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Necessary and sufficient conditions for the existence of m-perfect processes associated with Dirichlet forms

Séminaire de probabilités (Strasbourg), tome 25 (1991), p. 374-406
<http://www.numdam.org/item?id=SPS_1991__25__374_0>

Necessary and sufficient conditions for the existence of m-perfect processes associated with Dirichlet forms

by

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1. Introduction and the main result

As is well known, a Hunt process associated with a Dirichlet form with "C_0"-regularity(i.e. with a regular Dirichlet form on a locally compact metrizable space) was first constructed by M. Fukushima [Fu2]. See also the fundamental work of Fukushima [Fu3] and Silverstein [Si]. In this paper we extend the result of Fukushima and Silverstein to Dirichlet forms without the assumption of C_0-regularity. We mention that there exist already publications concerning the existence of strong Markov processes associated with non-regular Dirichlet forms, see the work of Fukushima [Fu1] and Silverstein [Si]. Moreover there are constructions of diffusion processes for Dirichlet forms in infinite dimensional spaces, see the papers by Albeverio and Høegh-Krohn [AH1]-[AH3], Albeverio and Röckner [ARö2], Fukushima [Fu4] and Kusuoka [Ku]. The authors of the above mentioned papers made use of the previous results for C_0-regular Dirichlet spaces by employing certain compactification methods.

There has been another treatment of the relationship between Markov process and Dirichlet spaces. In this treatment one assumes that there exists already certain strong Markov processes and then one investigates the related Dirichlet spaces. See the work of Dynkin [D1] [D2], Fitzsimmons [Fi1] [Fi2], Fitzsimmons and Getoor [FG], Fukushima [Fu 5], Bouleau-Hirsch [BoH]. For other work on Dirichlet forms see also Dellacherie-Meyer [DM Chap. XIII], Kunita-Watanabe [KW], Knight [Kn].[1])

Our approach differs from all the above mentioned treatments. We construct directly a strong Markov process along the same line of the construction used in [Fu3] Chapter 6. By so doing we obtain necessary and sufficient conditions for the existence of a certain right process (we call it an m-perfect process, see Def. 1.2 below) associated with a given Dirichlet space without the assumption of C_0-regularity. Our construction relies on the refinement of the semigroup via quasi-continuous kernels (see [AM1]). In fact we construct quasi-continuous kernels in a general framework, which can be used even in situations where there are no underlying Dirichlet forms (this is related to previous work by Getoor [G1] and Dellacherie–Meyer [DM Chap. IX]). In this connection we mention another related work of Kaneko [Ka] who constructed Hunt processes by quasi-continuous kernels with respect to $C_{r,p}$-capacity.

Our work is also an extension of a result of Y. LeYan [Le1-2] who obtained a characterization of the semigroup associated with Hunt processes. In fact our argument for the necessity of the condition (1.9) (see Th. 1.8 below) comes from an idea of [Le1-2]. Some of our results have been announced in [AM2].

We now introduce some concepts and related results which are necessary for describing our main result.

Let X be a metrizable topological space with Borel sets $\mathcal{X}$. A cemetery point $\Delta \notin X$ is adjoined to X as an isolated point of $X_\Delta := X \cup \{\Delta\}$. Let $(X_t) = (\Omega, M, M_t, X_t, \theta_t, P_x)$ be a strong Markov process with state space $(X, \mathcal{X})$ and life time $\zeta := \inf\{t \geq 0 : X_t = \Delta\}$ (c.f. e.g. [BG]). We denote by $(P_t)_{t\geq 0}$ the transition function of (X_t) and by $(R_\alpha)_{\alpha>0}$ the resolvent of (X_t), i.e.

$$P_t f(x) = E_x[f(x_t)] \tag{1.1}$$

and

$$R_\alpha f(x) = E_x\left[\int_0^\infty e^{-\alpha t} f(x_t)dt\right] \tag{1.2}$$

provided the above right hand sides make sense. σ_A denotes the hitting time of a subset A of X_Δ, i.e.

$$\sigma_A = \inf\{t > 0 \; : \; X_t \in A\} \; . \tag{1.3}$$

1.1 Definition (X_t) is called a perfect process if it satisfies the following properties:

(i) Normal property:

$$P_x(X_0 = x) = 1, \; \forall x \in X_\Delta \tag{1.4}$$

(ii) Right continuity: $t \longmapsto X_t(w)$ is right continuous from

$$[0, \infty) \text{ to } X_\Delta, \; P_x \text{ a.s.}, \; \forall x \in X_\Delta \; . \tag{1.5}$$

(iii) Left limit up to ζ: $\lim_{s\uparrow t} X_s(w) =: X_{t-}(w)$ exists in X

$$\text{for all } t \in (0, \zeta(w)), P_x \text{ a.s. }, \; \forall x \in X \; . \tag{1.6}$$

(iv) Strengthened fine continuity of resolvent: $R_1 f(X_{t-}) I_{\{t<\zeta\}}$ is P_x-indistinguishable from

$R_1 f(X_t)_- I_{\{t<\zeta\}}$ $\forall x \in X, \; f \in b\mathcal{X}$. (1.7)

Here and henceforth $b\mathcal{X}$ denotes all bounded $\mathcal{X}$-measurable functions,

$R_1 f(X_t)_- I_{\{t<\zeta\}} := \lim_{s\uparrow t} R_1 f(X_s) I_{\{t<\zeta\}}$ (we always make the convention that $Z_{0-} = Z_0$ for an arbitrary process $(Z_t)_{t\geq 0}$).

Remarks on the Definition 1.1

(i) A strong Markov process satisfying (1.4) and (1.5) is called a right process with Borel transition semigroup (see [Sh] Def. (8.1), see also [G2] (9.7); but in [Sh] and [G2] it is also assumed that X is a Radon space).

(ii) A special standard process (see [G2] (9.10)), in particular, a Hunt process always satisfies (1.6) and (1.7).

To sum up the above remarks, we have the following inclusions among different classes of processes (c.f. [G2] pp. 55):

(Feller) $\subset$ (Hunt) $\subset$ (special standard) $\subset$ (perfect) $\subset$ (right).

In what follows we assume that m is a σ-finite Borel measure on X.

1.2 Definition

(i) (X_t) is said to be <u>m-tight</u> if there exists an increasing sequence of compact sets $\{K_n\}_{n\geq 1}$ of X such that

$$P_x\left\{\lim_n \sigma_{X-K_n} \geq \zeta\right\} = 1, \ m \text{ a.e. } x \in X \tag{1.7}$$

(ii) (X_t) is called an <u>m-perfect process</u> if it is a perfect process and is m-tight.

Due to an idea of T.J. Lyons and M. Röckner [LR], we proved in [AMR1] the following proposition.

1.3 Proposition Suppose that X is a polish space, then any strong Markov process (X_t) satisfying (1.5) and (1.6) is m-tight.

For the proof of Proposition 1.3 see [AMR1].

Remark

(i) It is evident from Proposition 1.3 that any perfect process is an m-perfect process provided the state space X is a Polish space.

(ii) We mention that for the special case of (X_t) being a standard process on a locally compact metrizable space, the conclusion of Proposition 1.3 can be derived from [BG] (9.3).

We now consider a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X,m)$ (see e.g. [Fu3] for the definition). We set

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + (f,g), \ \forall\, f,g \in \mathcal{F} \ .$$

Here and henceforth $(\cdot,\cdot)$ denotes the inner product of $L^2(X,m)$. In the sequel we always regard $\mathcal{F}$ as a Hilbert space equipped with the inner product $\mathcal{E}_1$. For a closed set $F \subset X$, we set

$$\mathcal{F}_F = \{f \in F : \ f = 0 \text{ m-a.e. on } X - F\} \ . \tag{1.9}$$

$\mathcal{F}_F$ is then a closed subset of $\mathcal{F}$.

1.4 Definition An increasing sequence of closed sets $\{F_k\}_{k\geq 1}$ of X is called an <u>$\mathcal{E}$-nest</u> if $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$.

A subset $B \subset X$ is said to be <u>$\mathcal{E}$-polar</u> if there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $B \subset \bigcap_{k\geq 1}(X - F_k)$. A function f on X is said <u>$\mathcal{E}$-quasi-continuous</u> if there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $F_{|F_k}$, the restriction of f to F_k, is continuous on F_k for each $k \geq 1$.

We remark that every $\mathcal{E}$-polar set is m-negligible (see Prop. 2.7).

We denote by $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ the semigroup and resolvent on $L^2(X,m)$ associated with $(\mathcal{E}, \mathcal{F})$ respectively. We set

$$\mathcal{H} = \{h : \ h = G_1 f \text{ with } f \in L^2(X,m), \ 0 < f \leq 1 \text{ m.a.e. } \} \tag{1.10}$$

$\mathcal{H}$ is non-empty because we assumed m to be σ-finite. For $h \in \mathcal{H}$ we now define the <u>h-weighted capacity</u> Cap_h as follows:

$$\text{Cap}_h(G) = \inf\{\mathcal{E}_1(f,f) : f \in \mathcal{F},\ f \geq h \text{ m.a.e. on } G\} \tag{1.11}$$

for an open set G and

$$\text{Cap}_h(B) = \inf\{\ \text{Cap}_h(G) :\ G \supset B,\ G \text{ open }\} \tag{1.12}$$

for an arbitrary set $B \subset X$.

In Section 2 we shall show that Cap_h is a Choquet capacity enjoying countable subadditivity. The importance of Cap_h is its connection to $\mathcal{E}$-nest stated in the following proposition.

1.5 Proposition An increasing sequence of closed sets $\{F_k\}$ of X is an $\mathcal{E}$-nest if and only if for some $h \in \mathcal{H}$ (hence for all $h \in \mathcal{H}$):

$$\text{Cap}_h(X - F_k) \downarrow 0, \quad \text{as } k \longrightarrow \infty .$$

For the proof of Proposition 1.5 see Prop. 2.5 of Section 2.

Denote by Cap the usual 1-capacity defined e.g. in [Fu3]. Obviously we have $\text{Cap}_h(B) \leq \text{Cap}(B)$ for every $B \subset X$. Consequently we have the following corollary of Proposition 1.5.

1.6 Corollary Every set $B \subset X$ with $\text{Cap}(B) = 0$ is an $\mathcal{E}$-polar set. Every nest $\{F_k\}$ in the sense of [Fu3] is an $\mathcal{E}$-nest. Every quasi-continuous function in the sense of [Fu3] is an $\mathcal{E}$-quasi-continuous function.

Let (X_t) be a Markov process with transition function $(P_t)_{t\geq 0}$. We say that (X_t) is <u>associated with $\mathcal{E}$</u>, if

$$T_t f = P_t f \text{ m.a.e. },\ \forall f \in L^2(X,m),\ t > 0 \tag{1.13}$$

The main result of this paper is the following.

1.7 Theorem Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(X,m)$. Then the following conditions (i) - (iii) are necessary and sufficient conditions for the existence of an m-perfect process (X_t) associated with $\mathcal{E}$.

(i) There exists an $\mathcal{E}$-nest $\{X_k\}_{k\geq 1}$ consisting of compact sets. (1.14)

(ii) There exists an $\mathcal{E}_1$-dense subset $\mathcal{F}_0$ of $\mathcal{F}$ consisting of $\mathcal{E}$-quasi-continuous functions. (1.15)

(iii) There exists a countable subset B_0 of $\mathcal{F}_0$ and an $\mathcal{E}$-polar set N such that

$$\sigma\{u : u \in B_0\} \supset \mathcal{X} \cap (X - N)\ . \tag{1.16}$$

Remarks on Theorem 1.7

(i) Concerning the existence of a certain reasonable Markov process associated with a given Dirichlet form, it is often assumed in the literature that:

There exists an $\mathcal{E}_1$-dense subset $\tilde{\mathcal{F}}$ of $\mathcal{F}$ consisting of continuous functions. (1.17)

We remark that (1.17) is not necessary for the existence of an m-perfect process. It is even not necessary for the existence of a diffusion process. Here is a counter example. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on a locally compact space X such that each single-point set of X is a set of zero capacity (e.g. the classical Dirichlet form associated with the Laplacian on $\mathbb{R}^d$ with $d \geq 2$). Let μ be a smooth measure which is nowhere Radon in the sense that $\mu(G) = +\infty$ for all non-empty open set $G \subset X$. (For the existence of such nowhere Radon smooth measures see [AM4]). We now consider the perturbed form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ defined as follows:

$$\mathcal{F}^\mu = \mathcal{F} \cap L^2(X, m),$$

$$\mathcal{E}^\mu(f,g) = \mathcal{E}(f,g) + \int_X fg\mu(dx) \; \forall f, g \in \mathcal{F}^\mu .$$

It has been proved that $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is again a Dirichlet form ([AM7] Th. 3.2). We can check that $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies all the conditions (1.14) - (1.16) ([AM7]). Hence Theorem 1.8 is applicable and there exists an m-perfect process associated with $(\mathcal{E}^\mu, \mathcal{F}^\mu)$. Moreover, if $(\mathcal{E}, \mathcal{F})$ satisfies local property, then so does $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ and there exists a diffusion process associated with $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ (see (ii) below). On the other hand, it is evident that (1.17) fails to be true for $(\mathcal{E}^\mu, \mathcal{F}^\mu)$. In fact, there is even no continuous functions (except the null function) in the domain $\mathcal{F}^\mu$ because μ is nowhere Radon.

