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On Semi-Martingales Associated with Crossings

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Introduction. Let \((X_t)_{t \geq 0}\) be a Brownian motion, \(X_0 = x\) almost surely, \(x < a < b\). Let \(\sigma_t\) be the last exit time of \(X\) before \(t\) from \((a,b)^c\), defined in sec. 1.1. We note that when \(b = \infty\), \(X_t - X_{\sigma_t} = (X_t - a)^+\) and by Tanaka's formula it follows that \(X_t - X_{\sigma_t}\) and hence \(X_{\sigma_t}\), are semi-martingales. It is easy to see from Theorem 1 of [6] that when \(b < \infty\), \(|X_t - X_{\sigma_t}|\) is a semi-martingale given by

\[(b-a)c(t) + |X_t - X_{\sigma_t}| = \int_0^t I_{(a,b)}(X_s)\Theta(s) \, dX_s + \frac{1}{2}(L(t,a)+L(t,b))\]

where \(c(t)\) is the numbers of crossings of \((a,b)\) in time \(t\), \(\Theta(s,w)\) is 1 during an upcrossing and -1 during a downcrossing and \(L(t,\cdot)\) is the local time of \(X\).

In the case of a continuous semi-martingale \((X_t, \mathcal{F}_t)\), where \(\mathcal{F}_t\) is the underlying filtration and \(\sigma_t\) as above, it is an immediate consequence of Tanaka's formula that \((X_{\sigma_t}, \mathcal{F}_t), (X_t - X_{\sigma_t}, \mathcal{F}_t)\) are semi-martingales (Theorem 2.1). In this case, time changing by \(\sigma_t\) does not change the underlying filtration. In this paper, as our main result we determine the martingale and bounded variation parts of \(|X_t - X_{\sigma_t}|\) (Theorem 4.1). In sec. 5, we state a few applications of this result. These include Levy's crossing theorem, an asymptotic relationship between local times and crossings of Brownian motion and a probabilistic approximation of the remainder term in the 2nd order Taylor expansion of a function.
1. Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((\mathcal{F}_t)_{t \geq 0}\) a filtration on it satisfying usual conditions. For a continuous adopted process \((X_t)_{t \geq 0}\) and \(a < b\), the upcrossing intervals \((\sigma_{2k}, \sigma_{2k+1}], k = 0,1,2,\ldots\), are defined by \(\sigma_0 = \inf \{s > 0 : X_s \leq a\}\), \(\sigma_{2k} = \inf \{s > \sigma_{2k-1}, X_s \leq a\}\) and \(\sigma_{2k+1} = \inf \{s > \sigma_{2k} : X_s \geq b\}\). As usual the infimum over the empty set is infinity. The downcrossing intervals \((\tau_{2k}, \tau_{2k+1}], k = 0,1,\ldots\) are similarly defined. Let \(\Theta(s) = \sum_{k=0}^{\infty} I(\sigma_{2k}, \sigma_{2k+1}] (s)\); \(\Theta^d(s) = \sum_{k=0}^{\infty} I(\tau_{2k}, \tau_{2k+1}] (s)\) and \(\Theta(s) = \Theta^u - \Theta^d\). The number of upcrossings in time \(t\), denoted by \(U(t)\) is defined as \(U(t) = \max \{k : \sigma_{2k+1} \leq t\}\). The number of downcrossings is similarly defined. \(C(t) = U(t) + D(t)\) is the total number of crossings. Let \(\tau = \inf \{s > 0 : X_s \notin (a,b)\}\).

Let \(\sigma_t = \max \{s < t : X_s \in (a,b)\} \) for \(t > \tau\) and \(\sigma_t = t - \tau\) for \(t \leq \tau\). \(\sigma_t\) is in general not a stop time, but is however \(\mathcal{F}_t\) measurable. Consequently \(X_{\sigma_t}\) is \(\mathcal{F}_t\) measurable.

2. The Semi-Martingale \(X_t - X_{\sigma_t}\)

From now on we fix a continuous \(\mathcal{F}_t\) - semi-martingale \(X_t = X_0 + M_t + V_t\) and \(a < b\). Let \(L(t,x,w)\) be a jointly \((t,x,w)\) measurable version of the local time of \(X\) which is continuous in \(t\) and right continuous in \(x\). For the existence of such versions see [10]. Let \(Y_t = X_t - X_{\sigma_t}\) and \(Z_t = X_{\sigma_t}\).

