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On Almost Sure Convergence of Modified Euler-Peano Approximation of Solution to an S.D.E. Driven by a Semimartingale

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1 Introduction

We consider the equation

$$Z_t = H_t + \int_0^t b(s-,\cdot,Z)dX_s \tag{1.1}$$

where X is an $\mathbb{R}^d$-valued semimartingale, H is an $\mathbb{R}^m$-valued r.c.l.l. process and where $b:[0,\infty)\times\Omega\times D([0,\infty),\mathbb{R}^m)\to\mathbb{R}^m\otimes\mathbb{R}^d$ is a functional assumed to satisfy

$$|b(s,w,\rho_1)-b(s,w,\rho_2)| \le A_s(w)|\rho_1-\rho_2|_s^* \tag{1.2}$$

for an increasing process A. Here $|\cdot|$ denotes Euclidian norm (on $\mathbb{R}^m$ or $\mathbb{R}^m\otimes\mathbb{R}^d$) and $|\rho|_s^* := \sup_{t\le s}|\rho_t|$. Bichteler [1] had shown that the ε_n-Euler-Peano approximation to (1.1) converges almost surely, if $\Sigma\varepsilon_n^2<\infty$. Bichteler pointed out that when $b(t,w,\rho)=f(\rho(t))$ and$f$ is a Lipschitz function, so that the equation is

$$Z_t = H_t + \int_0^t f(Z_{s-})dX_s \tag{1.3}$$

the Eular-Peano scheme yields a *pathwise formula* for the solution Z *i.e.* the path $Z(t,w)$ for a fixed w can be obtained as an explicit functional of the paths $H(s,w)$ and $X(s,w)$. This is important for statistical applications. In this article we show that a modified Eular-Peano scheme yields a pathwise formula for (1.1) as well. It is well known that Picards successive approximation method converges a.s. , see Bichteler [1], Karandikar [3,4], Schwartz [7], Meyer [6]. In Karandikar [3,4] a modification is suggested that yields pathwise formula. However, in Picard's method, to get the n^{th} approximation, we need to compute $1^{st},2^{nd},\ldots,(n-1)^{th}$ approximation. However in the Euler-Peano method, to compute ε - approximation , we do not need to compute ε'- approximation for any other ε'.

The suggested approximation: Fix $\varepsilon>0$. The ε-approximation $Y\equiv Y^\varepsilon$ of Z is defined as follows.

Let stop times τ_i and processes W^i be defined inductively by:

$$\tau_0 = 0 \text{ and } W_t^0 \equiv H_0 \tag{1.4}$$

and having defined τ_j, W^j for $j \le i$, let

$$\tau_{i+1} = \inf\{t > \tau_i \ : \ |H_t - H_{\tau_i} + b(\tau_i, \cdot, W^i)(X_t - X_{\tau_i})| \ge \epsilon \\ \text{or } |b(t, \cdot, W^i) - b(\tau_i, \cdot, W^i)| \ge \epsilon\} \tag{1.5}$$

and

$$\begin{aligned} W_t^{i+1} &= W_t^i \text{ for } t < \tau_{i+1} \\ &= W_{\tau_i}^i + H_{\tau_{i+1}} - H_{\tau_i} + b(\tau_i, \cdot, W^i)(X_{\tau_{i+1}} - X_{\tau_i}) \text{ for } t \ge \tau_{i+1}. \end{aligned} \tag{1.6}$$

Thus, W^{i+1} is a process that has jumps at $\tau_1, ..., \tau_{i+1}$ and is constant on the intervals $[0, \tau_1), \cdots, [\tau_j, \tau_{j+1}), \cdots [\tau_i, \tau_{i+1}), [\tau_{i+1}, \infty)$. Let us piece together these processes $W^i, i = 1, 2, \ldots$ to define a step process $S \equiv S^\epsilon$ as follows.

$$S_t = W_{\tau_i}^i \text{ for } \tau_i \le t < \tau_{i+1}.$$

Now define $Y \equiv Y^\epsilon$ by $Y_0 = H_0$ and

$$Y_t = S_{\tau_i} + H_t - H_{\tau_i} + b(\tau_i, \cdot, W^i)(X_t - X_{\tau_i}) \text{ for } \tau_i < t \le \tau_{i+1}. \tag{1.7}$$

The main result of this article is the following.

