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On Newton's method for stochastic differential equations

SHIGETOKU KAWABATA AND TOSHIO YAMADA

1. Introduction.

The aim of this paper is to propose a formulation of Newton-Kantorovich's method for Ito-type stochastic differential equations. This note has three sources;

(1) Newton's method on Banach space by L.V. Kantorovich[6],
(2) Chaplygin-Vidossich's method for ordinary differential equations [3] [7],
(3) Newton's method for random operators by A.Bharucha-Reid and R. Kannan [2].

As is well known, S.A. Chaplygyn[3] introduced a process for the approximation of solutions for non-linear Cauchy problems for ordinary differential equations;

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{1.1}$$

consisting of the iterative solution of a sequence of linear Cauchy problems;

$$\begin{gathered} u'_{n+1} = f(t,u_n(t)) + f_x(t,u_n(t))(u_{n+1}(t) - u_n(t)) \\ u_{n+1}(t_0) = x_0. \end{gathered} \tag{1.2}$$

At the end of seventies, G.Vidossich [7] has shown that the Chaplygin sequence is exactly the Newton sequence for the operator;

$$F(x)(t) = x(t) - x_0 - \int_{t_0}^{t} f(s,x(s))\,ds \tag{1.3}$$

For stochastic initial value problems;

$$\begin{gathered} dX(t) = \sigma(t,X(t))dB(t) + b(t,X(t))dt, \qquad 0 \le t \le T \\ X(0) = \xi, \end{gathered} \tag{1.4}$$

one may propose heuristically an analogue of Chaplygin's method in the following iterative scheme;

$$\begin{aligned} X_0(t) &= \xi, \\ X_{n+1}(t) &= X(0) + \int_0^t \sigma(s,X_n(s))\,dB(s) + \int_0^t b(s,X_n(s))\,ds \\ &\quad + \int_0^t \sigma_x(s,X_n(s))(X_{n+1}(s) - X_n(s))\,dB(s) \\ &\quad + \int_0^t b_x(s,X_n(s))(X_{n+1}(s) - X_n(s))\,ds \end{aligned} \tag{1.5}$$

We shall show in this paper that the above sequence is the Newton sequence for the stochastic operator;

$$\begin{aligned} F(Z)(t) = Z(t) - Z(0) &- \int_0^t \sigma(s,Z(s))\,dB(s) \\ &- \int_0^t b(s,Z(s))\,ds \end{aligned} \tag{1.6}$$

We will also discuss the local as well as the global convergence of the sequence to the solution of the equation (1.4). Our investigation is motivated by the paper by Bharucha-Reid and Kannan [2], where they have developed a probabilistic analogue of Newton-Kantorovich's method for solutions of random operator equations. Applications of their theory are being considered by their school [1], although no explicit application to solutions of Ito-type stochastic differential equations seems exist. To avoid complicated notations, we deal in the present paper with one dimensional case only, but one may generalize the results obtained in this paper to multi-dimensional case without any difficulty.

2. Preliminaries.

Let $\sigma(t,x)$ and $b(t,x)$ be defined on $[0,\infty)\times R^1$ and Borel measurable. We consider following Ito-type stochastic differential equation;

$$X(t) = X(0) + \int_0^t \sigma(s, X(s))\, dB(s) + \int_0^t b(s, X(s))\, ds\ . \tag{2.1}$$

By a probability family space with an increasing family of σ-fields which is denoted as $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, we mean a probability space $(\Omega, \mathcal{F}, P)$ with right continuous increasing system $\mathcal{F}_t$ of sub-σ fields of $\mathcal{F}$, each containing all P-null sets.

Definition (2.1) By a solution of the equation (2.1), we mean a probability space with an increasing family of σ-fields $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and a family of stochastic processes $\{X(t), B(t)\}$ defined on it such that

(1) with probability one, $X(t)$ and $B(t)$ are continuous in t and $B(0) = 0$,
(2) $X(t)$ and $B(t)$ are $\mathcal{F}_t$-measurable,
(3) $B(t)$ is a $\mathcal{F}_t$-martingale such that

$$E[(B(t) - B(s))^2 / \mathcal{F}_s] = t - s, \qquad t \geq s, \tag{2.2}$$

(4) $X(t)$ and $B(t)$ satisfy

$$X(t) = X(0) + \int_0^t \sigma(s, X(s))\, dB(s) + \int_0^t b(s, X(s))\, ds, \tag{2:3}$$

where the integral by $dB(s)$ is understood in the sense of Ito integral.

Condition A We say that $\sigma(t,x)$ and $b(t,x)$ satisfy the Condition A, if

(1) $\sigma(t,x)$ and $b(t,x)$ are continuous in (t,x) and differentiable with respect to x, moreover $D_x\sigma(t,x) = \sigma_x(t,x)$ and $D_x b(t,x) = b_x(t,x)$ are continuous with respect to x.
(2) there exist positive constants K and M such that,

$$|\sigma(t,x)|^2 \leq K(1+x^2), \tag{2.4}$$

$$|b(t,x)|^2 \leq K(1+x^2), \tag{2.5}$$

$$|\sigma_x(t,x)| \leq M, \tag{2.6}$$

and

$$|b_x(t,x)| \leq M. \tag{2.7}$$

Remark 2.1. Since the inequalities (2.6) and (2.7) imply the global Lipschitz condition for $\sigma(t,x)$ and $b(t,x)$, then with the conditions (2.4) and (2.5), there exists a solution $X(t)$ of the equation (2.1) defined on $[0,T]$, such that

$$\sup_{t\in[0,T]} E[|X(t)|^2] < +\infty, \tag{2.8}$$

where $T < +\infty$ is an arbitrally given positive number. Furthermore a solution with the property (2.8) is pathwise unique. (see for e.g., [4] and [5]). In the following in this paper, we assume $E[|X(0)|^2] < +\infty$.

