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# ON A CONJECTURE OF F.B. KNIGHT
# TWO CHARACTERIZATION RESULTS RELATED TO PREDICTION PROCESSES

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ABSTRACT. In this paper, we answer a question raised by F.B. Knight on the characterization of processes whose prediction process is of pure jump type. Another characterization result corresponding to a conjecture of Knight is also obtained.

**1. Introduction.** In [2], F.B. Knight posed the question : for which cadlag processes (or, in his set-up, for which probabilities on the space of cadlag paths), is the prediction process a pure-jump process ? He also put forth a conjecture as a possible answer. In this paper, we completely characterize processes for which the prediction processes evolves purely through jumps (that do not accumulate in finite time). The necessary and sufficient condition that we get turns out to be a little more restrictive than that conjectured by F.B. Knight. However, in the course of our analysis, we show that processes of the type suggested by Knight can also be characterized by a property of their prediction process, slightly less obvious than the pure-jump property.

**2. Preliminaries.** We start with a summary of the notion of prediction process, as introduced by F.B. Knight. All this is adapted from [2].

Let $\Omega$ denote the space of all $\overline{\mathbb{R}}$-valued right-continuous functions on $[0,\infty[$ with left-limits on $(0,\infty)$, and let $\{X_t, t \geq 0\}$ be the coordinate process defined on $\Omega$ by $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \Omega$. As usual, let $\mathcal{F}^0 = \sigma\{X_t, t \geq 0\}$, and, for $t \geq 0$, $\mathcal{F}_t^0 = \sigma\{X_s, s \leq t\}$, $\mathcal{F}_{t+}^0 = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^0$. Also, let $\theta_t$, $t \geq 0$ denote the usual shift operators defined as $\theta_t\omega(\cdot) = \omega(t+\cdot)$, $\omega \in \Omega$, $t \geq 0$. $\mathbb{P}$ will denote the space of all probabilities on $(\Omega, \mathcal{F}^0)$, and for $P \in \mathbb{P}$, $\mathcal{F}^P$ is the $P$-completion of $\mathcal{F}^0$. For $t \geq 0$, $\mathcal{F}_t^P$, (*resp.* $\mathcal{F}_{t+}^P$) is the augmentation of $\mathcal{F}_t^0$ (*resp.* $\mathcal{F}_{t+}^0$) by $P$-null sets in $\mathcal{F}^P$.

Let $\rho$ denote a homeomorphic mapping between $\overline{\mathbb{R}}$ and $[0,1]$. Then the topology on $\Omega$ generated by the functions $\int_0^t \rho(\omega(s))\,ds$ $(t \geq 0)$ makes it a metrizable Luzin space with $\mathcal{F}^0$ as its Borel $\sigma$-field, and $\mathbb{P}$, equipped with the topology of weak convergence, also becomes a metrizable Luzin space with its Borel $\sigma$-field $\mathcal{P}$ coinciding with that generated by all mappings $P \longmapsto P(S)$, $S \in \mathcal{F}^0$.

F.B. Knight has shown that, for each $P \in \mathbb{P}$, there is a process $Z_t^P$, $t \geq 0$ on $(\Omega, \mathcal{F}^0, P)$, taking values in $(\mathbb{P}, \mathcal{P})$ with the following properties :

(i) *For any* $P \in \mathbb{P}$, *any* $(\mathcal{F}_{t+}^P)$*-stopping time* $T$ *and any* $S \in \mathcal{F}^0$,

$$Z_t^P(S) = P\,[\theta_T^{-1}S \mid \mathcal{F}_{T+}^P] \quad P\text{-a.s. on } \{T < \infty\} \quad .$$

(ii) *For any* $P \in \mathbb{P}$, *the trajectory* $t \longmapsto Z_t^P(\omega)$ *is cadlag (in* $\mathbb{P}$*) for* $P$*-a.e.* $\omega$.