Let $\Sigma\epsilon_n^2 < \infty$. Then S^{ϵ_n} converges uniformly in $t \in [0, T]$ a.s. for every T. Further, define $Z^n \equiv Y^{\epsilon_n}$. Then Z_t^n also converges uniformly in $t \in [0, T]$ to Z_t a.s. for every T as well as for any locally bounded predictable process f, $\int_0^t f dZ^n \to \int_0^t f dZ$ uniformly in $t \in [0, T]$ a.s. for every T.

Let us note that the w-path $Y^\epsilon(t, w)$ is defined explicitly as a functional of the w-paths $H(t, w), X(t, w)$ and $b(t, w, \rho)$. Thus $Z^n \equiv Y^{\epsilon_n}$ is defined 'pathwise' and hence so is Z, as Z^n converges a.s. to Z.

Also, note that we need to evaluate $b(t, w, \rho)$ only for piecewise constant fucntions ρ. This could be important, say when

$$b(t, w, \rho) = \tilde{b}(t, \rho_t, \int_0^t \rho_s dU_s(w)) \tag{1.8}$$

for a Lipschitz function $\tilde{b}$ and an increasing process U.

Another pathwise formula for solution Z of (1.2) was obtained in [4], and it was also shown there that Euler-Peano method yields a.s. approximation.

However, unless b satisfies an additional condition, it does not yield a pathwise formula. The functional b given by (1.8) does not satisfy this additional condition.

The main tool in proving the result stated above is the notion of 'dominating process' which is a modification of Metivier-Pellaumail's notion of 'control process'. This was introduced in [4] and it was shown that along with the Metivier-Pellaumail inequality, it is a very effective tool for studying approximation questions in stochastic analysis. It is a tool for establishing convergence in Emery topology as well as a.s. convergence. In section 2, we will discuss 'dominating processes'. The proof of the main theorem is given in section 3.

All processes we consider are defined on a fixed complete probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a filtration $(\mathcal{F}_t)$ assumed to satisfy usual conditions. The notions of predictable, stoptime, martingale etc. will be with reference to this filtration.

$\mathcal{V}^+$ will denote the class of r.c.l.l. increasing (V_t) with $V_0 \geq 0$. Also,

$$\mathcal{V} = \{V = V_1 - V_2;\ \ V_1, V_2 \in \mathcal{V}^+\}.$$

For $U \in \mathcal{V}, |U|_t(w)$ will denote total variation of $s \to U_s(w)$ on $[0,t]$. Note that $|U| \in \mathcal{V}$.

$\mathcal{M}^2_{loc}$ will denote the class of locally square integrable martingales. For $M \in \mathcal{M}^2_{loc}$, $[M,M]$, $< M, M >$ will respectively denote the quadratic variation process and predictable quadratic variation process of M.

$\mathcal{I}$ will denote the class of predictable process f that are locally bounded.

The following is a consequence of Metivier-Pellaumail inequality [5].

Let $M \in \mathcal{M}^2_{loc}$ and τ be a stop time. Then

$$E|M|^{*2}_{\tau-} \leq 4E\{[M,M]_{\tau-} + < M, M >_{\tau-}\}. \tag{1.9}$$

2 Dominating process of a semimartingale

Definition :*Let X be a semimartingale. A process $V \in \mathcal{V}^+$ is said to be a dominating process of X, written as $X \lll V$, if for some decomposition*

$$X = M + A, M \in \mathcal{M}^2_{loc}, A \in \mathcal{V} \tag{2.1}$$

of X, V^1 defined by

$$V^1_t := V_t - 2(< M, M >_t + [M,M]_t)^{1/2} - |A|_t$$

is an increasing process.

Recall that every semimartingale X admits a decomposition as in (2.1). Hence, every semimartingale admits a dominating process V. One can take

$$V_t := 2(< M, M >_t + [M,M]_t)^{1/2} + |A|_t.$$

where M, A are as in (2.1). Also, given finitely many semimartingal $X^1, ..., X^d$, one can choose a common dominating process : $V = V^1 + \cdots + V^d$, where $X^i \lll V^i$.

The following is an easy consequence of (1.9). Let X be a semimartingale and let $X \lll V$. Then for all stop times τ,

$$E|X|^{*2}_{\tau-} \leq 2EV^2_{\tau-} \tag{2.2}$$

The following lemma can be proved easily. (See [4]).