(ii) The application of Theorem 1.8 to infinite dimensional spaces and to quantum field theory will be discussed in subsequent papers. Here we mention that an m-perfect process is a diffusion (i.e. $P_x\{X_t$ is continuous in $t \in [0, \zeta)\} = 1$, for $q.e. x \in X$) if and only if the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the local property in the sense of [Fu3]. Hence Theorem 1.7 extends the results of the existence of diffusion processes for Dirichlet forms in infinite dimensional spaces ([AH1-3], [ARö2], [Ku]), on one hand. On the other hand Theorem 1.7 provides us with a mathematical tool for constructing strong Markov processes with discontinuous sample paths, having also applications in quantum field theory.

(iii) By requiring $\mathcal{F}_0$ (in (1.15)) consisting of strictly $\mathcal{E}$-quasi-continuous functions we obtain necessary and sufficient conditions for the existence of Hunt processes associated with Dirichlet forms. See [AM8] for details in this connection.

(iv) By introducing a dual h-weighted capacity and employing the Ray-Knight compactification method, it is possible to obtain an analogue result of Theorem 1.7 for non-symmetric Dirichlet forms satisfying the sector condition. This will be discussed in subsequent papers.

Before concluding this introduction, we present some more concepts and related results.

1.8 Definition (c.f. [Fu5])

(i) Let $(T_t)_{t>0}$ be the semigroup associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ and $(P_t)_{t\geq 0}$ be the transition semigroup of a perfect process (X_t). We say that (X_t) is properly associated with $\mathcal{E}$, if

$P_t f$ is an $\mathcal{E}$-quasicontinuous version of $T_t f$ for all $t > 0$ and $f \in L^2(X, m)$ (1.18)

(ii) Let (X_t) be a perfect process with state space $(X, \mathcal{X})$ and life time ζ, and let $S \in \mathcal{X}$. We say that S is (X_t)-invariant if

$$P_x \{X_t \in S \text{ and } X_{t-} \in S \text{ for all } t < \zeta\} = 1, \ \forall x \in S .$$

(iii) Let (X_t) and (Y_t) be two perfect processes on $(X, \mathcal{X})$. We say that (X_t) and (Y_t) are m-equivalent if there is a set $S \subset X$ with $m(X - S) = 0$ such that

(a) S is both (X_t)-invariant and (Y_t)-invariant;

(b) The transition semigroups of (X_t) and (Y_t) restricted to S are the same.

We now state the following results, to be further discussed below.

1.9 Proposition Let (X_t) be an m-perfect process. If (X_t) is associated with $(\mathcal{E}, \mathcal{F})$, then (X_t) is properly associated with $(\mathcal{E}, \mathcal{F})$.

1.10 Proposition Let (X_t) and (Y_t) be two symmetric m-perfect processes on $(X, \mathcal{X})$. Then (X_t) and (Y_t) are m-equivalent if and only if they are associated with a common Dirichlet form $(\mathcal{E}, \mathcal{F})$.

The above two propositions can be proved by employing the results of Proposition 7.3 in Section 7 and following the argument of [Fu5]. We omit their detailed proofs in this paper.

By virtue of Proposition 1.10 and Proposition 1.11 we can strenghten the statement of Theorem 1.8 as follows.

1.11 Theorem There is a one to one correspondence between the family of m - equivalence classes of symmetric m-perfect processes and the family of Dirichlet forms satisfying conditions (1.14) - (1.16). The correspondence is given by the relationship (1.18).

Titles of the remaining sections

2. h-weighted capacity
3. $\mathcal{E}$-quasicontinuity
4. Sufficiency of the conditions (1.14) - (1.16)
5. Necessity of the condition (1.14)
6. Necessity of the condition (1.15)
7. Necessity of the condition (1.16)

   Appendix: Construction of the process

2. h-weighted capacities

Throughout Sections 2 - 4 we assume that a Dirichlet form $(\mathcal{E},\mathcal{F})$ is given on $L^2(X,m)$. Let $(T_t)_{t>0}$ be the associated semigroup and $(G_\alpha)_{\alpha>0}$ the corresponding resolvent. Following [Fu3], we say that an element $u \in L^2(X,m)$ is α-excessive if u satisfies

$$u \geq 0, e^{-\alpha t}T_t u \leq u \text{ m.a.e., } \forall t > 0 . \tag{2.1}$$

In the following three lemmas we state some results on α-excessive functions without proof. These results are well known in the context of regular Dirichlet spaces, but their proofs in fact do not rely on the regularity assumption. See [Fu3] Section 3 for details.

2.1 Lemma (c.f. [Fu3], Theorem 3.2.1) The following statements are equivalent to each other (for $u \in \mathcal{F}$ and $\alpha > 0$).
(i) u is α-excessive.
(ii) $u \geq 0$, $\beta G_{\beta+\alpha} u \leq u$ m.a.e., $\forall \beta > 0$.
(iii) $\mathcal{E}_\alpha(u,\nu) \geq 0$, $\forall \nu \in \mathcal{F}$, $\nu \geq 0$ m.a.e. .

2.2 Lemma ([Fu3] Lemma 3.3.2) Let u_1 and u_2 be α-excessive functions in $L^2(X,m)$, $\alpha > 0$. If $u_1 \leq u_2$ m.a.e. and $u_2 \in \mathcal{F}$, then $u_1 \in \mathcal{F}$ and $\mathcal{E}_\alpha(u_1,u_1) \leq \mathcal{E}_\alpha(u_2,u_2)$.
For an α-excession function $h \in \mathcal{F}$ and $B \subset X$ an open set, we put

$$\mathcal{L}_{h,B} = \{f \in \mathcal{F} : f \geq h \text{ m.a.e. on } B\} \tag{2.2}$$

2.3 Lemma (c.f. [Fu3] Lemma 3.1.1, see also [R] Lemma 3.1) Let $h \in \mathcal{F}$ be α-excessive, $\alpha > 0$ and $B \subset X$ be open. Then we have the following assertions.
(i) There exists a unique element $h_B \in \mathcal{L}_{h,B}$ such that

$$\mathcal{E}_\alpha(h_B,h_B) = \inf\{\mathcal{E}_\alpha(u,u) \ : \ u \in \mathcal{L}_{h,B}\} .$$

(ii) $\mathcal{E}_\alpha(h_B,u) \geq 0$, $\forall u \in \mathcal{F}$, $u \geq 0$ m.a.e. on B. In particular, h_B is α-excessive.
(iii) $0 \leq h_B \leq h$ m.a.e. and $h_B = h$ m.a.e. on B.
(iv) h_B is the unique element of $\mathcal{L}_{h,B}$ satisfying

$$\mathcal{E}_\alpha(h_B,\ u - h_B) \geq 0,\ \forall u \in \mathcal{L}_{h,B} .$$

Let $\mathcal{H}$ be defined by (1.10). For $h \in \mathcal{H}$, we consider now the h-weighted capacity Cap_h defined by (1.11) and (1.12).

2.4 Proposition Cap $_h$ is a Choquet capacity, i.e.

(i) $$A \subset B \Longrightarrow \text{Cap}_h(A) \leq \text{Cap}_h(B)\ ,$$

(ii) $$A_n \uparrow \Longrightarrow \text{Cap}_h(\bigcup_n A_n) = \sup_n \text{Cap}_h(A_n)\ ,$$

(iii) $$A_n \text{ compact, } A_n \downarrow \Longrightarrow \text{Cap}_h\left(\bigcap_n A_n\right) = \inf_n \text{Cap}_h(A_n)\ .$$

Moreover, Cap $_h$ is countably subadditive, i.e.,

(iv) $$\text{Cap}_h\left(\bigcup_n A_n\right) \leq \sum_n \text{Cap}_h(A_n)$$

Proof Apply Lemma 2.3 and follow the argument of [Fu3] Lemma 3.1.2 and Theorem 3.1.1. ■

2.5 Proposition

(i) An increasing sequence F_k of closed sets is an $\mathcal{E}$-nest if and only if Cap $_h(X - F_k) \downarrow 0$.
(ii) A subset $N \subset X$ is an $\mathcal{E}$-polar set if and only if Cap $_h(N) = 0$.

Proof The assertion (ii) is a direct consequence of the assertion (i). We now prove (i). Let $\{F_k\}$ be an increasing sequence of closed sets and $\mathcal{F}_{F_k}$ be specified by (1.9). Then every element $u \in \mathcal{F}$ is uniquely decomposed by $u = (u - u_k) + u_k$ with $(u - u_k) \in \mathcal{F}_{F_k}$ and u_k being orthogonal to $\mathcal{F}_{F_k}$ with respect to the inner product $\mathcal{E}_1$. It is easy to check that $\{u_k\}$ is an $\mathcal{E}_1$-Cauchy sequence. Denote by u_∞ the limit of $\{u_k\}$ in $\mathcal{F}$. Then

$$\mathcal{E}_1(u_\infty, \nu) = 0,\ \forall \nu \in \bigcup_k \mathcal{F}_{F_k} \tag{2.3}$$

In particular, for $h \in \mathcal{H}$ we have $h_k = h_{X-F_k}$ with h_{X-F_k} being specified by Lemma 2.3. Suppose now $\{F_k\}$ is an $\mathcal{E}$-nest. Then $\bigcup \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$ with respect to $\mathcal{E}_1$-norm. Consequently by (2.3) we know the limit h_∞ of $\{h_k\}$ is zero, which in turn implies

$$\text{Cap}_h(X - F_k) = \mathcal{E}_1(h_k, h_k) \downarrow 0\ .$$

Conversely, suppose that Cap $_h(X - F_k) \downarrow 0$. Then for an arbitrary $u \in \mathcal{F}$, we have

$$\lim_k \mathcal{E}_1(h_k, u) \leq [\ \text{Cap}_h(X - F_k)]^{\frac{1}{2}}\ \|u\|_{\mathcal{E}_1} \longrightarrow 0\ .$$

On the other hand, suppose that $h = G_1 f$ with $0 < f \leq 1$, $f \in L^2(X, m)$, we have

$$\mathcal{E}_1(h_k, u) = \mathcal{E}_1(u_k, h) = \int_X u_k f m(dx)$$

Therefore if $u \in G_1 g$ for some nonnegative $g \in L^2(X, m)$, then by Fatou's lemma,

$$\int u_\infty f m(dx) \leq \liminf_k \int u_k f m(dx) = 0\ .$$

Consequently $u_\infty = 0$ m.a.e. and $(u - u_k)$ converges to u in $\mathcal{E}_1$-norm. From this we conclude that $\bigcup \mathcal{F}_{F_k}$ is a form core of $\mathcal{F}$ and hence complete the proof. ■

2.6 Corollary Let $h_1, h_2 \in \mathcal{H}$. Then Cap_{h_1} and Cap_{h_2} are equivalent in the sense that for any decreasing sequence of subsets $\{A_k\}$ of X, $\text{Cap}_{h_1}(A_k) \downarrow 0$ if and only if $\text{Cap}_{h_2}(A_k) \downarrow 0$.

Proof This Corollary is a clear consequence of Prop. 2.5. ∎

The following proposition shows that any $\mathcal{E}$-polar set is m-negligible.

2.7 Proposition Let $\{F_k\}$ be an $\mathcal{E}$-nest. Then

$$m(X - \bigcup_{k\geq 1} F_k) = 0$$

Proof By the definition of $\mathcal{E}$-nest, $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ where $\mathcal{F}_{F_k}$ is defined by (1.4), which in turn implies that $\bigcup \mathcal{F}_{F_k}$ is dense in $L^2(X,m)$. From (1.4) we know that $f = 0$ m.a.e. on $N := (X - \bigcup F_k)$ for each $f \in \bigcup \mathcal{F}_{F_k}$, and consequently $f = 0$ m.a.e. on N for each $f \in L^2(X,m)$. Thus $m(N) = 0$ because m is σ-finite on X. ∎

3. $\mathcal{E}$-quasicontinuity

Given an $\mathcal{E}$-nest $\{F_k\}$, we introduce the notation

$$C(\{F_k\}) = \{f : \ f_{|F_k} \text{ is continuous for each } k\} \ . \tag{3.1}$$

A function f is $\mathcal{E}$-quasi-continuous if and only if there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $f \in C(\{F_k\})$.