*(iii) All the processes* $\{Z_t^P,\ t \geq 0\}$, $P \in \mathbb{P}$, *are homogeneous strong Markov processes with* the same *transition function* $q$ *defined on* $[0,\infty[\times\mathbb{P}\times\mathcal{P}$ *by*

$$q(t,P,A) = P[\omega : Z_t^P \in A], \quad t \geq 0,\ P \in \mathbb{P},\ A \in \mathcal{P}\ .$$

*More explicitly, for every* $(\mathcal{F}_{t+}^P)$*-stopping time* $T$

$$P[\,Z_{T+t}^P \in A \,|\, \mathcal{F}_{T+}^P] = q(T, Z_T^P, A) \quad P\text{-a.s. on } \{T < \infty\} \quad .$$

*(iv) For any* $P \in \mathbb{P}$, *for* $P$*-a.e.* $\omega$, *for all* $t \geq 0$, *the distribution of* $X_0$ *under the law* $Z_t^P(\omega)$ *is the unit mass at the point* $X_t(\omega)$.

**3. Step processes with exponential waiting.** To begin with, let $\Omega_s$ (for *step*) denote the set of all $\omega \in \Omega$ which are step functions, with jump times (assumed to be finite for simplicity) $0 = t_0 < \ldots < t_n \uparrow \infty$. It is quite easy to check that $\Omega_s \in \mathcal{F}^\circ$. Let $\mathbb{P}_0$ be the set of all laws $P \in \mathbb{P}$ carried by $\Omega_s$. Since $\Omega_s$ is stable under the shift, for any $P \in \mathbb{P}_0$ one has $P\{Z_t^P \in \mathbb{P}_0\, \forall t \geq 0\} = 1$.

The jump times are uniquely determined by the function $\omega$, and we denote by $T_n(\omega)$, $n \geq 0$ the successive jump times, by $S_n$ $(n \geq 1)$ the differences $T_n - T_{n-1}$ between jump times, and by $J_n$ $(n \geq 1)$ the jumps $X_{T_n} - X_{T_{n-1}}$. Clearly, any law $P \in \mathbb{P}_0$ is uniquely determined by the family $\mu_0^P, \nu_n^P, \mu_n^P$, where $\mu_0^P$ is the $P$-distribution of $X_0$, $\nu_n^P$ is a (regular) conditional distribution under $P$ of $S_n$ given $X_0$, $S_i$, $J_i$ $(i \leq n-1)$, and $\mu_n^P$ is a (regular) conditional distribution under $P$ of $J_n$ given $X_0$, $S_i$, $(i \leq n)$, $J_i$ $(i \leq n-1)$.

We are specially interested in the subclass $\mathbb{P}_1$ of those probabilities $P \in \mathbb{P}_0$ under which the conditional laws $\nu_n^P$ are *exponential*. In that case, we denote by $\lambda_n^P(X_0, S_1, \ldots, S_{n-1}, J_1, \ldots, J_{n-1})$ the parameter of this exponential law, which completely determines $\nu_n^P$. A moment's reflection tells us that $P[Z_t^P \in \mathbb{P}_1\, \forall t \geq 0] = 1$ for any $P \in \mathbb{P}_1$. Indeed we have

THEOREM 3.1. *Let* $P \in \mathbb{P}_1$. *Then* $P[Z_t^P \in \mathbb{P}_1\, \forall t \geq 0] = 1$. *Moreover, if we denote by* $Z$ *the measure* $Z_t^P(\omega)$ *to abbreviate notation, we have on* $\{T_{n-1}(\omega) \leq t < T_n(\omega)\}$

$$\mu_0^Z = \delta_{X_{T_{n-1}}(\omega)}$$

$$\lambda_k^Z(X_{T_{n-1}}, s_1, \ldots, s_{k-1}, j_1, \ldots, j_{k-1}) =$$
$$\lambda_{n+k-1}^P(X_0, S_1, \ldots, S_{n-1}, s_1 + t - T_{n-1}, s_2, \ldots, s_{k-1}, J_1, \ldots, J_{n-1}, j_1, \ldots, j_{k-1})$$
$$\mu_k^Z(X_{T_{n-1}}, s_1, \ldots, s_k, j_1, \ldots, j_{k-1}) =$$
$$= \mu_{n+k-1}^P(X_0, S_1, \ldots, S_{n-1}, s_1 + t - T_{n-1}, s_2, \ldots, s_k, J_1, \ldots, J_{n-1}, j_1, \ldots, j_{k-1})\ .$$

The proof of this theorem being an immediate consequence of the definition of prediction process and the so-called "memoryless" property of the exponential distribution, we do not write the proof here. Instead, we remark that for $P \in \mathbb{P}_1$, eventhough $\{Z_t^P,\ t \geq 0\}$ is $P$-a.s. continuous on each interval $[T_{n-1}, T_n[$, it does not necessarily remain constant on that interval. Therefore, contrary to the comments made on page 39 of [2], the prediction process isn't a pure–jump process for $P \in \mathbb{P}_1$. However, the processes $\mu_0^{Z_t^P}$ and $\lambda_1^{Z_t^P}$ do remain constant on each interval $[T_{n-1}, T_n[$, and in particular $\{Z_t^P\text{-law of } (X_0, S_1),\ t \geq 0\}$ is, for $P \in \mathbb{P}_1$, a pure–jump process with $\{T_n,\ n \geq 0\}$ as its successive times of jump.