Lemma 2.1 *(a) Let X, Y be semimartingales and let $X \lll U, Y \lll V$. Let $Z = X + Y$. Then*

$$\exists\, W \text{ such that } Z \lll W \text{ and } W_t \leq U_t + V_t\ \forall\, t. \tag{2.3}$$

(b). Let $f \in \mathcal{I}$ and $X \lll U$ as above. Let

$$\theta_t(f,U) := \sqrt{2}\{(\int_0^t |f|^2 dU^2)^{1/2} + \int_0^t |f| dU\}. \tag{2.4}$$

Then

$$\exists\, D \text{ such that } (\int f dX) \lll D \text{ and } D_t \leq \theta_t(f,U). \tag{2.5}$$

The following notions of convergence play an important role in a.s. convergence results.

For processes f^n, f say that $f^n \xrightarrow{o} f$ if

$$\sum_{n=1}^{\infty} |f^n - f|_t^{*2} < \infty \quad \forall t \quad \text{a.s.}$$

For semimartingales X^n, X, say that $X^n \xrightarrow{*} X$ if

$$\exists\, V^n : (X^n - X) \lll V^n \text{ and } \sum_{n=1}^{\infty} (V_t^n)^2 < \infty \ \forall t \text{ a.s..}$$

It is clear that $f^n \xrightarrow{o} f$ implies that $|f^n - f|_t^* \to 0$ a.s. for every t.

The following properties one proved in [4]. Here, X^n, X are semimartingles and $f^n, f \in \mathcal{I}$.

$$X^n \xrightarrow{*} X \quad \text{implies} \quad X^n \xrightarrow{o} X \tag{2.6}$$

$$f^n \xrightarrow{o} f, X^n \xrightarrow{*} X \quad \text{implies} \quad \int f^n dX^n \xrightarrow{*} \int f dX. \tag{2.7}$$

It is proved in [4] that semimartingales X^n converge to X in Emery topology (see [2]) if and only $\exists\, V^n : (X^n - X) \lll V^n$ with $V_t^n \to 0$ in probability for every t. Thus $X^n \xrightarrow{*} X$ implies $X^n \to X$ in Emery toplogy. Moreover, $Y^n \to Y$ in Emery topology implies that for a suitable subsequence $X^k = Y^{n_k}$, one has $X^k \xrightarrow{*} Y$. This enables one to prove results on Emery topology via $\xrightarrow{*}$.

Using (2.1) and (2.4), one gets the following. Let $f \in \mathcal{I}, X \lll V$ and τ be a stop time. Then

$$E|\int f dX|_{\tau-}^{*2} \leq 2E\theta_{\tau-}^2(f,V) \tag{2.8}$$

Further,

$$E|\int f dX|_{\tau-}^{*2} \leq 4E(1+V_{\tau-}) \int_0^{\tau-} |f|^2 d(V^2 + V) \tag{2.9}$$

If $X = (X^1, \cdots, X^d)'$ is an $\mathbb{R}^d$ valued semimartingale, $X^j \lll V$, and $f = (f^{ij})$, where $f^{ij} \in \mathcal{I}, 1 \leq i \leq m, 1 \leq j \leq d, Y = \int f dX$ is defied by $Y = (Y^1, \cdots, Y^m)'$ and

$$Y^i = \sum_{j=1}^{d} \int f^{ij} dX^j.$$

Now

$$\begin{aligned} E|\int f dX|_{\sigma-}^{*2} &= \sum_{i=1}^{m} E|\sum_{j=1}^{d} \int f^{ij} dX^j|_{\sigma-}^{*2} \\ &\leq 2d \sum_{i=1}^{m} \sum_{j=1}^{d} E\theta_{\sigma-}^2(f^{ij}, V) \\ &\leq 2d^2 m E\theta_{\sigma-}^2(|f|, V). \end{aligned}$$

where $|f|^2 = \sum_{ij} |f^{ij}|^2$. One also can use the bound $\theta_t(f, V) \le 3|f|_t^* V_t$ to get

$$\begin{aligned} E|\int f dX|_{\sigma-}^{*2} &\le 2d \sum_{i=1}^{m} \sum_{j=1}^{d} 9\, E|f^{ij}|_{\sigma-}^{*2} V_{\sigma-}^2 \\ &\le 18dE|f|_{\sigma-}^2 V_{\sigma-}^2. \end{aligned} \tag{2.10}$$

Or one can use (2.9) to get

$$E|\int f dX|_{\sigma-}^{*2} \le 4dE(1 + V_{\sigma-}) \int_0^{\sigma-} |f|^2 d(V^2 + V). \tag{2.11}$$

3 The main result

We need to assume that (1.2) holds for an adapted process A and that for each $\rho \in D([0,\infty), \mathbb{R}^n)$, $b(s, w, \rho)$ is an adapted, r.c.l.l. process. Thus, for an r.c.l.l. adapted process Z,

$$F(Z)_t := b(t, w, Z(w)) \tag{3.1}$$

is itself an r.c.l.l. adapted process.