3.1 Proposition ([Fu3] Th. 3.1.2(i)) Let S be a countable family of $\mathcal{E}$-quasi-continuous functions. Then there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $S \subset C(\{F_k\})$.

Proof The proposition follows easily by applying Theorem 2.5 and following the argument of [Fu3] Th. 3.1.2(i). ∎

The following proposition is an analogue of [Fu3] Lemma 3.1.5. But our proof is slightly different from that of [Fu3] because we make no assumption that $\mathcal{F}$ contains an $\mathcal{E}_1$-dense subset consisting of continuous functions.

3.2 Proposition Let $f \in \mathcal{F}$ be $\mathcal{E}$-quasi-continuous. Then for $h \in \mathcal{H}$,

$$\text{Cap}_{h}\{x \in X \ : \ |f(x)| > \lambda\} \leq \frac{1}{\lambda^2}\mathcal{E}_1(f,f), \ \forall \lambda > 0 \ . \tag{3.2}$$

Proof Let $\{F_k\}$ be an $\mathcal{E}$-nest such that $f \in C(\{F_k\})$. For $\lambda > 0$, we set

$$G_k = \{|f| > \lambda\} \bigcup (X - F_k) \ .$$

Then G_k is an open set and $G_k \supset \{|f| > \lambda\}$. Let

$$f_k = \frac{|f|}{\lambda} + h_{(X-F_k)} \ .$$

Then $f_k \in \mathcal{F}$ and $f_k \geq h$ m.a.e. on G_k. Hence

$$\text{Cap}_{h}(G_k) \leq \mathcal{E}_1(f_k, f_k) \leq \left(\sqrt{\frac{1}{\lambda^2}\mathcal{E}_1(f,f)} + \sqrt{\text{Cap}_{h}(X - F_k)}\right)^2$$

Letting $k \longrightarrow \infty$, we obtain

$$\text{Cap}_{h}\{|f| > \lambda\} \leq \lim_{k} \ \text{Cap}_{h}(G_k) \leq \frac{1}{\lambda^2}\mathcal{E}_1(f,f) \ .$$

■

We say that f_n converges to f <u>$\mathcal{E}$-quasi-uniformly</u> if there exists an $\mathcal{E}$-nest F_k such that f_n converges to f uniformly on each F_k.

3.3 Proposition (c.f. [Fu3] Th. 3.1.4) Let $\{f_n\}$ be a sequence of $\mathcal{E}$-quasi-continuous functions such that $f_n \in \mathcal{F}$ and f_n converges to $f \in \mathcal{F}$ in $\mathcal{E}_1$-norm. There exists then a subsequence $\{f_{n_i}\} \subset \{f_n\}$ and an $\mathcal{E}$-quasi-continuous function $\tilde{f}$ such that $\tilde{f} = f$ m.a.e. and f_{n_i} converges to $\tilde{f}$ $\mathcal{E}$-quasi-uniformly.

Proof Apply Propositions 3.1, 3.2 and 2.5, and follow the argument of [Fu3] Th. 3.1.4. ■

3.4 Proposition Let $h \in \mathcal{H}$ be $\mathcal{E}$-quasi-continuous, $\{F_k\}$ be an $\mathcal{E}$-nest such that $h \in C(\{F_k\})$, and $\{\delta_k\}$ be a decreasing sequence of positive numbers such that $\delta_k \downarrow 0$. There exists then an $\mathcal{E}$-nest $\{F'_k\}$ such that $F'_k \subset F_k$ and $h \geq \delta_k$ on each F'_k.

Proof We set $F'_k = \{h \geq \delta_k\} \bigcap F_k$. Then $\{F'_k\}$ is an increasing sequence of closed sets. Let

$$g_k = (h \wedge \delta_k) + h_{X-F_k},\ \forall k \geq 1$$

Then $g_k \in \mathcal{F}$ and $g_k \geq h$ m.a.e. on $X - F'_k$. We have

$$\text{Cap}_{\,h}(X - F'_k) \leq \mathcal{E}_1(g_k, g_k) \leq 2\mathcal{E}_1(h \wedge \delta_k, h \wedge \delta_k) + 2\ \text{Cap}_{\,h}(X - F_k) \longrightarrow 0\ .$$

Hence by Prop. 2.5 $\{F'_k\}$ is an $\mathcal{E}$-nest with the required properties. ∎

The following Corollary is immediate.

3.5 Corollary Let $h \in \mathcal{H}$ be $\mathcal{E}$-quasi-continuous. Then $\{h = 0\}$ is an $\mathcal{E}$-polar set. ∎

Following [Fu3], we say that an $\mathcal{E}$-nest $\{F_k\}$ is regular if for each k, supp $(I_{F_k} \cdot m) = F_k$.

3.6 Proposition (c.f. [Fu3] Lemma 3.1.3) Let $\{F_k\}$ be an $\mathcal{E}$-nest. Suppose that for each k, the relative topology of F_k is secondly countable. Set $F'_k =$ supp $(I_{F_k} \cdot m)$. Then $\{F'_k\}$ is a regular $\mathcal{E}$-nest.

Proof The proof is easily obtained by applying Theorem 2.5 and following the argument of [Fu3] Lemma 3.1.3. ∎

Also the following Proposition is easily shown:

3.7 Proposition (c.f. [Fu3] Lemma 3.1.4) Let $\{F_k\}$ be a regular $\mathcal{E}$-nest and $f \in C(\{F_k\})$. If $f \geq 0$ m.a.e. on an open set G, then $f \geq 0$ on

$$\left(\bigcup_{k\geq 1} F_k\right) \bigcap G\ .$$

∎

4. Sufficiency of the conditions (1.14)-(1.16)

Let us now assume that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfying conditions (1.14) and (1.15) is given on $L^2(X, m)$. In particular, we fix an $\mathcal{E}$-nest $\{X_k\}$ consisting of compact sets.

We shall say that a property holds $\mathcal{E}$ q.e. (abbreviation for $\mathcal{E}$-quasi-everywhere), if it holds outside an $\mathcal{E}$-polar set.

4.1 Lemma

(i) Each element $f \in \mathcal{F}$ admits an $\mathcal{E}$-quasi-continuous version.

(ii) Let f be $\mathcal{E}$-quasi-continuous and $f \geq 0$ m.a.e. on an open set G, then $f \geq 0$ $\mathcal{E}$ q.e. on G.

Proof (i) follows from (1.14) and Proposition 3.3. To prove (ii), we assume that $f \in C(\{F_k\})$ for some $\mathcal{E}$-nest $\{F_k\}$. Set $F'_k = F_k \bigcap X_k$, then $\{F'_k\}$ is an $\mathcal{E}$-nest such that each F'_k is secondly countable. By Proposition 3.6 we may construct a regular $\mathcal{E}$-nest $\{F''_k\}$ such that $F''_k \subset F'_k \subset F_k$. Obviously $f \in C(\{F''_k\})$. The desired assertion thus follows from Proposition 3.7. ■

Let us denote by $Y = \bigcup_{k\geq 1} X_k$ and $\mathcal{Y} = \mathcal{X} \bigcap Y$. By Proposition 2.7, we have $m(X-Y) = 0$. Hence we may identify $L^2(Y,m)$ with $L^2(X,m)$ in an obvious way.

4.2 Lemma

(i) The relative topology of Y is secondly countable.

(ii) $(Y, \mathcal{Y})$ is a Lusinian measurable space.

(iii) $L^2(Y, \mathcal{Y})$ is separable.

(iv) $\mathcal{F}$ equipped with the $\mathcal{E}_1$-norm is separable.

Proof This follows easily from the fact that Y is a σ-compact metrizable space. ■

A nonnegative function k is called a <u>kernel on $X \times \mathcal{Y}$</u> if $k(x,\cdot)$ is a measure on $(Y, \mathcal{Y})$ for each fixed $x \in X$ and $k(\cdot, A)$ is $\mathcal{X}$ measurable for each fixed $A \in \mathcal{Y}$. We shall write kf for $\int_Y f(y)k(\cdot, dy)$ provided it makes sense.

Recall that $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ is the Markovian semigroup and resolvent associated with $(\mathcal{E}, \mathcal{F})$ respectively.

4.3 Proposition For each $t > 0$, there exists a kernel $\tilde{P}_t$ on $X \times \mathcal{Y}$ such that

(i) $\tilde{P}_t f$ is an $\mathcal{E}$-quasi-continuous version of $T_t f$ for each $f \in L^2(Y, m)$.

(ii) $\tilde{P}_t(x,y) \leq 1, \ \forall x \in X$.

For each $\alpha > 0$, there exists a kernel $\tilde{R}_\alpha$ on $X \times \mathcal{Y}$ such that

(iii) $\tilde{R}_\alpha f$ is an $\mathcal{E}$-quasi-continuous version of $G_\alpha f$ for each $f \in L^2(Y, m)$.

(iv) $\alpha \tilde{R}_\alpha(x,y) \leq 1, \ \forall x \in X$.

The kernel $\tilde{P}_t$ (resp. $\tilde{R}_\alpha$) is $\mathcal{E}$-q.e. unique in the sense that if there is another kernel k on $X \times \mathcal{Y}$ satisfying (i) (resp. (iii)), then $k(x,\cdot) = \tilde{P}_t(x,\cdot)$ (resp. $= \tilde{R}_\alpha(x,\cdot)$) for $\mathcal{E}$ q.e. $x \in X$.

Proof We prove only the case of T_t. By spectral calculus (c.f. [Fu3] Lemma 1.3.3) $T_t f \in \mathcal{F}$ for all $f \in L^2(X,m)$ and

$$\mathcal{E}(T_t f, T_t f) \leq \frac{1}{2t}\{(f,f) - (T_t f, T_t f)\}, \ \forall f \in L^2(X,m) \tag{4.1}$$

For each $f \in L^2(X,m) \simeq L^2(Y,m)$, choose an $\mathcal{E}$-quasi-continuous version $\tilde{T}_t f$ of $T_t f$. By (4.1), Proposition 3.2 and Lemma 4.1(ii) we see that $\tilde{T}_t$ is a quasi-linear positive map from $L^2(Y,m)$ to $qC(X)$ ($qC(X)$ denotes the collection of all $\mathcal{E}$-quasi-continuous functions $\mathcal{E}$q.e. on X) in the sense of [AM1] Def. 1.2 (with respect to some h-weighted capacity). Applying [AM1] Th. 4.2 we obtain a kernel k on $X \times \mathcal{Y}$ satisfying (i). Take a sequence of positive functions f_n in $L^2(Y,m)$ such that $0 \leq f_n \uparrow 1$. We have $kf_n \leq 1$ $\mathcal{E}$-q.e. by Lemma 4.1 (ii). Hence we may find an $\mathcal{E}$-polar set N such that $kf_n(x) \leq 1$ for all $x \in X - N$ and $n \geq 1$. Set

$$\tilde{P}_t(x,A) = \begin{cases} k(x,A), & \forall x \in X - N, \ A \in \mathcal{Y}, \\ 0, & \forall x \in N \ . \end{cases}$$

Then $\tilde{P}_t$ is a desired kernel. The $\mathcal{E}$-q.e. uniqueness follows also from [AM1] Th. 4.2. ■

With the quasi-continuous Markovian kernels $\{\tilde{P}_t\}_{t>0}$ and $\tilde{R}_1$ in hand, we can now follow the argument used in [Fu3] Chap. 6 to construct an m-perfect process associated with a given Dirichlet form satisfying conditions (1.14) - (1.16). We state the final result below and postpone the detailed proof to the Appendix.

4.4 Proposition Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(X,m)$ satisfying conditions (1.14) - (1.16). There exists then an m-perfect process (X_t) associated with $\mathcal{E}$ in the sense of (1.13).

