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Stochastic Integral Equations for The Random Fields.

by

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1. One simple aspect of the stochastic integral equation is that it transforms a fundamental process like the Brownian motion into another process of more compound natures. Standing at this viewpoint, we are inevitably led to consider a stochastic integral equation for the processes with multidimensional variables, namely for the random fields. Since there is no natural order in the multidimensional space R^p, we readily see that such equations should be treated in the framework of the noncausal stochastic calculus (cf., [1]). What we are going to show in this note is the first attempt on this subject. We will limit our discussions in the case of linear equations of Fredholm type and we will show some results concerning the question of existence and uniqueness of solutions.

In what follows, we fix a probability space $(\Omega,\mathcal{F},P)$ and we understand by the random fields, such as $f(t,\omega)$ or $L(t,s,\omega)$ $(t=(t_1,t_2,\cdots,t_p),\ s=(s_1,s_2,\cdots,s_p)\in R^p,\ \omega\in\Omega)$, the real functions, measurable in (t,ω) or in (t,s,ω) with respect to an appropriate σ-field like $\mathcal{F}\times\mathcal{B}_{R^p}$ (or $\mathcal{F}\times\mathcal{B}_{R^p}\times\mathcal{B}_{R^p}$ respectively), such that ;

$$\int_\Delta f^2(t,\omega)dt<+\infty,\quad \iint_{\Delta\times\Delta}L^2(t,s,\omega)dtds<+\infty\quad \text{(P-a.s.)}$$

where $dt=dt_1\times\cdots\times dt_p$, and $\Delta=[0,1]^p$ the unit cube in R^p.

2, Set Up. Let $Z(t,\omega)$ $((t,\omega)\in R^p\times\Omega)$ be such that the derivative, $\dot{Z}(t,\omega)$ $=\dfrac{\partial^p}{\partial t_1\cdots\partial t_p}Z(t,\omega)$ is well defined as a generalized random field on the Schwartz space $\mathcal{S}(R^p)$, and let $\{\phi_n\}_{n=1}^\infty$ be a complete orthonormal system of functions in the real Hilbert space $L^2(\Delta)$.

Definition 1. The stochastic integral of a random field $f(t,\omega)$, with respect to the fundamental pair $(Z,\{\phi_n\})$, $\int_\Delta f(t,\omega)d_\phi Z(t)$, is defined as being the limit in probability of the series, $\sum_{n=1}^\infty(f,\phi_n)(\phi_n,\dot{Z})$.

(Remark 1) (i) The functions of the basis $\{\phi_n\}$ are supposed to be arranged in a fixed order and the summation $\sum_n$ should be taken in this order.

(ii) Notice that, $\int_\Delta f(t,\omega)d_\phi Z(t)=\lim_{m\to\infty}\int_\Delta f(t,\omega)\dot{Z}_m^\phi(t)dt$ where

$$\dot{Z}_m^{\phi}(t) = \sum_{m \geq n} (\phi_n, \dot{Z})\phi_n(t).$$

The linear stochastic integral equation we are going to study is as follows,

$$(1); \quad x(t,\omega) = f(t,\omega) + \alpha \int_\Delta L(t,s,\omega)X(s)ds + \beta \int_\Delta K(t,s,\omega)X(s)d_\phi Z(s),$$

where $f(t,\omega)$, $L(t,s,\omega)$ and $K(t,s,\omega)$ are some random fields and α, β are constants.

For the simplicity of discussions, we will fix once for all, another c.o.n.s. $\{\psi_n\}$ in an arbitrary way and we set the next assumption (A) which concerns a regularity of the random kernels, K, L.

(A); There exists a positive sequence $\{\epsilon_n\}$ such that,

$$\text{(A,1)} \quad \{\epsilon_n \epsilon_m \gamma_{m,n}\} \in l^2 \ \ (P\text{-a.s}), \quad \gamma_{m,n} = \int_\Delta \psi_m(t)\psi_n(t)d_\phi Z(t),$$

$$\text{(A,2)} \quad \{k'_{m,n}\}, \{l'_{m,n}\} \in l^2 \ \ (P - a.s.) \quad \text{where} \quad k'_{m,n} = k_{m,n}/\epsilon_m \epsilon_n \ ,$$
$$l'_{m,n} = l_{m,n}/\epsilon_m, \quad k_{m,n} = (K, \psi_m \otimes \psi_n), \ l_{m,n} = (L, \psi_m \otimes \psi_n).$$

We will call such sequence $\{\epsilon_n\}$ the admissible weight.

It is worthwhile to notice that if $\{\epsilon_n\}$, $\{\eta_n\}$ are admissible weights then the sequences, $\{(\epsilon \wedge \eta)_n\}$, $\{(\epsilon \vee \eta)_n\}$, given by $(\epsilon \wedge \eta)_n = \min\{\epsilon_n, \eta_n\}$, $(\epsilon \vee \eta)_n = \max\{\epsilon_n, \eta_n\}$, are also admissible weights.