3. The Gâteaux derivative.

Let $\mathcal{L}_T$ be the set of $\varphi : [0,\infty)\times\Omega \to R$, such that (i) φ is $\mathcal{F}_t$-adapted and continuous with respect to t, (ii) $E[\sup_{0\le s\le T} |\varphi(s,\omega)|^2] < \infty$. Then $\mathcal{L}_T$ is a Banach space with the norm

$$||\varphi||^2 = E[\sup_{0\le s\le T} |\varphi(s,\omega)|^2].$$

Consider the following operator F defined on $\mathcal{L}_T$;

$$\begin{aligned} F(Z) = F(Z)(t) = Z(t,\omega) - Z(0,\omega) - \int_0^t \sigma(s,Z(s,\omega))\,dB(s) \\ - \int_0^t b(s,Z(s,\omega))\,ds \quad 0\le t\le T, \quad Z\in\mathcal{L}_T \end{aligned} \tag{3.1}$$

LEMMA(3.1). *Under the condition A the operator F maps the space $\mathcal{L}_T$ into itself.*

PROOF: Let a process Z belong to $\mathcal{L}_T$. It is obvious by the definition of F that $F(Z)(t)$ $0\le t\le T$, is $\mathcal{F}_t$-adapted and continuous in t. To prove that

$$E[\sup_{0\le t\le T} |F(Z)(t)|^2] < +\infty \quad \textit{holds},$$

we first observe that

$$\begin{aligned} E[\sup_{0\le t\le T} |F(Z)(t)|^2] \le 3E[\sup_{0\le t\le T} |Z(t,\omega) - Z(0,\omega)|^2] \\ + 3E[\sup_{0\le t\le T} |\int_0^t \sigma(s,Z(s,\omega))\,dB(s)|^2] \\ + 3E[\sup_{0\le t\le T} |\int_0^t b(s,Z(s,\omega))\,ds|^2] \quad \textit{holds}. \end{aligned} \tag{3.2}$$

By Doob's martingale inequlity and Schwarz's inequlity, we get from the above (3.2) that

$$\begin{aligned} E[\sup_{0\le t\le T} |F(Z)(t)|^2] \le 6E[\sup_{0\le t\le T} |Z(t,\omega)|^2] + 6E[|Z(0,\omega)|^2] \\ + 12E[|\int_0^T \sigma(s,Z(s,\omega))\,dB(s)|^2] \\ + 3TE[\int_0^T |b(s,Z(s,\omega))|^2\,ds] \end{aligned} \tag{3.3}$$

holds.

Noting that

$$E[|\int_0^T \sigma(s, Z(s,\omega))\, dB(s)|^2] = E[\int_0^T \sigma^2(s, Z(s,\omega))\, ds\ ],$$

we can conclude from (3.3) with (2.4) and (2.5) in the condition A that

$$\begin{aligned} E[\sup_{0\le t\le T} |F(Z)(t)|^2] &\le 6E[\sup_{0\le t\le T} |Z(t,\omega)|^2] \\ &+ 6E[|Z(0,\omega)|^2] + 12KE[\int_0^T [1+|Z(t,\omega)|^2]\, dt] \\ &+ 3TKE[\int_0^T [1+|Z(t,\omega)|^2]\, dt] \\ &\le 6E[\sup_{0\le t\le T} |Z(t,\omega)|^2] + 6E[|Z(0,\omega)|^2] \\ &+ 12KT[\ 1 + E[\sup_{0\le t\le T} |Z(t,\omega)|^2]] \\ &+ 3KT^2[\ 1 + E[\sup_{0\le t\le T} |Z(t,\omega)|^2]] < +\infty. \end{aligned} \tag{3.4}$$

q.e.d.

Now we are in a position to introduce the Gâteaux derivative of the operator F.

Definition 3.1 Let Z belong to $\mathcal{L}_T$. If for any $h \in \mathcal{L}_T$,

$$\lim_{u\downarrow 0} \frac{1}{u}[\, F(Z+uh) - F(Z)\,]$$

exists in norm convergence sense in the space $\mathcal{L}_T$, we call the limit the Gâteaux derivative of the operator F at Z. This limit element in $\mathcal{L}_T$ will be denoted by

$$dF(Z;h) = dF(Z;h)(t), \quad 0 \le t \le T.$$

LEMMA 3.2. *For any $Z \in \mathcal{L}_T$, there exists the Gâteaux derivative of the operator F at Z and it satisfies*

$$\begin{aligned} dF(Z;h) &= dF(Z;h)(t) \\ &= h(t,\omega) - h(0,\omega) - \int_0^t \sigma_x(s, Z(s,\omega))h(s,\omega)\, dB(s) \\ &- \int_0^t b_x(s, Z(s,\omega))h(s,\omega)\, ds. \end{aligned} \tag{3.5.}$$

PROOF: By the definition of the operator F, we observe that

$$\begin{aligned} &\frac{1}{u}[\ F(Z+uh)(t) - F(Z)(t)\ ] \\ &= \frac{1}{u}[\ uh(t,\omega) - \int_0^t [\sigma(s, Z(s,\omega)+uh(s,\omega)) - \sigma(s, Z(s,\omega))]\, dB(s) \\ &- \int_0^t [b(s, Z(s,\omega)+uh(s,\omega)) - b(s, Z(s,\omega))]\, ds] \\ &= h(t,\omega) - \int_0^t \sigma_x(s, Z(s,\omega))h(s,\omega)\, dB(s) \\ &- \int_0^t b_x(s, Z(s,\omega))h(s,\omega)\, ds + R(t,\omega), \ \ say. \end{aligned}$$