5. Necessity of the condition (1.14)

For proving the necessity of the conditions (1.14) – (1.16), in Section 5 – 7 we assume that an m–perfect process $(X_t) := (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \Theta_t, P_x)$ with life time ζ is associated with a given Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$.
We fix a function $\varphi \in L^2(X,m) \cap L^1(X,m)$ such that $0 < \varphi \leq 1$. Set $\mu(dx) = \varphi(x)m(dx)$ and

$$h(x) = E_x\left[\int_0^\infty e^{-t}\varphi(X_t)dt\right], \quad \forall x \in X \tag{5.1}$$

For an arbitrary subset $B \subset X$, we have defined

$$\sigma_B(\omega) = \inf\{t > 0 : \ X_t(\omega) \in B\} \tag{1.3}$$

We now set for $B \subset X$

$$U_B(x) = E_x\left[\int_0^{\sigma_B} e^{-t}\varphi(X_t)dt\right] \tag{5.2}$$

5.1 Lemma (c.f [FuO], [AM5]) Let B be an open subset of X, then $U_B \in \mathcal{F}$ and

$$\mathcal{E}_1(U_B, u) = (\varphi, u), \quad \forall u \in \mathcal{F}_{X-B} \tag{5.3}$$

(Recall $\mathcal{F}_{X-B}$ is defined by (1.9)).

Proof Let us define for a nonnegative Borel function f:

$$V_n f(x) = E_x \left[\int_0^\infty e^{-t-n\int_0^t I_B(X_s)ds} f(X_t)dt \right] \tag{5.4}$$

Then $V_n\varphi(x) \uparrow U_B(x)$ pointwise. By virtue of the Markovian property and Fubini's Theorem we have

$$V_n\varphi(x) = E_x \left[\int_0^\infty e^{-t}\varphi(X_t)dt \right] - nE_x \left[\int_0^\infty e^{-t}(V_n\varphi)I_B(X_t)dt \right] \tag{5.5}$$

Consequently $V_n\varphi \in \mathcal{F}$ and

$$\mathcal{E}_1(V_k\varphi, V_j\varphi) = (\varphi, V_j\varphi) - (k(V_k\varphi)I_B, V_j\varphi) \tag{5.6}$$

By the symmetry property we have

$$(k(V_k\varphi)I_B, V_j\varphi) = (V_j(k(V_k\varphi)I_B, \varphi) \tag{5.7}$$

Writing $B_t = \int_0^t I_B(X_s)ds$, we have for $j \geq k$,

$$\begin{aligned} V_j(k(V_k\varphi)I_B)(x) &= E_x \left[\int_0^\infty e^{-s-jB_s} \left(E_{X_s} \int_0^\infty e^{-t-kB_t}\varphi(X_t)dt \right) kdB_s \right] \\ &= E_x \left[\int_0^\infty e^{-(j-k)B_s} \left(\int_s^\infty e^{-t-RB_t}\varphi(X_t)dt \right) kdB_s \right] \\ &= E_x \left[\int_0^\infty e^{-t}\varphi(X_t) \left(e^{-RB_t} \int_0^t e^{-(j-R)B_s} kdB_s \right) dt \right]. \end{aligned}$$

Noticing that

$$1 \geq e^{-kB_t} \int_0^t e^{-(j-k)B_s} kdB_s \to 0 \text{ when } j \geq k \to \infty,$$

we conclude frome (5.7) that

$$(k(V_k\varphi)I_B, V_j\varphi) \to 0 \text{ when } j \geq k \to \infty.$$

Thus from (5.6) we see that $\{V_n\varphi\}_{n\geq 1}$ forms an $\mathcal{E}_1$-Cauchy sequence which implies $U_B \in \mathcal{F}$. Moreover, from (5.5) we see that

$$\mathcal{E}_1(V_n\varphi, u) = (\varphi, u), \ \forall u \in \mathcal{F}_{X-B}.$$

Letting $n \to \infty$ we obtain (5.3). ∎

5.2 Lemma Let B be an open subset of X. Set

$$\tilde{h}_B(x) = E_x\left[e^{-\sigma_B} h(X_{\sigma_B}) I_{\{\sigma_B<\zeta\}}\right]. \tag{5.8}$$

Then $\tilde{h}_B$ is a version of h_B being specified by Lemma 2.3.

Proof By the strong Markovian property we see that

$$\tilde{h}_B(x) = h(x) - U_B(x)$$

where U_B is defined by (5.2). Thus $\tilde{h}_B \in \mathcal{F}$ and

$$\mathcal{E}_1(\tilde{h}_B, u) = 0, \ \forall u \in \mathcal{F}_{X-B}. \tag{5.9}$$

It is obvious that $\tilde{h}_B = h$ on B, which together with (5.9) and Lemma 2.3 (iv) implies $\mathcal{E}_1\left(\tilde{h}_B - h_B, \tilde{h}_B - h_B\right) = 0.$ ■

5.3 Proposition Let B be an open subset of X. Then

$$\mathrm{Cap}_h(B) = E_\mu\left[e^{-\sigma_B} h(X_{\sigma_B}) I_{\{\sigma_B<\zeta\}}\right].$$

Proof

$$E_\mu\left[e^{-\sigma_B} h(X_{\sigma_B}) I_{\{\sigma_B<\zeta\}}\right] = (\varphi, \tilde{h}_B)$$
$$= \mathcal{E}_1(h, h_B) = \mathcal{E}_1(h_B, h_B) = \mathrm{Cap}_h(B).$$

Notice that up to now we didn't use m–tightness of (X_t). We now make use of m–tightness to prove the necessity of the condition (1.14). ■

5.4 Proposition $(\mathcal{E}, \mathcal{F})$ satisfies (1.14)

Proof Let $\{X_k\}_{k\geq 1}$ be an increasing sequence of compact sets of X satisfying

$$P_x\left\{\lim_k \sigma_{X-X_k} \geq \zeta\right\} = 1, \ m-\text{a.e. } x \in X \tag{5.9}$$

Set $G_k = X - X_k$, then

$$P_x\left\{\lim_k \sigma_{G_k} \geq \zeta\right\} = 1, \ m-\text{a.e. } x \in X$$

which implies $\tilde{h}_{G_k}(x) \downarrow 0$ m.a.e. and consequently

$$\mathrm{Cap}_k(G_k) \downarrow 0.$$

The proof is completed by applying Proposition 2.5. ■

6. Necessity of the condition (1.15)

Let φ, μ, h be specified as at the beginning of the previous section. For our purpose we now introduce another stopping time τ_A:

$$\tau_A = \inf\{0 \le t < \zeta : X_t \in A \text{ or } X_{t-} \in A\} \bigwedge \zeta. \tag{6.1}$$

(we make the convention that $\inf \phi = \infty$).

It is easy to check that $\tau_A \le \sigma_A \wedge \zeta$ and if A is an open set, then $\tau_A = \sigma_A \wedge \zeta$.

A subset $A \subset X$ is called finely polar if there exists a Borel set $\tilde{A} \supset A$ such that $P_\mu(\tau_{\tilde{A}} < \zeta) = 0$. We shall show in the next section that the concepts of "finely polar" and "$\mathcal{E}$–polar" are equivalent provideed (X_t) is m–tight. In this section we want to avoid employing m–tightness of (X_t). Nevertheless, we can see immediately that a finely polar set is necessarily m–negligible, because $P_\mu(\tau_A < \zeta) = 0$ implies $P_\mu(\sigma_A < \zeta) = 0$ and (X_t) is m–symmetric.

Let us define for $f \in b\mathcal{X}$,

$$\|f\| = E_\mu\left[\sup_{t\ge 0}\left(\int_t^\infty e^{-s}\varphi(X_s)ds\right)(|f(X_t)| \vee |f(X_{t-})|)\right]. \tag{6.2}$$

Obviously for a Borel set $A \subset X$, we have

$$\|I_A\| = E_\mu\left[\int_{\tau_A}^\infty e^{-s}\varphi(X_s)ds\right]. \tag{6.3}$$

6.1 Lemma

(i) Let A be an open set of X, then $\mathrm{Cap}_h(A) = \|I_A\|$.

(ii) Let $f \in b\mathcal{X}$. If $\|f\| = 0$, then $f = 0$ except for a finely polar set.

Proof (i) follows from (6.3), Lemma 5.3 and the fact that $\tau_A = \sigma_A \wedge \zeta$ for an open set A. To prove (ii), let $\|f\| = 0$ and define $A_n = \{|f| > \frac{1}{n}\}$. Then (6.3) shows that

$$E_\mu\left[\int_{\tau_{A_n}}^\infty e^{-s}\varphi(X_s)ds\right] \le n\|f\| = 0,$$

which implies $P_\mu\{\tau_{A_n} < \zeta\} = 0$.

Let $A = \{|f| > 0\}$. We have

$$P_\mu\{\tau_A < \zeta\} \le \sum_{n\ge 1} P_\mu\{\tau_{A_n} < \zeta\} = 0.$$

■

Set

$$\tilde{C} = \{f \in b\mathcal{X} : f(X_t) \text{ is right continuous and } f(X_{t-}) \text{ is left continuous on } [0,\zeta) P_\mu \text{ a.s.}\} \tag{6.4}$$

6.2 Lemma Let $\{f_n\} \subset \tilde{C}$. If $f_n(x) \downarrow 0$ for all $x \in X - N$, with some finely polar set N, then $\|f_n\| \downarrow 0$.

Proof Let $\omega \in \Omega$ be such that $f_n(X_t(\omega))$ is right continuous and $f_n(X_t(\omega))$ is left continuous on $[0, \zeta(\omega))$ for all $n \geq 1$ and such that $\tau_N(\omega) = \zeta(\omega)$. For an arbitrary $T > 0$, we have

$$\sup_{0 \leq t \leq T} \left(\int_t^\infty e^{-s} \varphi(X_s) ds \right) (|f_n(X_t)| \vee |f_n(X_{t-})|)(\omega) \downarrow 0, \ n \uparrow \infty.$$

Consequently

$$\limsup_{n \to \infty} \|f_n\| \leq e^{-T} \sup_x f_1(x),$$

which proves the lemma.

∎

Denote by bC the set of all bounded continuous functions on X. Then bC is a subset of $\tilde{C}$. Let $\bar{C}$ be the $\|\cdot\|$ short closure of bC.

6.3 Lemma $\tilde{C} \subset \bar{C}$.

Proof Applying Daniell's theorem ([DM] III.35) we see from Lemma 6.1 that any positive bounded linear functional on $(\tilde{C}, \|\cdot\|)$ admits an integral representation with some finite Borel measure on X. From the fact that $\tilde{C}$ is a vector lattice and $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, we have that any bounded linear functional on $\tilde{C}$ is a difference of two positive bounded linear functionals on $\tilde{C}$. Consequently each bounded linear functional on $\tilde{C}$ admits an integral representation in terms of some finite signed Borel measure on X. If the lemma were not true, then by a version of the Hahn–Banach Theorem (see e.g. [Scha] Chap. II 3.2) there would be a non-zero bounded linear functional on $\tilde{C}$ vanishing on bC, which would contradict the integral representation. ∎

We are now in a position to prove the necessity of the condition (1.15).

6.4 Proposition $(\mathcal{E}, \mathcal{F})$ satisfies (1.15).

Proof Let $\mathcal{F}_0' = \{R_1 f : f \in b\mathcal{X} \bigcap L^2(X,m)\}$. Then $\mathcal{F}_0'$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and $\mathcal{F}_0' \subset \tilde{C}$. The proposition will be proved by showing that for each $f \in \tilde{C}$, there exists an $\mathcal{E}$-quasi-continuous function $\tilde{f} \in \tilde{C}$ such that $\|f - \tilde{f}\| = 0$. Let $f \in \tilde{C}$. By Lemma 6.2 we can take a sequence $\{f_n\} \subset bC$ such that $\|f_n - f\| \to 0$. Without loss of generality we assume that $\{f_n\}$ is uniformly bounded and $\|f_n - f_{n+1}\| < 2^{-2n}$. Let

$$\begin{aligned} A_n &= \{x : |f_n(x) - f_{n+1}(x)| > 2^{-n}\} \\ B_n &= \cup_{m \geq n} A_m \end{aligned}$$

We have from Lemma 6.1 (i),

$$\begin{aligned} \mathrm{Cap}_h(B_n) &\leq \sum_{m \geq n} \mathrm{Cap}_h(A_m) = \sum_{m \geq n} \|I_{A_m}\| \\ &\leq \sum_{m \geq n} \|2^m |f_m - f_{m+1}|\| \leq 2^{-n+1} \end{aligned}$$

It is easy to see that $\{f_n\}$ converges uniformly on each $(X - B_n)$. Let $\tilde{f}(x) = \lim_n f_n(x)$ on $\bigcup_{n \geq 1}(X - B_n)$ and $\tilde{f}(x) = f(x)$ on $\bigcap_{n \geq 1} B_n$. Then $\tilde{f}$ is $\mathcal{E}$-quasi-continuous. Moreover, one can check that $\|\tilde{f} - f\| = 0$ and consequently $\tilde{f}$ is an $\mathcal{E}$-quasi-continuous version of f by Lemma 6.1 (ii). ∎

Remark In proving Proposition 6.4 we did not make use of m-tightness of (X_t).