Theorem 2.1. The process \(Y_t\) is an \(\mathcal{F}_t\) semi-martingale and we have

\[
Y_t = \int_{\tau_t}^t I(a,b] (X_s) dX_s + \frac{1}{2} (L(t,a) - L(t,b)) - (b-a)(U(t)-D(t)) \quad (1)
\]
Proof. The proof is immediate from Tanaka's formula and the following pathwise identity:

\[(X_t - X_0) + (b-a)(U(t)-D(t)) + (X_t - X_{\sigma_t})\]

\[= (X_t - a)^+ - (X_0 - a)^+ - (X_t - b)^+ + (X_0 - b)^+ \quad (2)\]

Remarks.

2.2 It is immediate from Theorem 2.1 that \(X_{\sigma_t}\) is also a semi-martingale whose components can be got by subtracting \((X_t)\) from both sides of eqn. (1).

2.3 The sum of the jumps of \(Y\) in time \(t\) is precisely \((b-a)(D(t)-U(t))\). Since \(|U(t) - D(t)| \leq 1\) this implies that \(Y\) (and hence \(Z\)) is a special semimartingale. Further the representation (1) of \(Y_t\) is unique (see [9]). The jump times of these processes are precisely the times of crossings of \((a,b)\) by \(X\) and \(|\Delta Y_s| = b-a\) or 0.

2.4 Equation (2) and hence Theorem 2.1 are still valid for a semi-martingale \((X_t)\) with \(\sum_{s \leq t} |\Delta X_s| < \infty \quad \forall \quad t\), almost surely.

Now \((b-a)(U(t)-D(t))\) is replaced by \(- \sum_{s \leq t} \Delta Y_s\) and \((X_t - a)^+, (X_t - b)^+\) are replaced by \((X_t - a)^+ - \sum_{s \leq t} \Delta (X_s - a)^+, (X_t - b)^+ - \sum_{s \leq t} \Delta (X_s - b)^+\) respectively.

3. Local times of \(X_t - X_{\sigma_t}\)

We now determine the local times of \(Y\) in terms of that of \(X\). We note that the process lives in \([0,b-a)\) during an upcrossing of \((a,b)\) and in \((- (b-a),0]\) during a downcrossing. Also \(Y_t = 0\) whenever \(X_t = a\) or \(b\). Let \(I(t,x)\) denote the local time of the \(Y\) process.
Lemma 3.1

(i) For $x \in [0, b-a)$,
\[
(Y_t-x)^+ = \int_0^t \mathbb{I}_{(a,b]}(X_s) \mathbb{I}_{(x,\infty)}(Y_{s-}) d\tau_s - (b-a-x)U(t) + \frac{1}{2}I(t,x) \tag{3}
\]

(ii) For $x \in (-b-a, 0]$,
\[
(Y_t-x)^- = -\int_0^t \mathbb{I}_{(a,b]}(X_s) \mathbb{I}_{(-\infty,x]}(Y_{s-}) d\tau_s + (b-a+x)D(t) + \frac{1}{2}I(t,x)
\]
\[
+ \frac{1}{2} \left( \int_0^t \mathbb{I}_{(\infty,x]}(Y_{s-}) L(ds,b) - \int_0^t \mathbb{I}_{(-\infty,x]}(Y_{s-}) L(ds,a) \right) \tag{4}
\]

Remark 3.2 Observe that in case $x < 0$, the 2nd term on the RHS of (4) is zero whereas when $x = 0$ it is $\frac{1}{2} (L(t,b)-L(t,a))$.

Proof. Tanaka's formula (see [4]) applied to $Y$ at the point $x \in [0,b-a)$ gives
\[
(Y_t-x)^+ = (Y_0-x)^+ + \int_0^t \mathbb{I}_{(x,\infty)}(Y_{s-}) d\tau_s
\]
\[
+ \sum_{0<s\leq t} \mathbb{I}_{(x,\infty)}(Y_{s-})(Y_{s-}-x)^-
\]
\[
+ \sum_{0<s\leq t} \mathbb{I}_{(-\infty,x]}(Y_{s-})(Y_{s-})^+ + \frac{1}{2}I(t,x)
\]
\[
= I_0 + I_1 + I_2 + I_3 + \frac{1}{2}I(t,x).
\]

Since $Y_0 = 0$, $I_0 = 0$. Using eqn. (2) for $Y_t$ and noting that the measures $L(ds,a)$, $L(ds,b)$, $D(ds)$ have no support on the set $s: Y_{s-} > x$ we get
\[
I_1(t) = \int_0^t \mathbb{I}_{(a,b]}(X_s) \mathbb{I}_{(x,\infty)}(Y_{s-}) d\tau_s - (b-a-x)U(t) + \frac{1}{2}I(t,x).
\]

Since the jumps of $Y$ occur at the crossing times $\sigma_{2k+1}$, $\tau_{2k+1}$ it is easy to see that almost surely for $x \in [0,b-a)$, $I_2(t) = x U(t)$, $I_3(t) = 0$. This proves the first part of the lemma. The proof of (4) is similar using the Tanaka formula for $(Y_t-x)^-$. 
The following theorem gives $I$ in terms of $L$.