It is easy to see that if $\tau_i < \infty$ then $\tau_i < \tau_{i+1}$. Let $\tau_\infty = \lim_i \tau_i$.

LEMMA 3.1 $\tau_\infty = \infty$ *a.s.*

PROOF : For an r.c.l.l. process B, define

$$(JB)_t := \sum_{i=0}^{\infty} B_{\tau_i} 1_{[\tau_i, \tau_{i+1})}(t). \tag{3.2}$$

Note that for $i \ge j$, $W^i_{\tau_j} = Y_{\tau_j}$ and hence

$$(JY)_t = (JW^i)_t = W^i_t \quad \text{for} \quad t < \tau_{i+1}. \tag{3.3}$$

Thus we have for $\tau_i < t \le \tau_{i+1}$

$$Y_t = Y_{\tau_i} + H_t - H_{\tau_i} + b(\tau_i, \cdot, JY)(X_t - X_{\tau_i}).$$

or in other words,

$$Y_{t\wedge\tau_k} = H_{t\wedge\tau_k} + \int_0^{t\wedge\tau_k} JF(JY)_- dX. \tag{3.4}$$

Moreover, by choice of $\{\tau_j\}$,

$$|JY - Y| \le \varepsilon \tag{3.5}$$

$$|J(F(JY)) - F(JY)| \le \varepsilon. \tag{3.6}$$

and on $\tau_{i+1} < \infty$

$$\text{either } |F(JY)_{\tau_{i+1}} - F(JY)| \ge \varepsilon \text{ or } |Y_{\tau_{i+1}} - Y_{\tau_i}| \ge \varepsilon. \tag{3.7}$$

Consider the equation

$$\tilde{Y}_t = H_t + \int_0^t JF(J\tilde{Y})_- dX. \tag{3.8}$$

Writing $G(B) = JF(JB)$, one sees that (3.8) admits a unique solution (which is r.c.l.l.). By (local) uniqueness of solution to (3.8), it follows that $\tilde{Y}_t = Y_t$ on $t \le \tau_k$, i.e.

$$P(\tilde{Y}_{t\wedge\tau_k} = Y_{t\wedge\tau_k}\ \forall\, t) = 1.$$

On the set $\tau_\infty < \infty$, at least one of the two limits $\lim\ F(JY)_{\tau_i}$ and $\lim Y_{\tau_i}$ does not exist (because of (3.7)), but both $\lim \tilde{Y}_{\tau_i}$ and $\lim F(J\tilde{Y})_{\tau_i}$ exist as, $F(J\tilde{Y})$ and $\tilde{Y}$ are r.c.l.l. Thus

$$\{\tau_\infty < \infty\} \subseteq \{\tilde{Y}_{t\wedge\tau_i} = Y_{t\wedge\tau_i}\ \forall\, t, \forall\, i\}^c$$

Hence $P(\tau_\infty < \infty) = 0$. ■

Thus it follows that Y is defined on $[0,\infty)$ and

$$Y_t = H_t + \int_0^t JF(JY)_- dX. \tag{3.9}$$

Lemma 3.2 *Let V be a dominating process for $X^i, 1 \le i \le d$. Let*

$$\tau_j = \inf\{t > 0 : A_t \ge j \quad or \quad V_t \ge j\}.$$

Then $\exists$ a constant C_j, depending only on j (and d) such that

$$E|Y - Z|^{*2}_{\tau_j -} \le C_j \varepsilon^2$$

where Z is the solution to (1.1) (and $Y \equiv Y^\varepsilon$).