7. Necessity of the condition (1.16)

Let (X_t) be the same as in the previous section.

7.1 Lemma Let $\{A_n\}_{n \geq 1}$ be a decreasing sequence of open sets of X such that $\bigcap_{n \geq 1} \bar{A}_n = B$. Then

$$\lim_{n \to \infty} \tau_{A_n} = \tau_B, \quad P_x^- \text{a.s.}, \; \forall x \in X.$$

Proof Let us set $\tau_\infty = \lim_n \tau_{A_n}$. Then $\tau_\infty \leq \tau_B$. On the other hand, set $\Omega_0 = \{\omega \in \Omega : X_t(\omega)$ is right continuous and has left limit on $[0, \zeta(\omega))\}$. Let $\omega \in \Omega_0$. If $\tau_\infty(\omega) < \zeta(\omega)$ and $\tau_{A_n}(\omega) = \tau_\infty(\omega)$ for some $n \geq 1$, then $X_{\tau_\infty}(\omega) \in B$. If $\tau_\infty(\omega) < \zeta(\omega)$ and $\tau_{A_n}(\omega) < \tau_\infty(\omega)$ for all $n \geq 1$, then $X_{\tau_\infty -}(\omega) \in B$. In both cases we have $\tau_B(\omega) \leq \tau_\infty(\omega)$. Hence $\tau_\infty = \tau_B$ P_x – a.s.

∎

In what follows we make full use of m-tightness of (X_t).

7.2 Proposition

$$\text{Cap}_h(A) = \|I_A\|, \ \forall A \in \mathcal{X}. \tag{7.1}$$

Proof In Lemma 6.1 we proved (7.1) for any open set A. Since X is a metric space, for an arbitrary closed set A we can always find a decreasing sequence of open sets $\{A_n\}$ such that $\bigcap_{n\geq 1} \bar{A}_n = A$. Thus by virtue of (6.3) and Lemma 7.1 we see that (7.1) holds also for any closed set A.

Let $Y = \bigcup_{n\geq 1} X_n$. Then Y is a σ–compact metric space. Cap_h restricted to the subsets of Y is still a Choquet capacity. Let A be an arbitrary Borel set of A. By Choquet Theorem (c.f. [DM] III 28.) we can find an increasing sequence of compact sets $\{K_n\}$ such that $K_n \subset A \cap Y$ and $\lim_n \text{Cap}_h(K_n) = \text{Cap}_h(A \cap Y)$. On account of the fact that $\text{Cap}_h(X - Y) = 0$, we see from the above that

$$\|I_A\| \geq \lim_n \|I_{K_n}\| = \text{Cap}_h(A),$$

which proves the Proposition because the inverse inequality is always true.

■

7.3 Proposition

(i) A set $A \subset X$ is finely polar if and only if A is $\mathcal{E}$–polar.

(ii) Any element $f \in \tilde{C}$ is $\mathcal{E}$–quasi–continuous.

(iii) For each closed set $A \subset X$, there exists an $\mathcal{E}$–quasi–continuous function $u_A \in \mathcal{F}$ such that $u_A = 0$ on A and $u_A > 0$ on $X - A$.

Proof (i) follows from (7.1) and (6.3). (ii) follows from (i) and the proof of Proposition 6.4. We now prove (iii). Let B be an open set of X and u_B be defined by (5.2). We see from the proof of Lemma 5.1 that u_B is $\mathcal{E}$–quasi–continuous, because $u_B(x) = \lim_n V_n\varphi(x)$, and $\{V_n\varphi\}_{n\geq 1}$ is an $\mathcal{E}_1$–Cauchy sequence of $\mathcal{E}$–quasi–continuous functions. Let now A be a closed set. We define

$$u_A(x) = E_x \int_0^{\tau_A} e^{-s}\varphi(X_s)ds. \tag{7.2}$$

Let $\{B_n\}$ be a decreasing sequence of open sets such that $\bigcap_{n\geq 1} \bar{B}_n = A$. By Lemma 7.1 we have $u_A(x) = \lim_n u_{B_n}(x)$. It is easy to check from (5.3) that $\{u_{B_n}\}$ is a $\mathcal{E}_1$–Cauchy sequence. Hence u_A is $\mathcal{E}$–quasi–continuous and $u_A \in \mathcal{F}$. Obviously we have $u_A = 0$ on A and $u_A > 0$ on $X - A$.

7.4 Proposition $(\mathcal{E}, \mathcal{F})$ satisfies (1.16).

Proof Let $\{A_n\}_{n\geq 1}$ be a countable family of open sets such that $\{A_n \cap Y : n \geq 1\}$ forms a basis for the relative topology of $Y := \bigcup_{n\geq 1} X_n$. Let $B_0 = \{u_{X-A_n} : n \geq 1\}$. Then B_0 satisfies condition (1.16).

■

Appendix. Construction of the process

This Appendix is devoted to the proof of Proposition 4.4.

Throughout this Appendix we assume that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfying (1.14) — (1.16) is given on $L^2(X, m)$. We shall freely employ the notations used in Section 4.

Let Θ be a countable family of open sets of X such that $X \in \Theta$ and

$$\{A \cap Y : A \in \Theta\} \text{ forms a basis of the relative topology of } Y . \tag{A.1}$$

Set

$$\Theta_1 = \{A : A = \bigcup_{i=1}^{n} A_i, \ A_i \in \Theta, \ n \geq 1\} \tag{A.2}$$

We fix an element $h \in \mathcal{H}$. For each $A \in \Theta_1$, choose an $\mathcal{E}$–quasi-continuous version $\tilde{h}_A$ of h_A, where h_A is specified by Lemma 2.3. In particular, we write $\tilde{h}$ for $\tilde{h}_X$.

Let B_0 be a countable set of $\mathcal{E}$–quasi-continuous functions in $\mathcal{F}$ satisfying (1.16). Without loss of generality we assume that all elements of B_0 are bounded and

$$\sigma\{u : u \in B_0\} \supset \mathcal{Y}. \tag{A.3}$$

By virtue of Lemma 4.2 (iv), we assume also that B_0 is $\mathcal{E}_1$–dense in $\mathcal{F}$.

We denote by Q and Q_+ the set of all rational numbers and all positive rational numbers respectively.

A.1 Lemma There exists a countable subset of bounded $\mathcal{E}$–quasi-continuous functions $\tilde{H} \subset \mathcal{F}$ such that

(i) $\tilde{H} \supset B_0 \cup \{\tilde{h}_A : A \in \Theta_1\}$.
(ii) $\tilde{H} \supset \bigcup_{t\in Q_+} \tilde{P}_t(\tilde{H})$, $\tilde{H} \supset \tilde{R}_1(\tilde{H})$.
(iii) $\tilde{H}$ is an algebra over Q.
(iv) $f \in \tilde{H}$ implies $|f| \in \tilde{H}$ and $f \wedge 1 \in \tilde{H}$.

Proof Apply [Fu3] Lemma 6.1.1. ■

A.2 Lemma There exists a regular $\mathcal{E}$–nest $\{F_k\}$ with $Y_1 := \bigcup_{n\geq 1} F_k$ satisfying the following properties.

(i) $\tilde{H} \subset C(\{F_k\})$. $F_n \subset X_k$, $\forall k \geq 1$.

(ii) $\inf\{\tilde{h}(x) : x \in F_k\} > 0$, $\forall k \geq 1$.

(iii) $0 \leq \tilde{h}_A(x) \leq \tilde{h}(x)$, $\forall x \in Y_1$, $A \in \Theta_1$
$\tilde{h}_A(x) = \tilde{h}(x)$, $\forall x \in A \cap Y_1$, $A \in \Theta_1$

(iv) There exists a sequence $\{t_k\} \subset Q_+$, $t_k \downarrow 0$ such that
$\tilde{P}_{t_k} u(x) \to u(x)$, $\forall u \in \tilde{H}$, $x \in Y_1$;
$\frac{1}{t_k}\{\tilde{R}_1 u(x) - e^{-t_k}\tilde{R}_1\tilde{P}_{t_k}u(x)\} \to u(x)$, $\forall u \in \tilde{H}$, $x \in Y_1$.

(v) $\tilde{P}_t\tilde{P}_s u(x) = \tilde{P}_{t+s}u(x)$, $\forall u \in \tilde{H}$, $x \in Y_1$, $t, s \in Q_+$.

(vi) $\tilde{P}_t\tilde{R}_1 u(x) = \tilde{R}_1\tilde{P}_t u(x)$, $e^{-t}\tilde{P}_t\tilde{R}_1 u(x) \leq \tilde{R}_1 u(x)$, $\forall x \in Y_1$, $t \in Q_+$, $u \in \tilde{H}_+ (\tilde{H}_+ := \{u \in \tilde{H} : u \geq 0\})$;

(vii) $e^{-t}\tilde{P}_t\tilde{h}_A(x) \leq \tilde{h}_A(x)$, $\forall x \in Y_1$, $A \in \Theta_1$

(viii) $\tilde{h}_A(x) \leq \tilde{h}_B(x)$, $\forall x \in Y_1$, $A, B \in \Theta_1$, $A \subset B$.

Proof By spectral calculus (c.f. [Fu3] Lemma 1.1.3.) we have $T_t u \to u$ and $\frac{1}{t}(G_1 u - e^{-t}$

$G_1 T_t u) \to u$ in $\mathcal{E}_1$–norm when $u \in \mathcal{F}$ and $t \downarrow 0$. Hence by Proposition 3.3 and Lemma 4.1 (ii), and taking the fact that $\tilde{H}$ is countable into account, we may take a sequence $\{t_k\} \subset Q_+$, $t_k \downarrow 0$, and an $\mathcal{E}$–polar set N such that (iv) holds for every $u \in \tilde{H}$ and $x \in X - N$. Let $\{F_{1,k}\}_{k\geq 1}$ be an $\mathcal{E}$–nest such that $N \subset \cup_k\{X - F_{1,k}\}$. By virtue of Prop. 3.4 we may take an $\mathcal{E}$–nest $\{F_{2,k}\}_{k\geq 1}$ such that $\tilde{h} \geq \frac{1}{k}$ on $F_{2,k}$ for each $k \geq 1$. Let $\{F_{3,k}\}_{k\geq 1}$ be an $\mathcal{E}$–nest such that $\tilde{H} \subset C(\{F_{3,k}\})$. Let

$$F'_k = \left(\bigcap_{i=1}^{3} F_{i,k}\right) \bigcap X_k, \quad k \geq 1.$$

Finally, by Proposition 3.6 we can find a regular $\mathcal{E}$–nest $\{F_k\}$ such that $F_k \subset F'_k$ for each k. Applying Proposition 3.7 we can check that $\{F_k\}$ is an $\mathcal{E}$–nest with the required properties. ■

A.3 Lemma

(i) There exists a Borel set $Y_2 \subset Y_1$ such that $X - Y_2$ is an $\mathcal{E}$–polar set and

$$\tilde{P}_t(x, Y - Y_2) = 0, \ \forall x \in Y_2, \ t \in Q_+.$$

(ii) Let

$$P_t(x, A) = \begin{cases} \tilde{P}_t(x, A), & \forall x \in Y_2, \ A \in \mathcal{Y} \\ 0, & \forall x \in Y - Y_2, \ A \in \mathcal{Y} \end{cases} \tag{A.4}$$

Then $\{P_t\}_{t\in Q_+}$ is a Markovian transition function on $(Y, \mathcal{Y})$. That is, P_t is a Markovian kernel on $(Y, \mathcal{Y})$ and

$$P_s P_t f(x) = P_{t+s} f(x), \ \forall t, s \in Q_+, \ f \in b\mathcal{Y} \tag{A.5}$$

($b\mathcal{Y}$ denotes all bounded $\mathcal{Y}$–measurable functions.)

Proof The Lemma follows by using Proposition 2.7, Lemma 4.1 (ii), (A.3) and following the argument of [Fu3] Lemma 6.1.4.