Note by the condition A that the functions $\sigma_x(t,x)$ and $b_x(t,x)$ are continuous with respect to x. Then we have

$$
\begin{aligned}
&R(t,\omega)\\
(3.6)\qquad &= -\frac{1}{u}[\int_0^t [\sigma_x(s,Z(s,\omega)+\theta uh(s,\omega)) - \sigma_x(s,Z(s,\omega))]uh(s,\omega)\,dB(s)\\
&+\int_0^t [b_x(s,Z(s,\omega)+\theta uh(s,\omega)) - b_x(s,Z(s,\omega))]uh(s,\omega)\,ds]
\end{aligned}
$$

where $\theta, 0<\theta<1$, depends on (s,ω,h) .

To complete the proof it suffices to show that

$$
(3.7)\qquad \lim_{u\downarrow 0} E[\sup_{0\le t\le T} |R(t,\omega)|^2] = 0, \quad \textit{holds.}
$$

By a similar way as in the proof of Lemma (3.1), we observe that

$$
\begin{aligned}
&E[\sup_{0\le t\le T} |R(t,\omega)|^2]\\
&\le 2E[\sup_{0\le t\le T} | \int_0^t (\sigma_x(s,Z(s,\omega)+\theta uh(s,\omega)) - \sigma_x(s,Z(s,\omega)))\\
&h(s,\omega)\,dB(s)|^2]\\
&+2E[\sup_{0\le t\le T} | \int_0^t (b_x(s,Z(s,\omega)+\theta uh(s,\omega)) - b_x(s,Z(s,\omega))h(s,\omega)\,ds|^2]\\
(3.8)\qquad &\le 8E[\int_0^T |\sigma_x(s,Z(s,\omega)+\theta uh(s,\omega)) - \sigma_x(s,Z(s,\omega))|^2 h^2(s,\omega)\,ds]\\
&+2E[(\int_0^T |b_x(s,Z(s,\omega)+\theta uh(s,\omega)) - b_x(s,Z(s,\omega))|^2\,ds)\\
&(\int_0^T h^2(s,\omega)\,ds)]\\
&= J_1 \;\; + J_2 \;, say.
\end{aligned}
$$

Since the function σ_x and the function b_x both are continuous with respect to x, it follows that

$$
\lim_{u\downarrow 0}[\; \sigma_x(s,Z(s,\omega)+\theta uh(s,\omega)) - \sigma_x(s,Z(s,\omega)) \;] = 0
$$

and also

$$
\lim_{u\downarrow 0}[\; b_x(s,Z(s,\omega)+\theta uh(s,\omega)) - b_x(s,Z(s,\omega))] = 0
$$

hold. Furthermore, we know by the condition A that

$$
|\sigma_x| \le M \; and \; |b_x| \le M \quad hold.
$$

Hence, Lebesgue's convergence theorem implies

$$
\lim_{u\downarrow 0}[\; J_1 \;+ J_2 \;] = 0.
$$

Thus by (3.8)

$$
\lim_{u\downarrow 0} E[\sup_{0\le t\le T} |R(t,\omega)|^2] = 0.
$$

The lemma is proved. *q.e.d.*

4. Stochastic analogue of Newton's method for stochastic differential equations.

First of all, we will discuss the existence of the inverse the Gâteaux derivative of F at Z which will be denoted by $dF^{-1}(Z)$.

LEMMA 4.1. *Let Z be a given element in $\mathcal{L}_T$. Let φ belong to $\mathcal{L}_T$ such that $\varphi(0,\omega) = 0$. Then, there exists one and only one element h in $\mathcal{L}_T$ such that,*

$$\varphi(t,\omega) = dF(Z;h)(t); \tag{4.1}$$

i.e.,

$$\varphi(t,\omega) = h(t,\omega) - \int_0^t \sigma_x(s, Z(s,\omega))h(s,\omega)\, dB(s) - \int_0^t b_x(s, Z(s,\omega))h(s,\omega)\, ds\,. \tag{4.2}$$

PROOF: Since the linear stochastic differential equation (4.2) satisfies the global Lipshitz condition for its diffusion coefficient as well as for its drift coefficient, then the existence and the pathwise uniqueness hold for the equation (4.2). From this fact, the lemma follows immediately. *q.e.d.*

LEMMA 4.2. *Let φ belong to $\mathcal{L}_T$, such that $\varphi(0,\omega) = 0$. Then, there exists a positive constant $L < +\infty$, which is independent of Z and also of $t \in [0,T]$,such that*

$$||dF^{-1}(Z)(\varphi)||_t^2 \le 3||\varphi||_t^2 e^{Lt}, \quad 0 \le t \le T \tag{4.3,}$$

where $||\varphi||_t^2$ stands for $E[\sup_{0\le s\le t} |\varphi(s,\omega)|^2]$.

