

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

MICHEL ÉMERY

On the Azéma martingales

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 66-87

http://www.numdam.org/item?id=SPS_1989__23__66_0

© Springer-Verlag, Berlin Heidelberg New York, 1989, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

On the Azéma Martingales

M. Emery^(*)

INTRODUCTION. Let $(X_t)_{t \geq 0}$ be a martingale such that $\langle X, X \rangle_t = t$. As shown by Meyer [5], the iterated integral

$$I_n(f) = \int_{0 < t_1 < \dots < t_n} f(t_1, \dots, t_n) dx_{t_1} \dots dx_{t_n}$$

can be defined for f square-integrable on the set $C_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n: 0 < t_1 < \dots < t_n\}$, and is an isometry from the Hilbert space $L^2(C_n)$ to its image $\chi_n = I_n(L^2(C_n))$. This subspace χ_n of $L^2(\Omega)$ is called the n^{th} chaos generated by X ; for $n = 0$, by convention, $L^2(C_0) = \mathbb{R}$, χ_0 is the space of constant random variables and I_0 is the obvious identification. These chaoses $(\chi_n)_{n \geq 0}$ are mutually orthogonal in $L^2(\Omega)$; hence they provide an orthogonal expansion for the random variables Z that belong to the Hilbert sum $\bigoplus_{n \geq 0} \chi_n$ (such random variables will be called chaos-decomposable). It may happen that this Hilbert space is the whole space L^2 ; such is the case, for instance, when X is a Brownian motion or a compensated Poisson process (and $\mathcal{F}$ the σ -field generated by X). When this happens, one says that X has the chaotic representation property. A problem raised by Meyer in [6] p. 262 is: Which martingales X have this property? We shall not answer this question, but merely add a few examples to the aforementioned ones.

Notice that this chaotic representation property is stronger than the predictable representation property, since every chaos-decomposable $Z \in L^2(\Omega)$ can always be written as $E[Z] + \int_0^\infty H_s dx_s$ for some predictable process H with $E \int_0^\infty H_s^2 ds < \infty$ (if $Z = I_n(f)$ for $n \geq 1$, just choose

$$H_t = \int_{0 < t_1 < \dots < t_{n-1} < t} f(t_1, \dots, t_{n-1}, t) dx_{t_1} \dots dx_{t_{n-1}};$$

this process has a predictable version, and any such version will do). Recall that, if the filtration $(\mathcal{F}_t)_{t \geq 0}$ we are working in is the one generated by X , then X has the predictable representation property iff its law is an extreme

(*) Part of this work was done in Strasbourg, during uncountable conversations with P.A. Meyer; part in Vancouver, while visiting U.B.C.

point of the set of all laws of martingales (see Jacod-Yor [4]).

Before starting, seeing what happens in the discrete-time case will be helpful: the time-axis $[0, \infty)$ is replaced by the finite set $\{0, 1, \dots, N\}$, we have a martingale $(X_n)_{0 \leq n \leq N}$, and, denoting by $Y_n = X_n - X_{n-1}$ its increments, the requirement $\langle X, X \rangle_t = t$ is naturally replaced by its discrete analogue $E[Y_n^2 | \mathcal{F}_{n-1}] = 1$. Supposing also that the initial value X_0 is deterministic, the n^{th} chaos χ_n is now the vector space generated by the random variables $\prod_{j \in J} Y_j$ where J ranges over all subsets of $\{0, \dots, N\}$ with n elements (χ_n is defined only for $n \leq N$). Now, supposing that the filtration is the one generated by X , the following turn out to be equivalent:

- (i) X has the predictable representation property;
- (ii) the filtration is dyadic: $\mathcal{F}_n$ essentially consists in 2^n atoms, each of them splitting into exactly two atoms of $\mathcal{F}_{n+1}$;
- (iii) for some Borel functions $\Phi_n: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$,

$$Y_n^2 = 1 + \Phi_n(Y_1, \dots, Y_{n-1})Y_n;$$
- (iv) X has the chaotic representation property.

PROOF. The equivalence between (i) and (ii) is well known; it can be obtained by replacing (i) by the law of X being extremal, and this amounts to the conditional law of X_n given $\mathcal{F}_{n-1}$ being supported by at most two points (see Dellacherie [2]). As the variance of this conditional law is 1, "at most" can be replaced by "exactly" in the previous sentence; and this means that the natural filtration of X is dyadic.