∎

We now proceed to construct Markov process. Let $Y_\Delta = Y \bigcup \{\Delta\}$ and $\mathcal{Y}_\Delta = \sigma\{\mathcal{Y}, \{\Delta\}\}$. Here Δ is adjoint to Y as an isolated point. We define

$$\begin{cases} P'_t(x,A) = P_t(x, A - \{\Delta\}) + (1 - P_t(x,Y) I_A(\Delta), \ \forall x \in Y, \ A \in \mathcal{Y}_\Delta \\ \quad P'_t(\Delta, A) = I_A(\Delta) \qquad\qquad , \ \forall A \in \mathcal{Y}_\Delta \end{cases} \tag{A.6}$$

$\{P'_t\}_{t\in Q_+}$ is then a Markovian transition function on $(Y_\Delta, \mathcal{Y}_\Delta)$ with $P'_t(x, Y_\Delta) = 1, \ \forall x \in Y_\Delta$. Set $\Omega_0 = (Y_\Delta)^{Q+}$ and consider the following objects:

$$\Theta_t : \Omega^0 \to \Omega^0, \text{ defined by } \Theta_t\omega = \{\omega_{t+s}\}_{Q_+} \text{ for } \omega = \{\omega\}_{Q_+} \text{ and } t \in Q_+; \tag{A.7}$$

$$X_t^0(\omega) = \omega_t, \quad \forall \omega \in \Omega_0, \ t \in Q_+; \text{(A.8)}$$

$$\mathcal{M} = \sigma\{X_s^0 : \ s \in Q_+\}, \ \mathcal{M}_t^0 = \sigma\{X_s^0 : \ s \leq t, \ s \in Q_+\}, \ \forall t \in Q_+. \tag{A.9}$$

Let $\{\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0, \Theta_t, P_x\}$ be a Markov process with state space $(Y_\Delta, \mathcal{Y}_\Delta)$, time parameter Q_+ and transition function $\{P'_t\}_{t\in Q_+}$. Let

$$\Omega_1 = \{\omega \in \Omega_0 : \ X_t^0(\omega) \in Y_2 \cup \Delta, \ \forall t \in Q_+\} \tag{A.10}$$

It is easy to check that

$$P_x(\Omega_1) = 1, \ \forall x \in Y_2 \tag{A.11}$$

Let us set, for any $t \geq 0$,

$$\mathcal{M}_t = \bigcap_{s \in Q_+, s > t} \mathcal{M}_s^0, \ \mathcal{M}'_t = \sigma(\mathcal{M}_t, \mathcal{N}) \tag{A.12}$$

where $\mathcal{N} = \{\Gamma \in \mathcal{M} : \ P_x(\Gamma) = 0, \ \forall x \in Y_2\}$.
Any function f on Y is extended to Y_Δ by setting $f(\Delta) = 0$. For $A \in \Theta_1$, we set $Z_t^A = e^{-t}\tilde{h}_A(X_t^0), \ t \in Q_+$. In particular, $Z_t^X = e^{-t}\tilde{h}(X_t^0)$.

A.4 Lemma Let $A \in \Theta_1, \ x \in Y_2$.
(i) $(Z_t^A, \mathcal{M}_t^0, P_x)_{t\in Q_+}$ is a supermartingale
(ii) $\lim_{t_k \in Q_+, t_k \downarrow t} E_x\left[Z_{t_k}^A\right] = E_x\left[Z_t^A\right], \ \forall t \in Q_+$

Proof (i) follows the Markov property of (X_t^0), (A.11) and Lemma A.2 (vii). (ii) follows from (i) and Lemma A.2 (iv). ∎

For Z_t^A defined as above, we set $\bar{Z}_t^A(\omega) = \lim_{s \in Q_+, s \downarrow t} Z_s^A(\omega)$ if the limit exists and $\bar{Z}_t^A(\omega) = 0$ otherwise. By virtue of Lemma A.4 $(\bar{Z}_t^A, \mathcal{M}'_t, P_x)_{t \geq 0}$ is then a right continuous nonnegative supermartingale. Moreover, if we set

$$\Omega_1^A = \{\omega \in \Omega_1 : \bar{Z}_t^A(\omega) \text{ is right continuous with left limits}\}, \tag{A.13}$$

then (c.f. ([Me] VI, T3)

$$P_x(\Omega_1^A) = 1, \ \forall x \in Y_2, \ A \in \Theta_1. \tag{A.14}$$

For an arbitrary subset $A \subset Y$, we define

$$\tau_A(\omega) = \inf\{t \in Q_+ : X_t^0(\omega) \in A\}. \tag{A.15}$$

A.5 Lemma Let $x \in Y_2$ and $A \in \Theta_1$, then

$$E_x\left[\bar{Z}_{\tau_A}^X I_{\{\tau_A < \infty\}}\right] = E_x\left[\bar{Z}_{\tau_A}^A I_{\{\tau_A < \infty\}}\right].$$

Proof For $\omega \in \Omega_1^X \cap \Omega_1^A$ satisfying $\tau_A(\omega) < \infty$, we may select $\{t_k\} \subset Q_+, \ t_k \downarrow \tau_A(\omega)$ such that

$$X_{t_k}^0(\omega) \in A \cap Y_2, \ \forall k \geq 1.$$

Applying Lemma A.2 (iii) we obtain $\bar{Z}_{\tau_A}^X(\omega) = \bar{Z}_{\tau_A}^A(\omega)$, which in turn implies the Lemma by virtue of (A.14). ∎

A.6 Lemma

(i) Let $A \in \Theta_1$, then

$$E_x\left[\bar{Z}_{\tau_A}^X I_{\{\tau_A < \infty\}}\right] \leq \tilde{h}_A(x), \ \forall x \in Y_2. \tag{A.16}$$

(ii) Let A be an arbitrary open set of X and $\tilde{h}_A$ be an arbitrary $\mathcal{E}$–quasi–continuous version of h_A, then

$$E_x\left[\bar{Z}_{\tau_A}^X I_{\{\tau_A < \infty\}}\right] \leq \tilde{h}_A(x), \ \mathcal{E} - \text{q.e. } x \in Y_2. \tag{A.17}$$

Proof Following the argument of [Fu3] Lemma 6.2.1 it can be shown that

$$E_x\left[\bar{Z}^A_{\tau_A} I_{\{\tau_A<\infty\}}\right] \leq \tilde{h}_A(x),\ \forall x \in Y_2,\ A \in \Theta_1 \tag{A.18}$$

The assertion (A.16) follows then from (A.18) and Lemma A.5. The assertion (A.17) follows from (A.16) by a similar argument of [Fu3] Lemma 6.2.2. ■

The following two lemmas are useful in proving the regularity of sample paths. We state them in a general context for their own interest.

A.7 Lemma Let $(X, \mathcal{X})$ be a measurable space such that each single-point set is measurable, and let H be a family of real valued functions on X such that $\sigma(f : f \in H) \supset \mathcal{X}$. Then H separates the points of X.

Proof Suppose that $f(x) = f(y)$ for all $f \in H$. We must have $\{x, y\} \subset A_x := \bigcap_{f\in H} f^{-1}[f(x)]$. But A_x is an atom of $\mathcal{X}$ and $\{x\} \in \mathcal{X}$. Hence $\{x, y\} \subset \{x\}$. That is, $x = y$. ■

A.8 Lemma Let $(X, \mathcal{X})$ be a measurable space. $\tilde{H}$ be a family of bounded $\mathcal{X}$–measurable functions such that $\tilde{H}$ satisfies Lemma A.1 (iii) and (iv), and such that $\tilde{H}$ contains a strictly positive element. Suppose that μ and ν are two finite measures on $(X, \mathcal{X})$ satisfying

$$\int f(x)\mu(dx) = \int f(x)\nu(dx),\ \forall f \in \tilde{H}.$$

Then μ and ν coincide on $\sigma(f : f \in \tilde{H})$.

Proof If $f \in \tilde{H},\ a \in Q_+$, then $f \wedge a \in \tilde{H}$. Let

$$f_n = n(f - f\wedge a) \wedge 1, \text{ then } f_n \in \tilde{H} \text{ and } f_n \uparrow I_{\{f>a\}}.$$

The proof of the lemma is completed by applying the monotone class theorem.

■

Let us introduce the following objects.

$$\tau_k = \tau_{X-F_k} := \inf\{t \in Q_+ :\ X^0_t \in X - F_k\} \tag{A.20}$$

where $\{F_k\}$ is specified by Lemma A.2.

$$\tau = \lim_{k\to\infty} \tau_k. \tag{A.21}$$

$$\zeta = \inf\{t \in Q_+ : X_t^0 = \Delta\} \tag{A.22}$$

$$\zeta_1 = \inf\{t \geq 0 : \bar{Z}_t^X = 0 \text{ or } \bar{Z}_{t-}^X = 0\} \tag{A.23}$$

$$\Omega_{11} = \{\omega \in \Omega_1^X : \bar{Z}_t^X = 0, \ \forall t \geq \zeta_1\} \tag{A.24}$$

$$\Omega_{12} = \bigcap_{u\in\tilde{H}_+} \{\omega \in \Omega_1^X : \{\tilde{R}_1 u(X_t^0)\}_{t\in Q_+} \text{ possesses both the right and left limits at each } t \geq 0\}, \tag{A.25}$$

where $\tilde{H}_+ = \{u \in \tilde{H} : u \geq 0\}$.

$$\Omega_{13} = \{\omega \in \Omega_1^X : \tau(\omega) \geq \zeta_1(\omega)\}. \tag{A.26}$$

$$\Omega_2 = \Omega_{11} \cap \Omega_{12} \cap \Omega_{13}. \tag{A.27}$$

The following is the key lemma concerning the regularity of sample paths.

A.9 Lemma (c.f. [Fu3] Lemma 6.2.3)

(i) There exists a Borel set $Y_3 \subset Y_2$ such that $X - Y_3$ is an $\mathcal{E}$–polar set and $P_x(\Omega_2) = 1, \ \forall x \in Y_3$

(ii) The following properties hold for $\omega \in \Omega_2$

(iia) $\zeta(\omega) = \zeta_1(\omega)$

(iib) $\{X_s^0(\omega)\}_{s\in Q_+}$ possesses at every $t < \zeta(\omega)$ the left and right limits inside Y_1 and $X_t^0(\omega) = \Delta$ for all $\zeta(\omega) \leq t \in Q_+$.

(iii) Set

$$X_t(\omega) = \lim_{s\in Q_+, s\downarrow t} X_s^0(\omega), \ \forall \omega \in \Omega_2 \ t \geq 0 \tag{A.28}$$

then

$$P_x(X_t = X_t^0, \ \forall t \in Q_+) = 1 \text{ and } P_x(X_0 = x) = 1, \ \forall x \in Y_3. \tag{A.29}$$

Proof (i) For $u \in \tilde{H}_+$, $\{e^{-t}\tilde{R}_1 u(X_t^0)\}_{t\in Q_+}$ is a nonnegative bounded $(\mathcal{M}_t^0, P_x)$ supermartingale for each $x \in Y_2$. Hence by [Me] $P_x(\Omega_{11} \cap \Omega_{12}) = 1$ for all $x \in Y_2$. Set $G_k = X - F_k$. From the proof of Theorem 2.5 (i) we know that $\tilde{h}_{G_k} \downarrow 0$ $\mathcal{E}$–q.e., which together with (A.17) implies that there exists a Borel set $Y_3 \subset Y_2$ such that $X - Y_3$ is $\mathcal{E}$–polar and $P_x(\Omega_{13}) = 1$ for all $x \in Y_3$.
(iia) follows from Lemma A.2 (ii) and the definitions of (A.10) and (A.24).
For proving (iib) , we observe that $\{\tilde{R}_1 u : u \in \tilde{H}_+\}$ separates the points of Y_1 by virtue of Lemma A.2 (iv) and Lemma A.7. Let $t < \zeta(\omega)$. By (iia) and (A.26), $\{X_s^0(\omega) : s \in Q_+, \ s \leq t_1\} \subset F_k$ for some $t_1 > t$ and $k \geq 1$. Following the argument of [Fu3] Lemma 6.2.3

(i) we obtain the first assertion of (iib). The last assertion of (iib) follows from (A.24), (A.10) and Lemma A.2 (ii).
(iii) can be proved by applying Lemma A.8 with a similar argument of [Fu3] Lemma 6.2.3 (ii) and (iii).

■

Let (X_t) be defined by (A.28) and Y_3 be specified by Lemma A.9. We define for $x \in Y_3$

$$\bar{P}_t f(x) = E_x\left[f(X_t)\right], \tag{A.30}$$

$$\bar{R}_1 f(x) = E_x\left[\int_0^\infty e^{-s} f(X_s) ds\right], \tag{A.31}$$

provided the right hand sides make sense.

A.10 Lemma

(i) $\bar{P}_t f$ is an $\mathcal{E}$–quasi–continuous version of $P_t f$, $\forall f \in L^2(X,m)$.
(ii) $\bar{R}_1 f$ is an $\mathcal{E}$–quasi–continuous version of $G_1 f$, $\forall f \in L^2(X,m)$
(iii) There exists a Borel set $Y_4 \subset Y_3$ such that $X - Y_4$ is $\mathcal{E}$–polar and

$$\tilde{R}_1 f(x) = \bar{R}_1 f(x), \ \forall f \in \tilde{H}, \ x \in Y_4 \tag{A.32}$$

Here $\tilde{R}_1$ is specified by Proposition 4.3 (iii) and $\tilde{H}$ is specified by Lemma A.1.