Proof : From (3.5) and the Lipschitz condition (1.2), it follows that

$$\begin{aligned} |F(JY) - F(Y)|^*_{\tau_j-} &\le A_{\tau_j-}|JY - Y|^*_{\tau_j-} \\ &\le j\varepsilon \end{aligned}$$

Along with (3.6), this gives

$$\begin{aligned} |JF(JY) - F(Y)|^*_{\tau_j-} &\le |JF(JY) - F(JY)|^*_{\tau_j-} + |F(JY) - F(X)|^*_{\tau_j-} \\ &\le \varepsilon + j\varepsilon \\ &= (j+1)\varepsilon. \end{aligned} \tag{3.10}$$

From (3.9) it follows that

$$\begin{aligned} Z_t - Y_t &= \int_0^t JF(JY)_- dX - \int_0^t F(Z)_- dX \\ &= \int_0^t JF(JY_- dX - \int_0^t F(Y)_- dX \\ &\quad + \int_0^t F(Y)_- dX - \int_0^t F(Z)_- dX \end{aligned}$$

Thus for any $\sigma \le \tau_j$

$$\begin{aligned} E|Z-Y|^{*2}_{\sigma-} &\le 2E|\int [JF(JY)-F(Y)]_{-}dX|^{*2}_{\sigma-} \\ &\quad +2E|\int [F(Y)-F(Z)]_{-}dX|^{*2}_{\sigma-} \\ &\le 36d(j+1)^2\varepsilon^2\cdot j^2 + 8d(1+j)E\int_0^{\sigma-}|Z-Y|^{*2}_{-}dU \end{aligned}$$

where $U = V^2 + V$. Thus for constants K_1, K_2 (depending only on j, d), we get

$$E|Z-Y|^{*2}_{\sigma-} \le K_1\varepsilon^2 + K_2 E\int_0^{\sigma-}|Z-Y|^{*2}_{-}dU \tag{3.11}$$

for all $\sigma \le \tau_j-$. Since $U_{\tau_j-} \le j^2 + j$, using an analogue of Gronwalls lemma (Lemma 29.1 in [4]) we get that for a constant C_j depending on j, K_1, K_2 and hence on j, d only,

$$E|Z-Y|^{*2}_{\tau_j-} \le C_j\varepsilon^2$$

(Here C_j can be explicitly evaluated). ∎

We are now in a position to prove the main result.

For a sequence $\{\varepsilon_n\}$, let $Z^n = Y^{\varepsilon_n}$, where Y^ε is defined in the introduction. Our main result is

Theorem 3.3 *Suppose $\Sigma\varepsilon_n^2 < \infty$. Then*

$$|Z^n - Z|^*_t \to 0 \quad a.s. \quad for\ all\ t \tag{3.12}$$

Further, for any $f \in \mathcal{I}$,

$$|\int f dZ^n - \int f dZ|^*_t \to 0 \quad a.s. \quad for\ all\ t. \tag{3.13}$$

Proof: By Lemma 3.2,

$$E|Z^n - Z|^{*2}_{\tau_j-} \le C_j\varepsilon_n^2.$$

Thus

$$E\sum_{n=1}^{\infty}|Z^n - Z|^{*2}_{\tau_j-} \le C_j\sum_{n=1}^{\infty}\varepsilon_n^2 < \infty.$$

Hence

$$\sum_{n=1}^{\infty}|Z^n - Z|^{*2}_{\tau_j} < \infty \text{ a.s..}$$

This implies $Z^n \xrightarrow{o} Z$ and in turn (3.12). For (3.13), let us write $\tilde{J}^n \equiv J^{\varepsilon_n}$; where we write J^ε for J defined earlier. Then

$$Z^n - Z = \int [\tilde{J}^n F(\tilde{J}^n Z^n) - F(Z^n)]_{-}dX + \int [F(Z^n) - F(Z)]_{-}dX. \tag{3.14}$$

Now (3.10) implies

$$\tilde{J}^n F(\tilde{J}^n Z^n) - F(Z^n) \xrightarrow{o} 0$$

and $Z^n \xrightarrow{o} Z$ and (1.3) implies

$$F(Z^n) - F(Z) \xrightarrow{o} 0.$$

Now (3.12) and (2.7) implies that $Z^n \xrightarrow{*} Z$. Then using (2.6) once again, we get $\int f dZ^n \xrightarrow{o} \int f dZ$ and hence (3.12). ∎

Remark:Since $|S^{\varepsilon_n} - Y^{\varepsilon_n}| \leq \varepsilon_n$, it follows That $S_t^{\varepsilon_n}$ converges to Z_t uniformly in $t \in [0,T]$ a.s. and this gives approximation of the solution by step processes.

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