We now proceed to show that the converse is also true, namely, that $\mathbb{P}_1$ is precisely the class of probabilities for which the prediction process has this property.

Following the notations already introduced, let us keep denoting, even for $P \in \mathbb{P}$, the $P$-distribution of $X_0$ by $\mu_0^P$ and put (to simplify notations) $\mu_t = \mu_0^{Z_t^P}$.

Let $P \in \mathbb{P}$ be fixed. Define $\tau_0 = 0$ and recursively $\tau_n = \inf\{t \geq \tau_{n-1} : \mu_t \neq \mu_{\tau_{n-1}}\}$. Clearly, $\{\tau_n\}$ is a non decreasing sequence of (possibly infinite) $(\mathcal{F}_{t+}^P)$-stopping times.

LEMMA 3.2. *If* $P\{0 < \tau_1 < \ldots, < \tau_n \uparrow \infty\} = 1$, *then* $P \in \mathbb{P}_0$, *and* $P\{T_n = \tau_n\} = 1$ *for all* $n$.

PROOF. By the basic property (iv) of section 2, the law of $X_0$ under $Z_t^P$ (which is constant in any interval $[\tau_{n-1}, \tau_n[$ ), is a unit mass at $X_t$. Now consider any real valued function $x(t)$, and the measure valued function $\varepsilon_{x(t)}$ : obviously if one of them is a step function so is the other, and they have the same jumps.

LEMMA 3.3. *If, besides the hypothesis of lemma 3.2., the prediction process also has the property that for all* $n$ *and all* $t \in [T_{n-1}, T_n[$

$$Z_t^P\text{-law of } S_1 = Z_{T_{n-1}}^P\text{-law of } S_1 \quad ,$$

*then, the* $P$*-conditional distributions of* $S_1$ *given* $X_0$ *and of* $S_k$ *given* $X_0, S_i, J_i$ $(1 \leq i \leq k-1)$, *must all be exponential.*

PROOF. Let $T$ be the stopping time $T_{k-1}$, and let $f$ denote any bounded $\mathcal{F}_{T+}^P$-measurable random variable. Then we have for $s, t > 0$

$$\begin{aligned}
E^P[f,\ S_k > s+t] &= E^P[f,\ S_k > t,\ S_1 \circ \theta_{T+t} > s] \\
&= E^P[f,\ S_k > t,\ Z_{T+t}^P\{S_1 > s\}] \\
\text{(by hypothesis)} \quad &= E^P[f,\ S_k > t,\ Z_T^P\{S_1 > s\}] \\
&= E^P[f,\ S_k > t,\ P[S_k > t \,|\, \mathcal{F}_{T+}^P]] \\
&= E^P[f,\ P[S_k > s \,|\, \mathcal{F}_{T+}^P]\, P[S_k > t \,|\, \mathcal{F}_{T+}^P]] \ .
\end{aligned}$$

Thus the multiplicative property of the exponential holds a.s. for given $s, t > 0$, and it is an easy matter to regularize the conditional distributions into true exponential laws. Note that if we hadn't assumed for simplicity that the number of jumps is infinite, a slightly awkward discussion of finiteness would be necessary, and the (random) parameter of the exponential laws could be $+\infty$.

Combining lemmas 3.2 and 3.3 and the remarks made before that, we get a characterization of the class $\mathbb{P}_1$ of jump processes with exponential waiting times, considered by Knight :

THEOREM 3.4. *Let $P \in \mathbb{P}$. Then the measure valued process* $\{Z_t^P\text{-law of }(X_0, S_1)\}$ $(t \geq 0)$ *is a pure-jump process (with no finite time accumulation of jumps) if and only if $P \in \mathbb{P}_1$. Moreover, in this case the successive jump times of this measure valued process are the same as those of $(X_t)$.*

**4. Step processes with pure-jump prediction.** In this section we formulate necessary and sufficient conditions on $P \in \mathbb{P}$ for its prediction process to be a pure-jump process.