Proof

(i) By (A.29) we have for all $f \in L^2(X,m)$,

$$\bar{P}_t f = E_\cdot\left[f(X_t)\right] = E_\cdot\left[f(X_t^0)\right] = \tilde{P}_t f, \ \forall t \in Q_+, \ x \in Y_3 \tag{A.33}$$

Let $t \in R_+$ be arbitrary. Then for all $f \in \tilde{H}$, $x \in Y_3, \bar{P}_t f(x) = E_x\left[f(X_t)\right] = \lim_{t' \downarrow t, \, t' \in Q_+} E_x\left[f(X_t')\right] = \lim_{t' \downarrow t, \, t' \in Q_+} \tilde{P}_{t'} f(x)$, which shows that $\bar{P}_t f$ is $\mathcal{E}$–quasi–continuous for all $f \in \tilde{H}$. Suppose that $F \in \mathcal{X}$, $m(F) < \infty$, then by monotone class theorem we see from the above that $\bar{P}_t(fI_F)$ is an $\mathcal{E}$–quasi–continuous version of $P_t(fI_F)$ for all $f \in b\mathcal{X}$. Using monotone convergence theorem we complete the proof of (i).
(ii) From (i) it is easy to see that $\bar{R}_1 f = \tilde{R}_1 f$ m.a.e. for all $f \in L^2(X,m)$. Consequently

$$e^{-t_k} \tilde{P}_{t_k} \tilde{R}_1 f(x) = e^{-t_k} \bar{P}_{t_k} \bar{R}_1 f(x) \quad \mathcal{E} - \text{q.e.},$$

since both sides are $\mathcal{E}$–quasi–continuous versions of a same element in $\mathcal{F}$.
Let $\{t_k\}$ be specified by Lemma A.2 (iv). We obtain by letting $t_k \downarrow 0$, for all $f \in \tilde{H}$

$$\tilde{R}_1 f(x) = \bar{R}_1 f(x) \quad \mathcal{E} - \text{q.e.}$$

From this (ii) follows.
(iii) This holds by virtue of the fact that $\tilde{H}$ is countable.

■

A.11 Lemma There exists a Borel set $S \subset Y_4$ and an $\mathcal{M}$-measurable set $\Omega' \subset \Omega_2$ such that
(i) $X - S$ is an $\mathcal{E}$-polar set;
(ii) $P_x(\Omega') = 1$, $\forall x \in S$;
(iii) If $\omega \in \Omega'$, then $X_t(\omega) \in S$ and $X_{t-}(\omega) \in S$ for all $0 \le t < \zeta(\omega)$.

Proof Since $X - Y_4$ is $\mathcal{E}$-polar, there exists an $\mathcal{E}$-nest$\{E_{4,k}\}_{k\ge 1}$ such that $E_{4,k} \subset F_k \cap Y_4$. Set $\tau_{4,k} = \inf\{t \in Q_+ : X_t^0 \in X - E_{4,k}\}$, $\Omega_3 = \{\omega \in \Omega_2 : \lim \tau_{4,k}(\omega) \ge \zeta(\omega)\}$. By a similar argument as in proving Lemma A.9 (i), we may find a Borel set $Y_5 \subset Y_4$ such that $X - Y_5$ is $\mathcal{E}$-polar and $P_x(\Omega_3) = 1$ for all $x \in Y_5$. In this way we have sequences $Y_4 \supset Y_5 \supset \ldots$, $\Omega_2 \supset \Omega_3 \supset \ldots$. Set $S = \bigcap_{k\ge 3} Y_k$, $\Omega' = \bigcap_{k \ge 2} \Omega_k$. S and Ω' then satisfy (i) — (iii) (c.f. [Fu3] Lemma 6.2.4).

■

As before, we set $S_\Delta = S \cup \{\Delta\}$ and $\mathcal{S}_\Delta = \mathcal{Y}_\Delta \cap S_\Delta$. Moreover we set

$$\Omega = \{\omega \in \Omega' : X_t(\omega) = X_t^0(\omega), \ \forall t \in Q_+\} \tag{A.34}$$

and denote the restriction of $(\mathcal{M}, \mathcal{M}_t^0, \mathcal{M}_t, (X_t^0)_{t\in Q_+}, (\Theta_t)_{t\in Q_+}, (X_t)_{t\ge 0}, \zeta, (P_x)_{x\in S_\Delta}$ to the set Ω by the same notation again. Furthermore, we define Θ_t for $t \ge 0$ by setting $\Theta_t\omega = \{\lim_{t'\in Q_+, t'\downarrow t} \omega_{t+s}\}_{s\in Q_+}$ for $\omega = \{\omega_s\}_{s\in Q_+}$. Θ_t is well defined for $\omega \in \Omega$ by virtue of Lemma A.9 (iib). We now consider the process

$$(X_t) := (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \Theta_t, P_x)_{x\in S_\Delta} \tag{A.35}$$

A.12 Lemma (X_t) is a strong Markov process on $(S, \mathcal{S})$ satisfying the conditions (1.4) — (1.6).

Proof It is evident from the above construction that (X_t) is a Markov process on $(S, \mathcal{S})$ satisfying (1.4) — (1.6). Also it is known from the construction that $(\mathcal{M}_t)_{t\ge 0}$ is right continuous. Let $\bar{P}_t$ be the transition function of (X_t) as specified by (A.30). We have from (A.33),

$$\bar{P}_s f \subset C(\{F_k\}, S), \ \forall f \in \tilde{H}, \ s \in Q_+, \tag{A.36}$$

where $C(\{F_k\}, S)$ denotes the restriction to S of functions in $C(\{F_k\})$. From (A.36) we conclude that

$$\lim_{t'\downarrow t} \bar{P}_s f(X_{t'}(\omega)) = \bar{P}_s f(X_t(\omega)), \ \forall s \in Q_+, \ f \in \tilde{H}, \tag{A.37}$$

because (X_t) is right continuous and $\{X_s(\omega) : 0 \le s \le t\} \in F_k \cap S$ for some $k \ge 1$ provided $t < \zeta(\omega)$. Again because (X_t) is right continuous we have

$$t \to \bar{P}_t f(x) \text{ is right continuous for fixed } f \in \tilde{H} \text{ and } x \in S. \tag{A.38}$$

From (A.37) and (A.38) we obtain the strong Markov property by a similar argument of [Fu3] Lemma 6.2.5 with $\tilde{H}$ in place of $C_\infty(X)$ and by virtue of Lemma A.8. ■

A.13 Lemma (X_t) satisfies the condition (1.7). That is, $\bar{R}_1 f(X_{t-})I_{\{t<\zeta\}}$ is P_x–indistingushable from $\bar{R}_1 f(X_t)_- I_{\{t<\zeta\}}$, $\forall x \in S,\ f \in b\mathcal{X}$. (A.39)
Here $\bar{R}_1$ is specified by (A.31).

Proof It follows from (A.32) and Lemma A.11 (iii) that (A.39) is true for $f \in \tilde{H}$ and $x \in S$. Let $f \in b\mathcal{X}$ and $x \in S$ be arbitrary. We have the following martingale decomposition:

$$e^{-t}\bar{R}_1 f(X_t) = M_t^{[f]} - \int_0^t e^{-s} f(X_s)ds\ P_x - \text{a.s.} \tag{A.40}$$

where $M_t^{[f]}$ is a right continuous P_x–martingale such that

$$M_t^{[f]} = E_x\left[\int_0^\infty e^{-s} f(X_s)ds|\mathcal{M}_t\right]\ P_x - \text{a.s.} \tag{A.41}$$

Suppose that $\{f_n\} \subset b\mathcal{X}$ is an increasing sequence of nonnegative functions satisfying (A.39). Set $f \equiv \lim_n f_n$. By virtue of (A.41) we can always find a subsequence $\{f_{n_k}\}$ such that

$$\lim_{k\to\infty} \sup_{0\le t\le\infty} |M_t^{[f_{n_k}]} - M_t^{[f]}| = 0\ P_x - \text{a.s.} \tag{A.42}$$

Consequently we have

$$\begin{aligned} e^{-t}\bar{R}_1 f(X_t)_- I_{\{t<\zeta\}} &= \lim_{t'\uparrow t}\lim_k e^{-t}\bar{R}_1 f_{n_k}(X_t')I_{\{t<\zeta\}} \\ &= \lim_k \lim_{t'\uparrow t} e^{-t}\bar{R}_1 f_{n_k}(X_{t'})I_{\{t<\zeta\}} \\ &= e^{-t}\bar{R}_1 f(X_{t-})I_{\{t<\zeta\}}\ P_x - \text{a.s.} \end{aligned}$$

Thus the proof is completed by a monotone class argument.

■

A.14 Lemma (X_t) is m–tight.

Proof Since $X - S$ is polar, we can find an $\mathcal{E}$–nest $\{S_k\}_{k\ge1}$ such that $S_k \subset F_k \cap S$ with $\{F_k\}$ being specified by Lemma A.2. Each S_k is compact since F_k is so. Let us set

$$\sigma_k(\omega) = \inf\{t > 0 : X_t \notin S_k\}.$$

It can be shown that (for the notations see Lemma A.6)

$$e^{-\sigma_k}\tilde{h}(X_{\sigma_k})I_{\{\sigma_k<\infty\}}(\omega) = \bar{Z}^X_{\tau_{X-S_k}} I_{\{\tau_{X-S_k}<\infty\}}(\omega),\ \forall\omega \in \Omega.$$

Consequently by (A.17)

$$E_x\left[e^{-\sigma_k}\tilde{h}(X_{\sigma_k})I_{\{\sigma_k<\infty\}}(\omega)\right] \le \tilde{h}_{X-S_k}(x),\ \mathcal{E}-\text{q.e. } x\in S. \tag{A.42}$$

From (A.42) and the fact that $\tilde{h}_{X-S_k}(x)\downarrow 0$, $\mathcal{E}$–q.e. we can prove the lemma.

■

To sum up Lemma A.12 — A.14, we conclude that (X_t) is an m–perfect process with state space $(S,\mathcal{S})$. By a similar argument as in [Fu3] Th. 4.13, we can now construct an m–perfect process $(\hat{X}_t)=(\hat{\Omega},\hat{\mathcal{M}},\hat{\mathcal{M}}_t,\hat{X}_t,\hat{\Theta}_t,P_x)$ with state space $(X,\mathcal{X})$ in such a way that

(i) S is $(\hat{X}_t)$ invariant and the restriction of $(\hat{X}_t)$ to S is (X_t). (A.36)

(ii) Each point $x\in X-S$ is a trap with respect to $(\hat{X}_t)$. (A.43)

$(\hat{X}_t)$ is then an m–perfect process associated with $\mathcal{E}$. In fact, if we denote by $\hat{P}_t$ the transition function of $(\hat{X}_t)$, then by Lemma A.10 (i) and the fact that $X-S$ is $\mathcal{E}$–polar, we can show that

$$\hat{P}_tf \text{ is an } \mathcal{E}\text{–quasi–continuous version of } T_tf,\ \forall t>0,\ f\in L^2(X,m) \tag{A.44}$$

In this way the proof of Proposition 4.4 is completed.

Footnote

[1]) For further recent work, especially on the infinite dimensional case, see also [ABrR], [AFHkL], [AHPRS1,2], [AK], [AKR], [AMR2], [ARöl–5], [FaR], [R], [Sch], [So1–3], [Tak].

Acknowledgements

We are very indebted to Prof. Dr. Masatoshi Fukushima who greatly encouraged our work, and carefully read the first version of the manuscripts of this paper and [AM1] suggesting many improvements. We are also very grateful to Prof. Dr. H. Airault, Dr. J. Brasche, Prof. Dr. R.K. Getoor, Prof. Dr. W. Hansen, Prof. Dr. M.L. Silverstein, Prof. Dr. S. Watanabe, Prof. Dr. R. Williams, Prof. Dr. J.A. Yan, Dr. T. Zhang and especially Prof. Dr. P.J. Fitzsimmons and Prof. Dr. Michael Röckner for very interesting and stimulating discussions. The second named author would like to thank Prof. Dr. W. Hansen and Prof. Dr. P.A. Meyer for their kind invitations to give talks in the Potential Theory Seminar in Bielefeld resp. the Probability Theory Seminar in Strasbourg. We also profited from meetings in Braga and Oberwolfach and are grateful to Professors Drs. M. De Faria, L. Streit resp. H. Bauer and M. Fukushima for kind invitations. The first named author would like to thank Prof. Dr. Hisao Watanabe and Prof. Dr. Takeyuki Hida for a kind invitation to visit Japan, with the support of the Japan Society for the Promotion of Science. During that visit he held seminars on topics related to this work at

Kyushu University (H. Watanabe, H. Kunita), Kumamoto (Hitsuda, Y. Oshima), Tokyo (S. Kotani), Nagoya (T. Hida), Kyoto/Osaka (M. Fukushima, N. Ikeda, S. Kusuoka), and received in this way much support and stimulation. Hospitality and / or financial support by BiBoS, A. von Humboldt Stiftung, DFG and Chinese National Sciences Foundation is also gratefully acknowledged.