To get (iii), just remark that, given $\mathcal{F}_{n-1}$, Y_n can assume exactly two values; hence it must solve some quadratic equation

$$Y_n^2 = \Psi_n(Y_1, \dots, Y_{n-1}) + \Phi_n(Y_1, \dots, Y_{n-1})Y_n.$$

But, since $E[Y_n | \mathcal{F}_{n-1}] = 0$ and $E[Y_n^2 | \mathcal{F}_{n-1}] = 1$, Ψ_n is identically 1.

Conversely, if (iii) holds, given $\mathcal{F}_{n-1}$, Y_n can take at most two values and the law

of X is extremal.

Last, to get the chaotic representation property (which we know is stronger than (i)), it suffices to notice that $L^2(F_n)$ is a vector space with 2^n dimensions, whereas the subspace χ_n has dimension $\binom{N}{n}$ (if $\dim \chi_n$ were smaller, some Y_p would be a linear combination of $Y_1, \dots, Y_{p-1}$); hence $\bigoplus_{n=0}^N \chi_n$, with dimension 2^N , must be the whole space. ■

Remark that, given $X_0 \in \mathbb{R}$ and the functions Φ_n , there is exactly one martingale X such that (iii) holds (uniqueness being understood in law). Indeed, if you know $X_0, \dots, X_{n-1}$ the quadratic equation (iii) gives you two possible values for Y_n ; since their product is -1 , they are real with opposite signs, so there is exactly one way of weighting them to get a mean equal to $0 = E[Y_n | F_{n-1}]$.

STRUCTURE EQUATIONS. The continuous time analogue of (iii) will be called a structure equation. It has the form

$$(SE) \quad d[X, X]_t = dt + \Phi_t dX_t;$$

this is, of course, a symbolic notation for the equation

$$[X, X]_t = t + \int_0^t \Phi_s dX_s$$

(by convention, all brackets and stochastic integrals are taken null at time zero).

In this equation, X is a martingale, Φ a predictable process, and the stochastic integral $\int \Phi dX$ is a martingale. Note that no further generality would be gained by assuming only that X and $M = \int \Phi dX$ are local martingales, or even stochastic integrals with respect to local martingales: In that case, the existence of the decomposition of the increasing process $[X, X]$ into $t + M$ shows that $[X, X]$ is locally integrable (see [3]); then the uniqueness implies that M is a local martingale. So it is reduced by some stopping times T_n ; and $E[[X, X]_{t \wedge T_n}] = E(t \wedge T_n)$. Taking increasing limits yields $E[[X, X]_t] = t$, and, on compact time intervals, X is a square-integrable martingale and M a uniformly integrable martingale (it is even in H^1 , since

$$\sup_{s \leq t} |M_s| \leq [X, X]_t + t).$$

Remark also that $\langle X, X \rangle_t = t$ is a consequence of the structure equation itself, since $[X, X]_t - t$ is a local martingale. The initial value X_0 plays no role in (SE); it would be possible to decide that it is zero; for the sake of further convenience we shall take for X_0 an arbitrary constant x_0 .

The structure equation is an obvious necessary condition for X to have the predictable representation property, since $[X, X]_t - t$, being a local martingale, must be an integral with respect to X . The similar necessary condition for an L^4 -martingale $(X_t)_{0 \leq t \leq 1}$ to have the chaotic representation property, would be

$$(SE') \quad [X, X]_1 = 1 + \sum_{n \geq 1} \int_{0 < t_1 < \dots < t_n < 1} f_n(t_1, \dots, t_n) dX_{t_1} \dots dX_{t_n}$$

for some functions $f_n \in L^2(C_n)$ with $\sum_n \|f_n\|^2 < \infty$; in such an equation, the functions f_n can be considered as the data and the martingale X as the unknown. We shall also deal with such equations in the sequel; but since they can be considered as a particular case of (SE), it will be convenient to use the latter when stating a few general properties.

The two basic examples of structural equations are obtained by taking a constant process Φ in (SE).

(a) Brownian case: $\Phi \equiv 0$. The equation is $[X, X]_t = t$, and shows that $[X, X]$ is continuous, hence also X . Being a continuous martingale with quadratic variation t , X is a Brownian motion.