PROOF: Let $h(t,\omega)$ be

$$h(t,\omega) = dF^{-1}(Z)(\varphi)(t,\omega), \quad 0 \le t \le T.$$

Then by (4.2), we get

$$\begin{aligned} ||h||_t^2 = E[\sup_{0\le s\le t} |h(s,\omega)|^2] &\le 3E[\sup_{0\le s\le t} |\varphi(s,\omega)|^2] \\ &+ 3E[\sup_{0\le s\le t} |\int_0^s \sigma_x(u, Z(u,\omega))h(u,\omega)\, dB(u)|^2] \\ &+ 3E[\sup_{0\le s\le t} |\int_0^s b_x(u, Z(u,\omega))h(u,\omega)\, du|^2] \end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$\begin{aligned} ||h||_t^2 &\le 3||\varphi||_t^2 + 12E[\int_0^t \sigma_x^2(s, Z(s,\omega))h^2(s,\omega)\, ds] \\ &+ 3E[(\int_0^t b_x^2(s, Z(s,\omega))\, ds) \int_0^t h^2(s,\omega)\, ds] \\ &\le 3||\varphi||_t^2 + 12M^2 E[\int_0^t h^2(s,\omega)\, ds] + 3M^2 T E[\int_0^t h^2(s,\omega)\, ds] \\ &\le 3||\varphi||_t^2 + L\int_0^t ||h||_s^2\, ds, \end{aligned} \tag{4.5}$$

where we have used (2.6) and (2.7) in the condition A, and L stands for $12M^2 + 3M^2T$.
By Gronwall's inequality, it follows from (4.5) that

$$\|h\|_t^2 = \| dF^{-1}(Z)(\varphi) \|_t^2 \leq 3\|\varphi\|_t^2 e^{Lt}, \quad 0 \leq t \leq T.$$

q.e.d.

We are now in a position to introduce the Newton sequence for the operator F.
Let

$$\begin{aligned} X_0(t) &= X(0,\omega), \\ X_{n+1}(t) &= X_n(t) - dF^{-1}(X_n)(F(X_n))(t) \quad n = 1,2,... \end{aligned} \tag{4.6}$$

We call $X_n(t), n = 1,2,...,$ the Newton sequence for the operator F.
It follows from (4.6) that the sequence satisfies that

$$\begin{aligned} X_{n+1}(t) = X_0(t) &+ \int_0^t \sigma(s, X_n(s))\, dB(s) \\ &+ \int_0^t b(s, X_n(s))\, ds + \int_0^t \sigma_x(s, X_n(s))(X_{n+1}(s) - X_n(s))\, dB(s) \\ &+ \int_0^t b_x(s, X_n(s))(X_{n+1}(s) - X_n(s))\, ds \end{aligned} \tag{4.7}$$

Thus the Newton sequence introduced in the above (4.6) is exactly the same sequence as the stochastic analogue of Chaplygin sequence (1.5) discussed in the introduction.

The following theorem concerns the convergence in local sense of the Newton sequence to the solution of the stochastic differential equation (2.1).

THEOREM 4.1. *Let $X(t)$ be the solution of the equation (2.1). Choose a positive number δ such that,*

$$120\delta M^2 e^{L\delta} = \alpha < 1 \tag{4.8}$$

holds. Then,

$$\lim_{n\to 0} E[\sup_{0\leq t\leq \delta} |X_n(t) - X(t)|^2] = 0 \tag{4.9}$$

holds with error bound,

$$\begin{aligned} &[E[\sup_{0\leq t\leq\delta} |X_n(t) - X(t)|^2]]^{1/2} \\ &\leq \frac{\beta^n}{1-\beta}[E[\sup_{0\leq t\leq\delta} |X_1(t) - X_0(t)|^2]]^{1/2}, \end{aligned} \tag{4.10}$$

where $\beta = \sqrt{\alpha}$.

PROOF: we will devide the proof in two steps. Without loss of generality, we can suppose that $\delta < 1$ holds.

First step. In this step we will show that

$$\|X_{n+1} - X_n\|_\delta^2 \leq \alpha\|X_n - X_{n-1}\|_\delta^2 \tag{4.11}$$

holds.

By the definition of the Newton sequence (4.6), we know

$$X_{n+1}(t) - X_n(t) = -dF^{-1}(X_n)(F(X_n))(t) \ . \tag{4.12}$$

Hence, Lemma 4.2 implies that

$$\|X_{n+1} - X_n\|_\delta^2 \ \leq \ 3e^{L\delta}\|F(X_n)\|_\delta^2 \tag{4.13}$$

holds.

For $n \geq 1$, we observe by (4.6)

$$\begin{aligned} F(X_n)(t) &= F(X_n)(t) - F(X_{n-1})(t) + F(X_{n-1})(t) \\ &= F(X_n)(t) - F(X_{n-1})(t) - dF(X_{n-1}; X_n - X_{n-1})(t) \ . \end{aligned}$$

Hence, we have

$$\begin{aligned} F(X_n)(t) &= \int_0^t \sigma(s, X_{n-1}(s))\, dB(s) - \int_0^t \sigma(s, X_n(s))\, dB(s) \\ &+ \int_0^t b(s, X_{n-1}(s))\, ds - \int_0^t b(s, X_n(s))\, ds \\ &+ \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, dB(s) \\ &+ \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, ds \\ &= \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, dB(s) \\ &+ \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, ds \\ &- \int_0^t \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s)))(X_n(s) - X_{n-1}(s))\, dB(s) \\ &- \int_0^t b(s, X_{n-1}(s) + \theta'(X_n(s) - X_{n-1}(s)))(X_n(s) - X_{n-1}(s))\, ds \end{aligned}$$

where $0 < \theta, \theta' < 1$.