Example In the case that $Z =$ the Brownian sheet and $\{\psi_n\}$ is such that all elements are uniformly bounded on Δ, then any positive l^2-sequence satisfies the condition (A,1).

Definition 2. We will say that a random field $g(t,\omega)$ admits a sequence $\{\epsilon_n\}$ as the weight (or shortly, $\{\epsilon_n\}$-smooth) if there exists an admissible weight $\{\epsilon_n\}$, such that ;

(t,1) the integral $\hat{g}_n = \int_\Delta g(t,\omega)\psi_n(t)d_\phi Z(t)$ exists for all $n \in \mathrm{N}$ and $\{\epsilon_n \hat{g}_n\} \in l^2 \ \ (P - a.s.)$.

$$\text{(t,2)} \quad \lim_{m \to \infty} \sum_{n=1}^{\infty} \{\epsilon_n(\hat{g}_n - \int_\Delta g(t,\omega)\psi_n(t)dZ_m^{\phi}(t))\}^2 = 0,$$

We will denote by $S(l^2)$ the totality of all such random fields that are $\{\epsilon\}$-smooth for some admissible weight $\{\epsilon_n\}$.

It is easy to check that if a $S(l^2)$-field $g(t,\omega)$ admits two sequences, $\{\epsilon_n\}$, $\{\eta_n\}$, as the weights, then it also admits the sequences $\{(\epsilon \wedge \eta)_n\}$, $\{(\epsilon \vee \eta)_n\}$, mentioned in Remark 1, as the weights.

(Remark 2). In the case that $Z =$ the Brownian sheet and the all elements of $\{\psi_n\}$ are uniformly bounded, we see that $S(l^2) \supset L^2(\Delta)$ and that every admissible sequence can be the weight for any $g(t,\omega) \in L^2(\Delta)$.

Associated to the notion of $S(l^2)$-fields, we introduce a linear stochastic transformation, $\mathcal{T}_\epsilon$ acting on $S(l^2)$, in such way that ; For a $g(t,\omega) \in S(l^2)$ admitting a $\{\epsilon_n\}$ as the weight, we set,

$$(2) \qquad (\mathcal{T}_\epsilon g)(t) = \sum_n \epsilon_n \hat{g}_n \psi_n(t), \quad \text{where} \quad \hat{g}_n(\omega) = \int_\Delta g(t,\omega)\psi_n(t) d_\phi Z(t).$$

We should notice that $\mathcal{T}_\epsilon g \in L^2(\Delta)$, $(P - a.s.)$.

3, Results.

Theorem 1. *For any $f(t,\omega) \in S(l^2)$ the following equation*

$$(3), \qquad X(t,\omega) = f(t,\omega) + \int_\Delta K(t,s,\omega) X(s) d_\phi Z(s),$$

has the unique $S(l^2)$-solution provided that the following condition (C) holds,

(C); the homogeneous equation, $X(t,\omega) = \int_\Delta K(t,s,\omega)X(s)d_\phi Z(s)$, does not have nontrivial $S(l^2)$-solutions.

(Proof) Let $\{\epsilon_n\}$ be an admissible weight for the random field $f(t,\omega)$. First we are going to show that the condition (C) is sufficient to assure the existence of a $S(l^2)$-solution X, which is unique in those admitting the same weight $\{\epsilon_n\}$.

Let X be an $\{\epsilon\}$-smooth solution of (3). Then, since $K(t,s,\omega) = \sum_{m,n} k_{m,n}\psi_m(t)\psi_n(s)$, we get the following relation (4) by virtue of the condition (t,2),

$$(4) \qquad X(t) = f(t) + \sum_{m,n} \epsilon_m \epsilon_n k'_{m,n} \psi_m(t) \hat{x}_n, \quad \text{where} \quad \hat{x}_n(\omega) = \int_\Delta X(t,\omega)\psi_n(t) d_\phi Z(t).$$

Multiplying by $\psi_l(t)$ and taking the stochastic integration over Δ on both sides of the equation (4), we obtain, under the assumption (A,2) the next relation,

$$(5) \qquad \hat{x}_l = \hat{f}_l + \sum_{m,n} \gamma_{l,m} k_{m,n} \hat{x}_n, \quad (\forall l \in N)$$

or equivalently,

(5)' $\quad \epsilon_l \hat{x}_l = \epsilon_l \hat{f}_l + \sum_{m,n} \epsilon_l \epsilon_m \gamma_{l,m} k'_{m,n} \epsilon_n \hat{x}_n.$

So if we set, $\quad \bar{K}(t,s,\omega) = \sum_{l,n} \epsilon_l \{\sum_m \epsilon_m \gamma_{l,m} k'_{m,n}\} \psi_l(t)\psi_n(s),$

then by virtue of the condition (t,1), we see that the kernel $\bar{K}(\cdot,\cdot,\omega)$ is of Hilbert-Schmidt type for almost all ω and that the field, $Y = (\mathcal{T}_\epsilon X)(t,\omega)$ satisfies the following random integral equation,

(6) $\quad Y(t) = (\mathcal{T}_\epsilon f)(t) + \int_\Delta \bar{K}(t,s,\omega) Y(s) ds.$