Added in proof: After we finished this work we learned at the Durham LMS-Symposium (July '90) from Prof. Dr. P.J. Fitzsimmons that every (nearly) m-symmetric right process is an m-special standard process, see [Fil]. This fact implies, using the inclusions between classes of processes mentioned before Def. 1.2, that our Theorem 1.7 gives a characterization of Dirichlet forms associated with symmetric Borel right processes. We are most grateful to Professor Fitzsimmons for pointing out this fact to us.

References

[AH1] Albeverio, S., Høegh-Krohn, R.: *Quasi-invariant measures, symmetric diffusion processes and quantum fields.* In: **Les méthodes mathématiques de la théorie quantique des champs**, Colloques Internationaux du C.N.R.S., no. 248, Marseille, 23–27 juin 1975, C.N.R.S., 1976.

[AH2] Albeverio, S., Høegh-Krohn, R.: *Dirichlet forms and diffusion processes on rigged Hilbert spaces.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 40, 1–57 (1977).

[AH3] Albeverio, S., Høegh-Krohn, R.: *Hunt processes and analytic potential theory on rigged Hilbert spaces.* Ann. Inst. Henri Poincaré, vol. XIII, no. 3, 269–291 (1977).

[ABrR] Albeverio, S., Brasche, J., Röckner, M.: *Dirichlet forms and generalized Schrödinger operators*, pp. 1-42 in Proc. Sønderborg Conf. "Schrödinger operators", Edts. H. Holden, A. Jensen, Lect. Notes Phys. 345, Springer, Berlin (1989).

[AFHKL] Albeverio, S., Fenstad, J.E., Høegh-Krohn, R., Lindstrøm, T: *Non standard methods in stochastic analysis and mathematical physics*, Academic Press, Orlando (1986)

[AHPRS1] Albeverio, S. Hida, T. Potthoff, J., Röckner, M., Streit, L.: *Dirichlet forms in terms of white noise analysis I - Construction and QFT examples*, Rev. Math. Phys. 1, 291-312 (1990).

[AHPRS2] Albeverio, S. Hida, T. Potthoff, J., Röckner, M., Streit, L.: *Dirichlet forms in terms of white noise analysis II - Closability and Diffusion Processes*, Rev. Math. Phys. 1, 313-323 (1990).

[AK] Albeverio, S., Kusuoka, S.: *Maximality of infinite dimensional Dirichlet forms and Høegh-Krohn's model of quantum fields*, to appear in Memorial Volume for Raphael Høegh-Krohn.

[AKR] Albeverio, S., Kusuoka, S., Röckner, M.: *On partial integration in infinite dimensional space and applications to Dirichlet forms*, J. London Math. Soc. (1990)

[AM1] Albeverio, S., Ma, Z.M.: *A note on quasicontinuous kernels representing quasi- linear positive maps*, BiBoS - Preprint 1990, to appear in Forum Math.

[AM2] Albeverio, S., Ma, Z.M.: *A general correspondence between Dirichlet forms and right processes*, BiBoS-Preprint 415 (1990).

[AM3] Albeverio, S., Ma, Zhi-Ming: *Nowhere Radon smooth measures, perturbations of Dirichlet forms and singular quadratic forms*, pp. 3-45 in Proc. Bad Honnef Conf.1988, ed. N. Christopeit, K. Helmes, M. Kohlmann Lect. Notes Control and Inform. Sciences 126, Springer, Berlin (1989).

[AM4] Albeverio, S., Ma, Z.M.: *Additive functionals, nowhere Radon and Kato class smooth measure associated with Dirichlet forms*, SFB 237 - Preprint.

[AM5] Albeverio, S., Ma, Z.M.: *Perturbation of Dirichlet forms - lower semiboundedness, closability and form cores*, SFB 237 - Preprint, to appear in J. Funct. Anal. (1990).

[AM6] Albeverio, S., Ma, Z.M.: *Local property for Dirichlet forms on general metrizable spaces*, in preparation.

[AM7] Albeverio, S., Ma, Z.M.: *Diffusion processes associated with singular Dirichlet forms*, to appear in Proc. Lisboa Conf., Ed. A.B. Cruzeiro, Birkhäuser, New York (1990).

[AM8] Albeverio, S., Ma, Z.M.: *Characterization of Dirichlet spaces associated with symmetric Hunt processes*, in preparation.

[AMR1] Albeverio, S., Ma, Z.M., Röckner, M.: in preparation.

[AMR2] Albeverio, S., Ma, Z.M., Röckner, M.: *Dirichlet forms and Markov fields - A report on recent developments*, to appear in Proc. Evanston Conf. "Diffusion Processes and Related Problems in Analysis", Ed. M. Pinsky, Birkhäuser, New York (1990).

[ARö1] Albeverio, S., Röckner, M.: *Classical Dirichlet forms on topological vector spaces—closability and a Cameron-Martin formula*, J. Funct. Anal., 88, 395-436 (1990).

[ARö2] Albeverio, S., Röckner, M.: *Classical Dirichlet forms on topological vector spaces - the construction of the associated diffusion process*, Prob. Theory and Rel. Fields, 83, 405-434 (1989).

[ARö3] Albeverio, S., Röckner, M.: *New developments in theory and applications of Dirichlet forms* (with M. Röckner), to appear in "Stochastic Processes, Physics and Geometry", Proc. 2nd Int. Conf. Ascona - Locarno - Como 1988, Ed. S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, R. Moresi, World Scient., Singapore (1990).

[ARö4] Albeverio, S., Röckner, M.: *Stochastic differential equations in infinite dimension: solutions via Dirichlet forms*, SFB 237 - Preprint.

[ARö5] Albeverio, S., Röckner, M.: *Infinite dimensional diffusions connected with positive generalized white noise functionals*, Edinburgh Preprint (1990), to appear in Proc. Bielefeld Conference "White Noise Analysis", Ed. T. Hida, H. H. Kuo, J. Potthoff, L. Streit.

[BG] Blumenthal, R.M., Getoor, R.K. (1968). *Markov Processes and Potential Theory*, Academic Press, New York.

[BoH1] Bouleau, N., Hirsch, F.: *Propriétés d'absolue continuité dans les espaces de Dirichlet et applictions aux équations différentielles stochastiques*, Sém. de Probabilités, XX, Lect. Notes Maths., Springer.

[BoH2] Bouleau, N., Hirsch, F.: *Formes de Dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener*, J. Funct. Anal. 69, 229–259 (1986).

[D1] Dynkin, E.B.: *Green's and Dirichlet spaces associated with fine Markov processes.* J. Funct. Anal. 47, 381–418 (1982).

[D2] Dynkin, E.B.: **Green's and Dirichlet spaces for a symmetric Markov transition function.** Lecture Notes of the LMS, 1982.

[DM] Dellacherie, C., Meyer, P.A.: *Probabilities and Potential*, Ch. I-XIII, Vol. A,B,C, North Holland Math. Studies: Hermann, Paris (1978); North Holland, Amsterdam (1982); Elsevier, Amsterdam (1988) (transl. of Probabilités et Potentiel, Hermann, Paris 1966–1988)

[Fi1] Fitzsimmons, P.J. (1988). *Markov processes and nonsymmetric Dirichlet forms without regularity.* To appear in J. Funct. Anal.

[Fi2] Fitzsimmons, P.J.: *Time Changes of Symmetric Markov Processes and a Feynman-Kac formula*, to appear in J. Theoretical Prob.

[FG] Fitzsimmons, P.J., Getoor, R.K. (1988). *On the potential theory of symmetric Markov processes.* Math. Annalen., **281**, 495–512.

[Fu1] Fukushima, M.: *Regular representations of Dirichlet forms.* Trans. Amer. Math. Soc. **155**, 455–473 (1971).

[Fu2] Fukushima, M.: *Dirichlet spaces and strong Markov processes.* Trans. Amer. Math. Soc. **162**, 185–224 (1971).

[Fu3] Fukushima, M.: *Dirichlet forms and Markov processes.* Amsterdam–Oxford–New York: North Holland 1980.

[Fu4] Fukushima, M.: *Basic properties of Brownian motion and a capacity on the Wiener space.* J. Math. Soc. Japan **36**, 161–175 (1984).

[Fu5] Fukushima, M.: *Potentials for symmetric Markov processes and its applications*, pp. 119–133 in Proc. 3d Japan–USSR Symp. Prob. Th., Ed. G. Mamyama, J.V. Prokhorov, Lecture Notes Math. **550**, Springer (1976).

[FuO] Fuskushima, M., Oshima, Y.: *On skew product of symmetric diffusion processes*, Forum Math. 2, 103-142 (1989).

[FaR] Fan Ruzong: *On absolute continuity of symmetric diffusion processes on Banach spaces*, Beijing Prepr. 1989, to appear in Acta Mat. Appl. Sinica

[G1] Getoor, R.K.: *On the construction of kernels*, Lect. Notes Maths. 465, 443-463 (1986)

[G2] Getoor, R.K.: *Markov processes: Ray processes and right processes.* Lecture Notes in Math. **440**, Berlin–Heidelberg–New York: Springer 1975.

[Ka] Kaneko, H.: *On (r,p)-capacities for Markov processes,* Osaka J. Math. **23** (1986) 325–336.

[Kn] Knight, F.: *Note on regularization of Markov processes*, Ill. J. Math. 9, 548 – 552 (1965).

[Ku] Kusuoka, S.: *Dirichlet forms and diffusion processes on Banach space.* J. Fac. Science Univ. Tokyo, Sec. 1A **29**, 79–95 (1982).

[KW] Kunita, H., Watanabe, T.: *Some theorems concerning resolvents over locally compact spaces*, Proc. Vth Berkeley Symp. Math. Stat. and Prob. Vol. II, Berkeley (1967).

[Le1] Le Jan, Y.: *Quasi-continuous functions and Hunt processes*, J. Math. Soc. Japan, **35**, 37–42 (1983).

[Le2] Le Jan, Y.: *Dual Markovian semigroups and processes*, pp. 47–75 in "Functional Analysis in Markov Processes", Ed. M. Fukushima, Lect. Notes Maths. **923** (1982)

[LR] Lyons, T., Röckner, M.: *A note on tightness of Capacities associated with Dirichlet forms*, Edinburgh Preprint (1990).

[Me] Meyer, P.A.: *Probability and Potentials*, Ginn Blaisdell, Massachussetts (1966)

[P] Parthasarathy, K.R.: *Probability measures on metric spaces.* New York–London: Academic Press 1967.

[R] Röckner, M.: *Generalized Markov fields and Dirichlet forms.* Acta Appl. Math. **3**, 285–311 (1985).

[Sch] Schmuland, B.: *An alternative compactification for classical Dirichlet forms on topological vector spaces*, Vancouver Preprint (1989).

[Scha] Schaefer, H.H.: *Topological vector spaces*, Mac Millan (1987)

[Sh] Sharpe, M.J.: *General Theory of Markov Processes,* Academic Press, New York (1988).

[Si] Silverstein, M.L.: *Symmetric Markov Processes.* Lecture Notes in Math. **426**. Berlin-Heidelberg–New York: Springer 1974.

[So1] Song Shiqui: *The closability of classical Dirichlet forms on infinite dimensional spaces and the Wiener measure*, Acad. Sinica Preprint, Beijing

[So2] Song Shiqui: *Admissible vectors and their associated Dirichlet forms*, Acad. Sinica Preprint, Beijing (1990)

[So3] Song Shiqui: *An infinite dimensional analogue of the Albeverio–Röckner's theorem on Fukushima's conjecture*, Acad. Sinica Preprint, Beijing (1989)

[Tak] Takeda, M.: *On the uniqueness of Markovian self-adjoint extensions of diffusion operators on infinite dimensional spaces*, Osaka J. Math. **22**, 733–742 (1985)

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