(b) Poisson case: $\Phi \equiv \alpha \neq 0$. The equation is now

$$[X, X]_t = t + \alpha(X_t - X_0);$$

dividing by α shows that X has finite variation, so it is the compensated sum of its jumps. These jumps verify

$$\Delta X_t^2 = \Delta [X, X]_t = \alpha \Delta X_t,$$

so each jump has amplitude α , and the martingale $Y_t = \frac{1}{\alpha} X_{\alpha^2 t}$, a compensated sum of unit jumps with $\langle Y, Y \rangle_t = t$, must be a compensated Poisson process $N_t - t$. So the solution to the structure equation is

$$X_t = X_0 + \alpha \left(N_{t/\alpha^2} - \frac{t}{\alpha^2} \right),$$

where N is a standard Poisson process.

In the latter example, the law of X depends continuously upon α (for the Skorohod topology on $[0, \infty)$); moreover, when α tends to zero, the Brownian case is the limit (in law) of the Poisson one. This makes it possible to see (SE) as some kind of differential equation: Φ being predictable, you may heuristically consider Φ as known a very short instant before X is, and, during this minute time-interval, solve the equation by considering Φ as fixed and using (b), or (a) if $\Phi = 0$.

GENERAL PROPERTIES OF STRUCTURE EQUATIONS.

PROPOSITION 1. Let a martingale X be a solution of (SE). Then

(i) When a jump of X occurs, it is equal to Φ_t ; the jump times are totally inaccessible.

(ii) The continuous and purely discontinuous parts of X are given by

$$dX^C = I_{\{\Phi = 0\}} dX; \quad dX^d = I_{\{\Phi \neq 0\}} dX.$$

(iii) If V is a neighbourhood of 0 in $\mathbb{R}$, let

$$Z_t = \int_0^t I_{\{\Phi_s \notin V\}} \frac{dX_s}{\Phi_s};$$

$$A_t = \int_0^t I_{\{\Phi_s \notin V\}} \frac{ds}{\Phi_s^2}.$$

Then $Z_t = N_{A_t} - A_t$ for some Poisson process N .

PROOF. (i) If T is any stopping time, (SE) implies

$$\Delta X_T^2 = \Delta[X, X]_T = \Phi_T \Delta X_T$$

and $\Delta X_T \in \{0, \Phi_T\}$: the jumps of X are equal to Φ .

If T is a bounded predictable stopping time,

$$E[\Delta X_T^2 | \mathcal{F}_{T-}] = E[\Delta[X, X]_T | \mathcal{F}_{T-}] = \Phi_T E[\Delta X_T | \mathcal{F}_{T-}] = 0;$$

hence $\Delta X_T = 0$ and X is quasi-left-continuous.

(ii) Let $C = \int I_{\{\Phi=0\}} dX$ and $D = \int I_{\{\Phi \neq 0\}} dX$. Since

$$[C, C] = \int I_{\{\Phi=0\}} d[X, X] = \int I_{\{\Phi=0\}} dt$$

is continuous, C is a continuous martingale. Since $\int \Phi dX = [X, X] - t$ has finite

variation, $\int \Phi dX^C = 0$, hence $\int \Phi^2 d\langle X^C, X^C \rangle = 0$, $\int I_{\{\Phi \neq 0\}} d\langle X^C, X^C \rangle = 0$ and

$D^C = \int I_{\{\Phi \neq 0\}} dX^C = 0$; so D is a purely discontinuous martingale. As

$C + D = X - X_0$, $C = X^C$ and $D = X^d$.

(iii) The quadratic variation of Z is

$$d[Z, Z]_t = I_{\{\Phi_t \neq 0\}} \frac{d[X, X]_t}{\Phi_t^2} = I_{\{\Phi_t \neq 0\}} \left(\frac{dt}{\Phi_t^2} + \frac{dX_t}{\Phi_t} \right) = dA_t + dZ_t ;$$

hence, if τ is the inverse of A (defined on the interval $[0, A_\infty[$),

$$d[Z, Z]_{\tau_t} = dA_{\tau_t} + dZ_{\tau_t} = dt + dZ_{\tau_t} ;$$

so, on $[0, A_\infty[$, the martingale Z_{τ_t} is a compensated Poisson process, $N_t - t$.