Thus, we have

$$
\begin{aligned}
&\|F(X_n)\|_\delta^2 \\
&\le 2E[\sup_{0\le t\le\delta} | \int_0^t (\sigma_x(s, X_{n-1}(s)) - \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s)))) \\
&(X_n(s) - X_{n-1}(s))\, dB(s)|^2] \\
&+ 2E[\sup_{0\le t\le\delta} | \int_0^t (b_x(s, X_{n-1}(s)) - b_x(s, X_{n-1}(s) + \theta(X_{n-1}(s) - X_n(s)))) \\
&(X_n(s) - X_{n-1}(s))\, ds|^2] \\
&\le 8E[\int_0^t (\sigma_x(s, X_{n-1}(s)) - \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s)))^2 \\
&(X_n(s) - X_{n-1}(s))^2\, ds] \\
&+ 2E[\int_0^t (b_x(s, X_{n-1}(s)) - b_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s))))^2\, ds \\
&\int_0^t (X_n(s) - X_{n-1}(s))^2\, ds]
\end{aligned}
$$

Hence, by (2.6) and (2.7) in the condition A, we observe that

$$
\begin{aligned}
\|F(X_n)\|_\delta^2 &\le 32M^2\delta\|X_n - X_{n-1}\|_\delta^2 + 8M^2\delta^2\|X_n - X_{n-1}\|_\delta^2 \\
&\le 40M^2\delta\|X_n - X_{n-1}\|_\delta^2\,, \quad (0 < \delta < 1).
\end{aligned} \tag{4.14}
$$

Combine (4.13) with (4.14). Then, we can conclude that (4.11) holds.

Second step. Put $\beta = \sqrt{\alpha}$. The inequality (4.11) implies

$$
\|X_{n+1} - X_n\|_\delta \le \beta\|X_n - X_{n-1}\|_\delta \tag{4.15}
$$

From this it follows immediately

$$
\|X_{n+1} - X_n\|_\delta \le \beta^n\|X_1 - X_0\|_\delta \tag{4.16}
$$

Since $\|\ \|_\delta$ is the norm of the Banach space $\mathcal{L}_\delta$, we get from (4.16) that

$$
\begin{aligned}
\|X_{n+p} - X_n\|_\delta &\le (\beta^{n+p-1} + \cdots + \beta^n)\|X_1 - X_0\|_\delta \\
&\le \frac{\beta^n}{1-\beta}\|X_1 - X_0\|_\delta\,.
\end{aligned} \tag{4.17}
$$

Hence, the sequence X_n $n = 1, 2, \ldots$, is a Cauchy sequence in the Banach space $\mathcal{L}_\delta$. Put $\tilde{X}(t)$ $0 \le t \le \delta$ the limit of the sequence X_n $n = 1, 2, \ldots$. Since the process $X_n(t)$ satisfies

$$
\begin{aligned}
X_n(t) = X(0) &+ \int_0^t \sigma(s, X_{n-1}(s))\, dB(s) + \int_0^t b(s, X_{n-1}(s))\, ds \\
&+ \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, dB(s) \\
&+ \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s))\, ds \quad 0 \le t \le \delta\,,
\end{aligned}
$$

then, the limit process $\tilde{X}(t)$ satisfies the equation (2.1);

$$\tilde{X}(t) = X(0) + \int_0^t \sigma(s, \tilde{X}(s))\, dB(s) + \int_0^t b(s, \tilde{X}(s))\, ds\,, \quad 0 \leq t \leq \delta.$$

Since the Pathwise uniqueness holds for the equation (2.1), we observe that

(4.18) $$\tilde{X}(t) = X(t) \textit{ holds.}$$

Hence we get (4.9). (4.10) follows from (4.17) and (4.18). *q.e.d.*

5. The convergence in the large of the Newton sequence.

In this section we assume for the coefficients the following condition B.

Condition B : We say that the coefficients $\sigma(t,x)$ and $b(t,x)$ satisfy the Condition B , if they satisfy the Condition A and moreover there exists a positive constant $N < +\infty$, such that

(5.1) $$|\sigma(t,x)| \leq N \quad \textit{and} \quad |b(t,x)| \leq N$$

hold for all t and x.

Under the condition B, we have the following theorem which concerns the convergence in the large.

THEOREM 5.1. *Let T be a fixed positive number. Then the Newton sequence $X_n(t)$ $n = 1, 2, \ldots$ defined by (4.6) converges in the large to the solution $X(t)$ of the equation (2.1) in the following sense;*

(5.2) $$\lim_{n\to\infty} E[\sup_{0\leq t\leq T} |X_n(t) - X(t)|^2] = 0,$$

if and only if

(5.3) $$\sup_n E[\sup_{0\leq t\leq T} |X_n(t)|^2] \ < +\infty$$

holds.

PROOF: The necessity is obvious. To prove the sufficiency, we will devide the proof in several steps. In the proof $K_2 < \infty$ stands for $\sup_n E[\sup_{0\leq t\leq T} |X_n(t)|^2]$

First step : Let T_1 be defined by

(5.4) $$T_1 = \sup\{t; t \in [0,T] \textit{ and } \lim_{n\to\infty} E[\sup_{0\leq s\leq t} |X_n(s) - X(s)|^2] = 0\}.$$

Then Theorem 4.1 implies

(5.5) $$0 \ < \ \delta \ \leq \ T_1 \ \leq \ T.$$

Second step : In the present step, we will show that

(5.6) $$\lim_{n\to\infty} E[\sup_{0\leq t\leq T_1} |X_n(t) - X(t)|^2] = 0$$

holds.

Let $\epsilon > 0$ be an arbitrary positive number. Choose S_0 such that

$$
(5.7) \qquad \begin{aligned} 0 &< S_0 < \min(T_1, 1) \\ (80M^2K_2 + 20N^2)S_0 &< \frac{\epsilon}{10}. \end{aligned}
$$

By the definition of T_1, we get

$$
\lim_{n\to\infty} E[\sup_{0\le t\le T_1 - S_o} |X_n(t) - X(t)|^2] = 0.
$$

Hence, for sufficiently large N_1, we observe that

$$
(5.8) \qquad E[\sup_{0\le t\le T_1 - S_o} |X_n(t) - X(t)|^2] \le \frac{\epsilon}{10}, \quad n \ge N_1
$$

holds.