Conversely if we set $\hat{x}_n = (Y,\psi_n)/\epsilon_n$ for an L^2–solution Y of (6), then we see that the $\{\hat{x}_n\}$ satisfies the equation (5) and so the field $X(t)$ defined through the relation (4) becomes an $S(l^2)$–solution of (3). As is easily seen, this correspondence between the $\{\epsilon\}$-smooth solution of (3) and the L^2–solution of (6) is one-to-one and onto. Thus the question of the existence and the uniqueness of the $\{\epsilon\}$-smooth solution is reduced to the same question about the L^2–solutions of (6). Hence, by a simple application of The Riesz-Schauder Theory, we confirm that the condition (C) is sufficient for the validity of the prescribed result.

Next, we are going to show that this solution X which has the $\{\epsilon_n\}$ as the weight is unique among all $\{\epsilon\}$- smooth fields. So let X' be another $S(l^2)$–solution of (3) having a different sequence $\{\eta\}$ as the weight. Then it satisfies a similar relation as (4) from which we see the field $f(t,\omega)$ is $\{\eta\}$-smooth. Since all $S(l^2)$-fields f, X and X' are $\{(\epsilon \wedge \eta)\}$-smooth, the field X' and X must coincide with each other as the unique $S(l^2)$-solution admitting the same sequence as the weight. *Q.E.D.*

Corollary, *If all elements of the c.o.n.s.* $\{\psi_n\}$ *are continuous and uniformly bounded over* Δ *and if almost all sample of the field* $f(t,\omega)$ *are continuous. Then the* $S(l^2)$*-solution of (3) is also almost surely sample-continuous.*

(Proof) Evident from the equality (4) and the fact that, $\sum_{m,n} |\epsilon_m k'_{m,n} \epsilon_n \hat{x}_n|$ $< +\infty \quad (P-a.s.)$ *Q.E.D.*

Now we are to give a result for the general case (1) in the next,

Proposition; *Let* $f(t,\omega) \in S(l^2)$ *then for almost all* α, $\beta \in R^1$ *the equation (1) has a unique* $S(l^2)$*-solution.*

(Proof) We notice that the condition (A,2) implies ;

$(LX)(t,\omega) = \int_\Delta L(t,s,\omega) X(s) ds \in S(l^2)$ for any random field X.

Let $\{\epsilon_n\}$ be a weight for the $f(t,\omega)$. Then following the same discussion as in the proof of Theorem 1, we see that any $S(l^2)$-solution X admitting the $\{\epsilon_n\}$ as weight, if exists, satisfies the following equation,

$$(7) \qquad Y(t,\omega) = \{\mathcal{T}_\epsilon(f+\alpha LX)\}(t,\omega) + \beta\int_\Delta \bar{K}(t,s,\omega)Y(s)ds$$

where $Y(t,\omega) = (\mathcal{T}_\epsilon X)(t,\omega)$.

Since the operator $\bar{K}(\omega)$, given by ;

$$(L^2(\Delta)\ni)Y \longrightarrow (\bar{K}Y)(t,\omega) = \int_\Delta \bar{K}(t,s,\omega)Y(s)ds\ (\in L^2(\Delta)),$$

is compact for almost all ω, we know that for all (but with at most countable exception) of β the operator $(I-\beta\bar{K})$ is invertible and for such β we get, by solving (7) in Y, the following expression,

$$(8); \qquad (\mathcal{T}_\epsilon X)(t) = f_1(t) + \alpha(L'_\beta X)(t)$$

where

$$f_1(t) = (I-\beta\bar{K})^{-1}(\mathcal{T}_\epsilon f)(t) \quad \text{and} \quad (L'_\beta X)(t) = \{(I-\beta\bar{K})^{-1}(LX)\}(t).$$

On the other hand we have the next relation which can be derived in a same way as in the derivation of the (4);

$$(9) \qquad X(t) = f(t) + \alpha(LX)(t) + \beta(K_1\,\mathcal{T}_\epsilon X)(t)$$

where

$$(K_1Y)(t) = \int_\Delta K_1(t,s,\omega)Y(s)ds, \quad K_1(t,s,\omega) = \sum_{m,n}(k_{m,n}/\epsilon_n)\psi_m(t)\psi_n(s).$$

Substituting the relation (8) into (9), we find that the solution X, if exists, satisfies the next

$$(10) \qquad X(t) = f_2(t) + \alpha(L''X)(t)$$

where

$$f_2(t) = f(t) + \beta(K_1f_1)(t), \ \text{and} \ (L''Y)(t) = \{(L+\beta K_1L'_\beta)Y\}(t) \ \ (Y\in L^2(\Delta))$$

The operator L'' being compact for almost all ω, the equation (10) has for almost all α a unique $S(l^2)$-solution. Moreover, it is immediate to see, following the same argument as in the proof of Theorem 1, that this solution does not depend on the choice of the weight $\{\epsilon_n\}$ for the $f(t,\omega)$. *Q.E.D.*

REFERENCES;

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