**Theorem 3.3**

(i) For $x \in (0, b-a)$, almost surely,
\[
I(t, x) = \int_0^t \theta^u(s)L(ds, a+x)
\]  
(5)

(ii) For $x \in (-b+a, 0)$, almost surely,
\[
I(t, x) = \int_0^t \theta^d(s)L(ds, b+x)
\]  
(6)

(iii) For $x = 0$, almost surely,
\[
I(t, 0) = L(t, a)
\]  
(7)

**Proof.**

(i) Let $x \in (0, b-a)$. Fix $k \geq 0$. Let $Y_1(t) = (Y_t - x)^+$ and $Y_2(t) = (X_t - (a+x))^+$. We note that, \( \forall t \in (\sigma_{2k}, \sigma_{2k+1}] \)
\[
\int_0^t I(s) dY_1(s) = \int_0^t I(s) dY_2(s)
\]  
(8)

By Tanaka's formula,
\[
\int_0^t I(s) dY_2(s) = \int_0^t I(s) (X_s)_I (x, \infty) (Y_s) dX_s
\]  
(9)

By eqn. (3), \( \forall t \in (\sigma_{2k}, \sigma_{2k+1}] \)
\[
\int_0^t I(s) dY_1(s) = \int_0^t I(s) (X_s)_I (x, \infty) (Y_s) dX_s
\]  
(10)

By eqn. (8) now implies that \( \forall t \geq 0 \),
\[
I(t \wedge \sigma_{2k+1}, x) - I(t \wedge \sigma_{2k}, x) = L(t \wedge \sigma_{2k+1}, a+x) - L(t \wedge \sigma_{2k}, a+x)
\]  
since $I(ds, x)$ is supported on the upcrossing intervals, the proof of (i) is complete.

(ii) Let $x \in (-b-a, 0)$. Then $Y_1(t) = (Y_t - x)^-$ and
\(Y_2(t) = (X_t-(b+x))^+\) agree on the downcrossing intervals.

Applying Tanaka's formula for \(Y_1\) and eqn. (4) to \(Y_2\) the proof is completed as in (i) above.

(iii) Let \(x = 0\). Proceeding as in case (i) we show that

\[
\int_0^t \theta^U(s) I(ds,0) = \int_0^t \theta^U(s) L(ds,a) = L(t,a).
\]

To complete the proof we show that \(\int_0^t \theta^d(s) I(ds,0) = 0\).

To see this we fix \(k\) and as in case (ii), compare the expressions for \((X_t-b)^-\) and \((Y_t)^-\) for \(t \in (\tau_{2k}, \tau_{2k+1})\) given by Tanaka's formula and eqn. (4) respectively. Using Remark 3.2 we see that

\[
L(t \wedge \tau_{2k+1}, b) - L(t \wedge \tau_{2k}, b) + (I(t \wedge \tau_{2k+1}, 0) - I(t \wedge \tau_{2k}, 0))
\]

whence \(I(t \wedge \tau_{2k+1}, 0) - I(t \wedge \tau_{2k}, 0) = 0\).

Remarks.

3.4 We recall from [10] that for the semi-martingale \((X_t)\) with \(-\sum_{s \leq t} \Delta X_s + X_t = X_0 + M_t + V_t\), where \(M\) and \(V\) are the continuous martingale and bounded variation parts respectively, the jumps of the local time \(L(t,x)\) is given by the formula: almost surely,

\[
L(t,x) - L(t,x-) = \int_0^t \mathbf{1}_{\{X_s = x\}} dV_s \quad (9)
\]

Using (9) it is easy to see that for \(x \in (0, b-a)\), \(I(t,x)\) is continuous at \(x\) if \(L(t,.)\) is continuous at \(a+x\).

The case \(x \in (-b-a, 0)\) is similar. When \(x = 0\), it is easy to see that \(I(t,0-) = L(t,b-) \neq L(t,a)\).

3.5 Let \(\overline{I}(t,x)\) denote the local time process of \(Z_t = X_{\sigma_t}\). The martingale, bounded variation part and the jumps of \(Z_t\) are easily calculated from eqn. (1). By using Tanaka's formula it is easily verified that \(\overline{I}(t,x) = L(t,x)\),
4. The Semi-Martingale \(|X_t-X_0^t|\)

We now determine the continuous martingale and the continuous bounded variation parts of \(|X_t-X_0^t|\). We note that the sum of the jumps up to time \(t\) is \(-(b-a)c(t)\).