On the other hand, we have

$$
(5.9) \qquad E[\sup_{T_1 - S_0\le t\le T_1} |X_n(t) - X(t)|^2] \le 3I_1 + 3I_2 + 3I_3,
$$

where

$$
\begin{aligned}
I_1 &= E[\sup_{T_1 - S_0\le t\le T_1} |X_n(t) - X_n(T_1 - S_0)|^2], \\
I_2 &= E[|X_n(T_1 - S_0) - X(T_1 - S_0)|^2], \\
I_3 &= E[\sup_{T_1 - S_0\le t\le T_1} |X(t) - X(T_1 - S_0)|^2].
\end{aligned}
$$

First, we will deal with I_1. We have

$$
(5.10) \qquad \begin{aligned}
X_n(t) - X_n(T_1 - S_0) &= \int_{T_1 - S_0}^{t} \sigma(s, X_{n-1}(s))\, dB(s) \\
&+ \int_{T_1 - S_0}^{t} b(s, X_{n-1}(s))\, ds + \int_{T_1 - S_0}^{t} \sigma_x(s, X_{n-1}(s))Y_n(s)\, dB(s) \\
&+ \int_{T_1 - S_0}^{t} b_x(s, X_{n-1}(s))Y_n(s)\, ds,
\end{aligned}
$$

where $Y_n(t) = X_n(t) - X_{n-1}(t)$.

By Doob's martingale inequality with Schwarz's inequality, it follows from (5.10) that

$$
\begin{aligned}
&E[\sup_{T_1 - S_0\le t\le T_1} |X_n(t) - X_n(T_1 - S_0)|^2] \\
&\le 16E[\int_{T_1 - S_0}^{T_1} |\sigma(s, X_{n-1}(s))|^2\, ds] + 4S_0E[\int_{T_1 - S_0}^{T_1} |b(s, X_{n-1}(s))|^2\, ds] \\
&+ 16E[\int_{T_1 - S_0}^{T_1} |\sigma_x(s, X_{n-1}(s))Y_n(s)|^2\, ds] \\
&+ 4S_0E[\int_{T_1 - S_0}^{T_1} |b_x(s, X_{n-1}(s))Y_n(s)|^2\, ds].
\end{aligned}
$$

Hence, by the condition B, we observe that

$$(5.11)\quad \begin{aligned} &E[\sup_{T_1-S_0\le t\le T_1}|X_n(t)-X_n(T_1-S_0)|^2] \\ &\qquad \le 16N^2S_0+4N^2S_0+16M^2(\int_{T_1-S_0}^{T_1} E[\sup_{0\le t\le T}|Y_n(t)|^2]\,ds) \\ &\qquad + 4M^2S_0\int_{T_1-S_0}^{T_1} E[\sup_{0\le t\le T}|Y_n(t)|^2]\,ds \end{aligned}$$

Here, note that by the condition (5.3) that

$$(5.12)\quad \begin{aligned} &E[\sup_{0\le t\le T}|Y_n(t)|^2] \le 2E[\sup_{0\le t\le T}|X_n(t)|^2] \\ &\qquad + 2E[\sup_{0\le t\le T}|X_{n-1}(t)|^2] \le 4K_2<+\infty, \quad n=1,2,\dots \end{aligned}$$

Then the inequalities (5.11) and (5.12) imply that

$$I_1 \quad \le \quad 80M^2K_2S_0+20N^2S_0.$$

Hence, by (5.7),

$$(5.13)\qquad I_1 \quad \le \quad \frac{\epsilon}{10}$$

holds.

Second, for I_2, we can choose a number N_2 such that

$$(5.14)\qquad I_2=E[|X_n(T_1-S_0)-X(T_1-S_0)|^2]\le\frac{\epsilon}{10}, \quad n\ge N_2$$

holds.

Finaly for I_3 , it is easily seen that

$$\begin{aligned} I_3 &\le 8E[\int_{T_1-S_0}^{T_1}\sigma^2(s,X(s))\,ds]+2S_0E[\int_{T_1-S_0}^{T_1}b^2(s,X(s))\,ds] \\ &\le 8N^2S_0+2N^2S_0. \end{aligned}$$

Hence by (5.7) we get

$$(5.15)\qquad I_3 \quad \le \quad \frac{\epsilon}{10}.$$

From the inequalities (5.8),(5.9),(5.13),(5.14) and (5.15), we can conclude that

$$(5.6)\qquad \lim_{n\to\infty}E[\sup_{0\le t\le T_1}|X_n(t)-X(t)|^2]=0$$

holds.

Third step : In this step, we shall show that $T_1=T$, using the method of reduction to absurdity.

Assume $T_1\ne T$ and let us find a contradiction.

By what has been proved in the second step, we can choose a sequence of positive numbers a_n, $n = 1, 2, \ldots$ such that

$$\begin{aligned} a_n \quad &\downarrow \quad 0 \ (n \to \infty) \\ E[|X_n(T_1) - X(T_1)|^2] \ &\leq \ a_n. \end{aligned} \tag{5.16}$$

We will devide the step in two substeps.

(i): First, we will find a positive number $h > 0$ such that

$$\begin{aligned} &T_1 + h \ \leq \ T, \\ &\lim_{n\to\infty} E[\sup_{T_1 \leq t \leq T_1+h} |Y_n(t)|^2] = 0, \quad \textit{where } Y_n(t) = X_n(t) - X_{n-1}(t), \end{aligned} \tag{5.17}$$

holds.