We define $\mathbb{P}_2$ as the subclass of $\mathbb{P}_1$ consisting of the laws $P$ for which

$$\lambda_n^P(x_0, s_1, \dots, s_{n-1}, j_1, \dots, j_{n-1}) \quad \text{and} \quad \mu_n^P(x_0, s_1, \dots, s_n, j_1, \dots, j_{n-1}\,;\,\cdot)$$

*depend only on* $(x_0, j_1, \dots, j_{n-1})$. This means that we can completely describe the process $\{X_t,\ t \geq 0\}$ by giving ourselves the discrete process $\{X'_n = X_{T_n}\}$, of positions at the successive jumps, and for each discrete time $n$ the interval between jumps $S_n = \lambda_n e_n$, $(e_k)$ being a sequence of independent exponential r.v.'s of parameter 1, and $\lambda_n$ being a positive r.v. which depends only on the process $\{X'_k\}$ up to time $n-1$.

A close look at the formula given in theorem 3.1 shows that for $P \in \mathbb{P}_2$ $\{Z_t^P,\ t \geq 0\}$ is a pure jump process with $T_n$, $n \geq 1$ as its successive jump times. We will now show that the converse is also true, namely that if the prediction process for $P$ is of pure-jump type (without accumulation of jumps, as always), then $P$ must belong to $\mathbb{P}_2$.

We denote by $0 = \tau_0 < \dots < \tau_n \uparrow \infty$ the successive jumps of the prediction process (assumed to be in infinite number for simplicity). Note that the meaning of $\tau_n$ isn't the same here as in lemma 3.2 where the jump was that of the initial measure of $Z_t^P$. Since the weaker condition for theorem 3.4 is clearly satisfied, $P$ belongs to $\mathbb{P}_1 \subset \mathbb{P}_0$, and we may denote by $(T_n)$ the sequence of jumps of $\{X_t,\ t \geq 0\}$.

LEMMA 4.1. *We have $P$-a.s. $T_n = \tau_n$ for all $n$.*

PROOF. According to the argument in lemma 3.2, the prediction process jumps at each $T_n$. Thus the union of the graphs $[T_n]$ of the jump stopping times is contained in $\cup_n [\tau_n]$. On the other hand, since $\{Z_t^P,\ t \geq 0\}$ is a pure-jump strong Markov process, its successive jump times $\tau_n$ are all totally inaccessible $(\mathcal{F}_{t+}^P)$ stopping times. Now, $\{X_t,\ t \geq 0)$ is also of pure-jump type, and it is a well known fact (see [1] for example) that if $\tau$ is any totally inaccessible $(\mathcal{F}_{t+}^P)$-stopping time, then its graph $[\tau]$ is contained in the union of the graphs $[T_n]$ of the jump stopping times. Thus we have the inverse inclusion, and the lemma follows.

We may now answer F.B. Knight's question :

THEOREM 4.2. *For a probability $P$ on $(\Omega, \mathcal{F}^0)$ to have a pure-jump prediction process $\{Z_t^P,\ t \geq 0\}$ (with no finite accumulation of jumps), it is necessary and sufficient that $P$ belong to $\mathbb{P}_2$. Moreover, in this case, the successive jump times of the prediction process a.s. coincide under $P$ with those of the coordinate process.*

PROOF. The only point we have to prove is that the pure-jump property of the prediction process implies $P \in \mathbb{P}_2$. For each index $k$ we define the following $\sigma$-fields

$$\mathcal{J}_k \ \text{ generated by } \ X_0, J_1, \dots, J_k \quad ; \quad \mathcal{S}_k \ \text{ generated by } \ S_1 \dots, S_k \quad ;$$

For $k = 0$ we may replace $\mathcal{J}_0$ by $\mathcal{F}^0_{0+}$, and take for $\mathcal{S}_0$ the trivial $\sigma$-field. Note that $\mathcal{J}_k \vee \mathcal{S}_k = \mathcal{F}^0_{T_k+}$ up to sets of measure 0 under $P$. We consider also for $k \geq 1$ the $\sigma$-field

$$\mathcal{T}_k \quad \text{generated by} \quad J_k, S_{k+1}, J_{k+1} S_{k+2}, J_{k+2} \dots .$$