**Theorem 4.1** For \(a < b\), we have almost surely,

\[
(b-a)c(t) + |X_t-X_0^t| = \int_0^t \Theta(s,w)I_{(a,b)}(X_s)dw + \frac{1}{2}(L(t,a)+L(t,b)-) \tag{10}
\]

**Proof.** Lemma 3.1 and Theorem 3.3 together give

\[
|X_t-X_0^t| = (X_t-X_0^t)^+ + (X_t-X_0^t)^-
\]

\[
= \int_0^t \Theta(s,w)I_{(a,b)}(X_s)dw - (b-a)c(t) + \frac{1}{2}\left(L(t,a)+L(t,b)-\right)
\]

where in the last equality we have used eqn. (9).

**Remark 4.2** We refer to [6] for an analogous result on crossings of closed intervals by a continuous martingale.

5. Applications

We now give some applications of the previous results. We mention only the results and refer the proofs to [5], [6] and [7].

Firstly we note that letting \(a \uparrow b\) in Theorem 4.1 eqn. (10) yields Levy's crossing theorem. We note that if \(\varepsilon_1 \leq \varepsilon \leq \varepsilon_2\) then \(\varepsilon_1 c_{\varepsilon_2}(t) \leq \varepsilon c_{\varepsilon}(t) \leq \varepsilon_2 c_{\varepsilon_1}(t)\) where
\( C_e(t) \) = number of crossings of \((b-e, b)\) in time \(t = C((b-e, b), t)\). Hence sufficient to let \(a \uparrow b\) along a sequence. This is done via the Borel-Cantelli lemma and an estimate due to Yor (Theorem 1, [10]). The following theorem (Levy's (down) crossing theorem) was first proved in the case of a continuous semi-martingale in El Karoui [3] where the discontinuous case is also discussed.

**Theorem 5.1** Let \((X_t)\) be a continuous semi-martingale. Then

(a) almost surely, \( \lim_{a \uparrow b} (b-a)C((a, b), t) = L(t, b-) \)

(b) If further \((X_t) \in H^p, p \geq 1\) then the above limits hold in \(H^p\).

Next let \((X_t)\) be a Brownian motion. We now state a result somewhat related to Theorem 5.1 above and whose proof can be found in [6], [7]. The crossing theorem say, that \((b-a)C(t) \sim L(t,a)\) as \(b \downarrow a\), the parameter \(t\) being fixed. It is an interesting fact that the same is true when we let \(t \to \infty\). We have the following theorem.

**Theorem 5.2** Let \((X_t)\) be a Brownian motion and \(a < b\). Then almost surely,

\[
\lim_{t \to \infty} \frac{L(t, a)}{C((a, b), t)} = \lim_{t \to \infty} \frac{E(L(t, a))}{E(C((a, b), t))} = (b-a)
\]

**Remark 5.3** The proof of the 2nd equality is immediate from Theorem 4.1 and Theorem 2.1

**Corollary 5.4** Let \(a < b, d < e\). Then almost surely,

\[
\lim_{t \to \infty} \frac{C((a, b), t)}{C((d, e), t)} = \lim_{t \to \infty} \frac{E(C((a, b), t))}{E(C((d, e), t))} = \frac{b-a}{e-d}
\]

We continue with a Brownian motion \((X_t)\). The following result gives the average sojourn time in \((a, b)\) per crossing.
Theorem 5.3 If \((X_t)\) is a Brownian motion and \(a < b\), then almost surely,

\[
\lim_{t \to \infty} \frac{\int_0^t I(a,b)(X_s)ds}{C((a,b),t)} = \lim_{t \to \infty} \frac{E \int_0^t I(a,b)(X_s)ds}{EC((a,b),t)} = (b-a)^2
\]

We refer to [8] for a proof of this result. The 2nd equality is an immediate consequence of Theorem 1, [5] which is also proved in [11]. We refer to [1] for a more general result in the context of Hunt processes and to [2] for related results involving recurrent diffusions. The following is a different generalization of Theorem 5.3 and can be thought of as a random approximation to the remainder term in a 2nd order Taylor expansion for a \(C^2\)-function. For the proof of this result see [6], [7].

Theorem 5.4 Let \((X_t)\) be a Brownian motion, \(a < b\), and \(f\) a \(C^2\)-function. Then almost surely,

\[
\lim_{t \to \infty} \frac{\int_0^t f''(|X_s-X_{\sigma_s}|)I(a,b)(X_s)ds}{C((a,b),t)} = \lim_{t \to \infty} \frac{E \int_0^t f''(|X_s-X_{\sigma_s}|)I(a,b)(X_s)ds}{EC((a,b),t)} = f(b-a)-f(0)-f'(0)(b-a).
\]

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References