By the definition of $Y_n(t)$, we have for $T_1 \leq t \leq T$,

$$\begin{aligned} Y_n(t) &= X_n(T_1) - X_{n-1}(T_1) \\ &+ \int_{T_1}^t (\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s)))\, dB(s) \\ &+ \int_{T_1}^t (b(s, X_{n-1}(s)) - b(s, X_{n-2}(s)))\, ds \\ &+ \int_{T_1}^t \sigma_x(s, X_{n-1}(s)) Y_n(s)\, dB(s) - \int_{T_1}^t \sigma_x(s, X_{n-2}(s)) Y_{n-1}(s)\, dB(s) \\ &+ \int_{T_1}^t b_x(s, X_{n-1}(s)) Y_n(s)\, ds - \int_{T_1}^t b_x(s, X_{n-2}(s)) Y_{n-1}(s)\, ds \end{aligned}$$

Thus, we get from the above that

$$\begin{aligned} &E[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2] \\ &\leq 7E[|X_n(T_1) - X_{n-1}(T_1)|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u)))\, dB(u)|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (b(u, X_{n-1}(u)) - b(u, X_{n-2}(u)))\, du|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (b_x(u, X_{n-1}(u)) Y_n(u)\, du|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (b_x(u, X_{n-2}(u)) Y_{n-1}(u)\, du|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (\sigma_x(u, X_{n-1}(u)) Y_n(u)\, dB(u)|^2] \\ &+ 7E[\sup_{T_1 \leq s \leq t} | \int_{T_1}^s (\sigma_x(u, X_{n-2}(u)) Y_{n-1}(u)\, dB(u)|^2]. \end{aligned}$$

Hence, by Doob's martingale inequality with Schwarz's inequality, we observe that

$$
\begin{aligned}
E[\sup_{T_1 \le s \le t} |Y_n(s)|^2] &\le 7E[|X_n(T_1) - X_{n-1}(T_1)|^2] \\
&+ 28E[\int_{T_1}^t |\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u))|^2 \, du] \\
&+ 7(t - T_1)E[\int_{T_1}^t |b(u, X_{n-1}(u)) - b(u, X_{n-2}(u))|^2 \, du] \\
&+ 7(t - T_1)M^2 E[\int_{T_1}^t |Y_n(u)|^2 \, du] + 7(t - T_1)M^2 E[\int_{T_1}^t |Y_{n-1}(u)|^2 \, du] \\
&+ 28M^2 E[\int_{T_1}^t |Y_n(u)|^2 \, du] + 28M^2 E[\int_{T_1}^t |Y_{n-1}(u)|^2 \, du]
\end{aligned} \tag{5.18}
$$

holds, where we have used the inequlities (2.6) and (2.7) in the condition A.

Note, by the condition A again, that

$$
|b(u, X_{n-1}(u)) - b(u, X_{n-2}(u))| \le M|X_{n-1}(u) - X_{n-2}(u)| \tag{5.19}
$$

and

$$
|\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u))| \le M|X_{n-1}(u) - X_{n-2}(u)| \tag{5.20}
$$

holds.

Then, the inequalities (5.18),(5.19) and (5.20) imply that

$$
\begin{aligned}
&E[\sup_{T_1 \le s \le t} |Y_n(s)|^2] \\
&\le 7E[|X_n(T_1) - X(T_1)|^2] + (56M^2 + 14(t - T_1)M^2)E[\int_{T_1}^t |Y_{n-1}(u)|^2 \, du] \\
&+ (7(t - T_1)M^2 + 28M^2)E[\int_{T_1}^t |Y_n(u)|^2 \, du]
\end{aligned} \tag{5.21}
$$

Choose $h > 0$, such that

$$
\eta = (56M^2 h + 14M^2 h^2)e^{7M^2h^2 + 28M^2h} < 1 \tag{5.22}
$$

Note that

$$
E[|X_n(T_1) - X_{n-1}(T_1)|^2] \le 2a_n + 2a_{n-1} \le 4a_{n-1} \tag{5.23}
$$

Then the (5.21) implies for $T_1 \le t \le T_1 + h$,

$$
\begin{aligned}
&E[\sup_{T_1 \le s \le t} |Y_n(s)|^2] \\
&\le 28a_{n-1} + (56M^2 + 14M^2h^2)|||Y_{n-1}||| \\
&+ (28M^2 + 7M^2h) \int_{T_1}^t E[\sup_{T_1 \le u \le s} |Y_n(u)|^2] \, ds
\end{aligned} \tag{5.24}
$$

where $|||\varphi|||$ stands for $E[\sup_{T_1 \le t \le T_1 + h} |\varphi(t)|^2]$.

Hence, by Gronwall's inequality, we observe that

$$(5.25)\qquad \begin{aligned} &E[\sup_{T_1 \le s \le t} |Y_n(s)|^2] \\ &\le \{28a_{n-1} + (56M^2h + 14M^2h^2)|||Y_{n-1}|||\}e^{(7M^2h+28M^2)(t-T_1)} \end{aligned}$$

Put

$$(5.26)\qquad \gamma_n = 28a_n e^{7M^2h^2+28M^2h}.$$

Then we get from the above inequality (5.25) that

$$(5.27)\qquad |||Y_n||| = E[\sup_{T_1 \le s \le T_1+h} |Y_n(s)|^2] \le (\gamma_{n-1} + \eta|||Y_{n-1}|||)$$

holds. Now we are in a position to prove

$$(5.17)\qquad \lim_{n\to\infty} E[\sup_{T_1 \le t \le T_1+h} |Y_n(t)|^2] = \lim_{n\to\infty} |||Y_n||| = 0.$$

Let $\epsilon > 0$ be an arbitrary positive number. Choose an positive integer N_1 such that,

$$(5.28)\qquad \gamma_{n-1} \quad \le \quad \frac{\epsilon}{2}(1-\eta), \quad n \ge N_1,$$

holds.