The theorem will follow if we prove that, for every $k \geq 1$, the $\sigma$-fields $\mathcal{T}_k$ and $\mathcal{S}_k$ are conditionally independent under $P$ given $\mathcal{J}_{k-1}$. For every $n \geq 1$ we denote by $W_n$ a random variable of the form

$$W_n = g(J_n, J_{n+1}, S_{n+1}, J_{n+2}, S_{n+2}, \dots) ,$$

where $g$ is a borel bounded function on $\mathbb{R}^{\mathbb{N}}$. Thus the property to be proved can be reduced to

$$(*) \qquad E^P[UV 1_{\{S_k > t\}} W_k] = E^P[U E^P[V 1_{\{S_k>t\}} \,|\, \mathcal{J}_{k-1}]\, E^P[W_k \,|\, \mathcal{J}_{k-1}]]$$

where $U$ and $V$ are two bounded random variables, measurable w.r.t. $\mathcal{J}_{k-1}$ and $\mathcal{S}_{k-1}$ respectively.

We begin with the case $k = 1$. Then $V$ can be omitted, and we may assume that $U$ is $\mathcal{F}^P_{0+}$-measurable. To prove

$$E^P[U 1_{\{S_1>t\}} W_1] = E^P[U P[S_1 > t \,|\, \mathcal{F}^P_{0+}]\, E^P[W_1 \,|\, \mathcal{F}^P_{0+}]] ,$$

we use the facts that $U 1_{\{S_1>t\}}$ is $\mathcal{F}^0_{t+}$-measurable and that, on the set $\{S_1 > t\}$, $W_1 \circ \theta_t = W_1$ :

$$\begin{aligned} E^P[U 1_{\{S_1>t\}} W_1] &= E^P[U 1_{\{S_1>t\}} E^{Z^P_t}[W_1]] \\ (\text{since } Z^P_t = Z^t_0 \text{ on } S_1 > t\,) \quad &= E^P[U 1_{\{S_1>t\}} E^{Z^P_0}[W_1]] \\ &= E^P[U 1_{\{S_1>t\}} E^P[W_1 \,|\, \mathcal{F}^P_{0+}]] \\ &= E^P[U P[S_1 > t \,|\, \mathcal{F}^P_{0+}]\, E^P[W_1 \,|\, \mathcal{F}^P_{0+}]] . \end{aligned}$$

We prove the general case by induction on $k$. The induction hypothesis implies that, for every bounded borel function $f$ on $\mathbb{R}$, $W_k f(J_{k-1})$ being $\mathcal{T}_{k-1}$-measurable,

$$E^P[W_k f(J_{k-1}) \,|\, \mathcal{J}_{k-2} \vee \mathcal{S}_{k-1}] = E^P[W_k f(J_{k-1}) \,|\, \mathcal{J}_{k-2}]$$

from which we deduce

$$E^P[W_k \,|\, \mathcal{J}_{k-1} \vee \mathcal{S}_{k-1}] = E^P[W_k \,|\, \mathcal{J}_{k-1}] .$$

After this remark, we proceed to the proof. Put $T = T_{k-1}$. Since $UV1_{\{S_k>t\}}$ is $\mathcal{F}^{\circ}_{(T+t)+}$-measurable and on $\{S_k > t\}$ we have $W_k = W_1 \circ \theta_{T+t}$, the l.h.s. of $(*)$ becomes

$$
\begin{aligned}
& = E^P[UV1_{\{S_k>t\}}E^{Z^P_{T+t}}[W_1]] \\
\text{since } Z^P_{T+t} = Z^P_T \text{ on } \{S_k > t\} \quad & = E^P[UV1_{\{S_k>t\}}E^{Z^P_T}[W_1]] \\
& = E^P[UV1_{\{S_k>t\}}E^P[W_k \,|\, \mathcal{J}_{k-1} \vee \mathcal{S}_{k-1}]] \\
\text{(remark above)} \quad & = E^P[UV1_{\{S_k>t\}}E^P[W_k \,|\, \mathcal{J}_{k-1}]] \\
& = E^P[U E^P[V 1_{S_k>t} \,|\, \mathcal{J}_{k-1}]\, E^P[W_k \,|\, \mathcal{J}_{k-1}]]
\end{aligned}
$$

which concludes the proof.

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