dots,n+\ell\}$. However, many of these permutations will give error terms which are

$$O\left(\epsilon^{(\beta-1)(K+1)}\right), \qquad n = 2K \text{ or } 2K+1.$$

Consider for definiteness $n = 2K$. Let us use s_i, $1 \le i \le \ell$ for the time variable of $dL_{s_i}^0$ corresponding to the ℓ factors $(L_t^0)^\ell$, and t_j $j = 1,\dots,2K$ for the time variables corresponding to the factors $L_t^{\epsilon x_{\pi_j}} - L_t^0$, where the permutation π of $\{1,\dots,n\}$ is naturally induced from $\tilde{\pi}$ by requiring $t_1 \le t_2 \le \cdots \le t_n$. We can check that unless our permutation $\tilde{\pi}$ is such that for each i we have

$$t_{2j} \le s_i \le t_{2j+1} \tag{7.3}$$

for some $j = j(i) = 0,1,\dots,K$, then we obtain an extra Δp factor, giving rise to a contribution which is $O\left(\epsilon^{(\beta-1)(K+1)}\right)$. On the other hand, as seen in the proof of Proposition 2, each permutation $\tilde{\pi}$ inducing π and satisfying the above conditions (7.3) gives rise to the analogue of (4.18):

$$(2c)^K \prod_{j=1}^K \Gamma^{(\beta-1)}(x_{\pi_{2j-1}},\ x_{\pi_{2j}}) \frac{1}{(\ell+K)!} E_0\left(\left(L_t^0\right)^{\ell+K}\right). \tag{7.4}$$

There are $\frac{(\ell+K)!}{K!}$ permutations $\tilde{\pi}$ satisfying the above conditions (7.3) and inducing π, since there are $\ell!$ ways to order the ℓ letters s_i, $1 \le i \le \ell$ and $\binom{\ell+K}{K}$ ways to partition an ordered sequence of ℓ objects into $K+1$ ordered groups.

Hence the total is

$$\sum_\pi (2c)^K \prod_{j=1}^K \Gamma^{(\beta-1)}(x_{\pi_{2j-1}},\ x_{\pi_{2j}}) \frac{1}{K!} E_0\left(\left(L_t^0\right)^{\ell+K}\right) \tag{7.5}$$

and the proof is finished as in the proof of Lemma 3—see especially (4.18), (4.19).

The proof of proposition 3 now follows in the same way that Theorem 2 followed from Lemma 3.

Proof of Theorem 5: This will follow, in analogy with the proof of theorem 4, from Lemma 5.7, p. 11 of Revuz and Yor [1990] and the fact that

$$\left(X_t\ ;\ \frac{1}{\epsilon^{(\beta-1)\frac{1}{2}}}\left(L_t^{\epsilon x} - L_t^0\right)\right) \stackrel{\mathcal{L}}{\Longrightarrow} \left(X_t\ ;\ 2\sqrt{c}B_{L_t^0}^{(\beta-1)}(x)\right)$$

as $\epsilon \to 0$ in the sense of weak convergence of processes in $D(\mathcal{R}_+) \times C(\mathcal{R}_+ \times \mathcal{R},\ \mathcal{R})$, and this follows from moment considerations analogous to the proof of proposition 3. More precisely, since X_t itself doesn't have moments of all order, we look at $\varphi(X_t)$ where φ is an invertable transformation from $\mathcal{R}$ to [-1,1].

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