We have by (5.27) that

$$(5.29)\qquad \begin{aligned} |||Y_{N_1+m}||| &\le \gamma_{N_1+m-1} + \eta|||Y_{N_1+m-1}||| \\ &\le \gamma_{N_1+m-1} + \eta\gamma_{N_1+m-2} + \eta^2|||Y_{N_1+m-2}||| \\ &\le \gamma_{N_1-1}(1+\eta+\eta^2+\cdots+\eta^m) + \eta^{m+1}|||Y_{N_1-1}||| \\ &\le \frac{\gamma_{N_1-1}}{1-\eta} + 4K_2\eta^{m+1}. \end{aligned}$$

Choose a positive integerN_2 such that

$$(5.30)\qquad 4K_2\eta^{m+1} \quad < \quad \frac{\epsilon}{2}, \quad m \ge N_2,$$

holds. Hence we can conclude that

$$|||Y_n||| \ < \ \epsilon, \quad n \ \ge \ N_1 + N_2,$$

holds.

(*ii*) : Here we will show that

$$(5.31)\qquad \lim_{n\to\infty} E[\sup_{T_1 \le t \le T_1+h} |X_n(t) - X(t)|^2] = 0.$$

holds.

Note that (5.16) and (5.17) hold. Then we can choose a sequence of positive numbers $\delta_n, n = 1, 2, \ldots$ such that

$$\begin{aligned}&\delta_n \downarrow 0 (n \to \infty)\\ &5E[|X_n(T_1) - X(T_1)|^2]\\ &+ (60M^2h + 15M^2h^2)E[\sup_{T_1 \le t \le T_1+h} |Y_n(t)|^2] \le \delta_n\end{aligned} \tag{5.32}$$

holds.

By the definition of the processes X_n and X, we have

$$\begin{aligned}&E[\sup_{T_1 \le s \le t} |X_n(s) - X(s)|^2]\\ &\le 5E[|X_n(T_1) - X(T_1)|^2]\\ &+ 5E[\sup_{T_1 \le s \le t} |\int_{T_1}^s (\sigma(u, X_{n-1}(u)) - \sigma(u, X(u)))\, dB(u)|^2]\\ &+ 5E[\sup_{T_1 \le s \le t} |\int_{T_1}^s (b(u, X_{n-1}(u)) - b(u, X(u)))\, du|^2]\\ &+ 5E[\sup_{T_1 \le s \le t} |\int_{T_1}^s (\sigma_x(u, X_{n-1}(u))Y_n(u)\, dB(u)|^2]\\ &+ 5E[\sup_{T_1 \le s \le t} |\int_{T_1}^s (b_x(u, X_{n-1}(u))Y_n(u)\, du|^2], \quad T_1 \le t \le T_1 + h.\end{aligned} \tag{5.33}$$

By (2.6) and (2.7) in the condition A, it follows from the above (5.33) that

$$\begin{aligned}&E[\sup_{T_1 \le s \le t} |X_n(s) - X(s)|^2]\\ &\le 5E[|X_n(T_1) - X(T_1)|^2]\\ &+ 20M^2 \int_{T_1}^t E[\sup_{T_1 \le u \le s} |X_{n-1}(u) - X(u)|^2]\, ds\\ &+ 5hM^2 \int_{T_1}^t E[\sup_{T_1 \le u \le s} |X_{n-1}(u) - X(u)|^2]\, ds\\ &+ 20hM^2 E[\sup_{T_1 \le u \le T_1+h} |Y_n(t)|^2]\\ &+ 5M^2h^2 E[\sup_{T_1 \le u \le T_1+h} |Y_n(t)|^2], \quad T_1 \le t \le T_1 + h.\end{aligned} \tag{5.34}$$

Note that

$$\begin{aligned}&E[\sup_{T_1 \le u \le s} |X_{n-1}(u) - X(u)|^2]\\ &\le 2E[\sup_{T_1 \le u \le T_1+h} |Y_n(u)|^2] + 2E[\sup_{T_1 \le u \le s} |X_n(u) - X(u)|^2], \quad T_1 \le s \le T_1 + h.\end{aligned}$$

Then, we observe from (5.34) that

$$\begin{aligned}&E[\sup_{T_1 \le s \le t} |X_n(s) - X(s)|^2]\\ &\le 5E[|X_n(T_1) - X(T_1)|^2] + 40M^2 \int_{T_1}^t E[\sup_{T_1 \le u \le s} |X_n(u) - X(u)|^2]\, ds\\ &+ 10M^2h \int_{T_1}^t E[\sup_{T_1 \le u \le s} |X_n(u) - X(u)|^2]\, ds\end{aligned}$$

$$+ (40M^2h + 10M^2h^2 + 20M^2h + 5M^2h^2)E[\sup_{T_1 \le t \le T_1+h} |Y_n(t)|^2]$$

$$\le \delta_n + (40M^2 + 10M^2h) \int_{T_1}^{t} E[\sup_{T_1 \le u \le s} |X_n(u) - X(u)|^2]\, ds, \quad T_1 \le t \le T_1 + h.$$

Hence, by Gronwall's inequality, we have

$$E[\sup_{T_1 \le s \le t} |X_n(s) - X(s)|^2] \le \delta_n e^{(40M^2h + 10M^2h^2)}, \quad T_1 \le t \le T_1 + h.$$

Thus we can conclude that

$$\lim_{n \to \infty} E[\sup_{T_1 \le t \le T_1+h} |X_n(t) - X(t)|^2] = 0.$$

But this contradicts the definition of T_1. *q.e.d.*

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