Inverting again, $Z_t = N_{A_t} - A_t$. ■

PROPOSITION 2. (Change of variable formula). Let M be a $\mathbb{R}^n$ -valued local martingale of the form $M = M_0 + \int H dX$, where H is a predictable $\mathbb{R}^n$ -valued process and X a martingale verifying (SE). For every $f \in C^2(\mathbb{R}^n, \mathbb{R})$,

$$f(M_t) - f(M_0) = \int_0^t U_s dX_s + \int_0^t V_s ds,$$

with

$$U_s = \sum_i \int_0^1 H_s^i D_i f(M_{s-} + \theta \Phi_s H_s) d\theta = \frac{f(M_{s-} + \Phi_s H_s) - f(M_{s-})}{\Phi_s}$$

and

$$V_s = \sum_{ij} \int_0^1 H_s^i H_s^j D_{ij} f(M_{s-} + \theta \Phi_s H_s) (1 - \theta) d\theta = \frac{f(M_{s-} + \Phi_s H_s) - f(M_{s-}) - \Phi_s H_s^T \nabla f(M_{s-})}{\Phi_s^2}$$

In this formula, the integrals $\int U dX$ and $\int V ds$ are formal semimartingales. If the semimartingale $f \circ M$ is special, then U is integrable with respect to dX , V to dt , and the formula holds in the ordinary sense.

[Recall Schwartz' definition of formal semimartingales [10]: this simply means that, if K is any strictly positive predictable process, small enough for all the following integrals to exist, for instance $K = (1 + U^2 + V^2)^{-1}$, then

$$\int K d(f \circ M) = \int (KU) dX + \int (KV) ds.$$

The reason why such formal integrals occur here is that $f \circ M$ may have very large jumps, too large to be the jumps of some local martingale. But the small jumps of $f \circ M$ can be compensated all right, and a formula involving only semimartingales would require cutting off the large jumps and dealing with them separately.]

Only the first given expressions for U and V are rigorous. The second ones, involving Φ in the denominator, are valid only for $\Phi_s \neq 0$; when $\Phi_s = 0$,

they should be replaced by their limits $U_s = H_s \nabla f(M_{s-})$ and

$$V_s = \frac{1}{2} \text{Hess } f(M_{s-})(H_s, H_s).$$

PROOF. First, the formula holds if f is a constant ($U = V = 0$) or a linear form ($U = \nabla f \cdot H, V = 0$).

Second, it holds for f a polynomial. Indeed, it suffices to check it for fg , knowing that it holds for f and g . With the obvious notations U^f, V^f , etc..., one has

$$d(fg \circ M) = (f \circ M_-)(U^g dX + V^g dt) + (g \circ M_-)(U^f dX + V^f dt) + U^f U^g d[X, X]$$

(the computation rules for formal semimartingales are naturally the usual ones).

Replacing $d[X, X]$ by $dt + \Phi dX$, we have to verify that

$$\begin{aligned} U^{fg} &= (f \circ M_-) U^g + (g \circ M_-) U^f + \Phi U^f U^g \\ V^{fg} &= (f \circ M_-) V^g + (g \circ M_-) V^f + U^f U^g. \end{aligned}$$

When $\Phi_s = 0$, these formulae reduce to the identities

$$\nabla(fg) = f \nabla g + g \nabla f$$

$$\text{Hess}(fg) = f \text{Hess} g + g \text{Hess} f + \nabla f \otimes \nabla g + \nabla g \otimes \nabla f;$$

and when $\Phi_s \neq 0$, to the identities

$$\begin{aligned} \frac{f(b)g(b) - f(a)g(a)}{\phi} &= f(a) \frac{g(b) - g(a)}{\phi} + g(a) \frac{f(b) - f(a)}{\phi} + \phi \frac{f(b) - f(a)}{\phi} \frac{g(b) - g(a)}{\phi} \\ \frac{f(b)g(b) - f(a)g(a) - h \nabla(fg)(a)}{\phi^2} &= f(a) \frac{g(b) - g(a) - h \nabla g(a)}{\phi^2} \\ &+ g(a) \frac{f(b) - f(a) - h \nabla f(a)}{\phi^2} + \frac{f(b) - f(a)}{\phi} \frac{g(b) - g(a)}{\phi}. \end{aligned}$$

Third, to prove the formula for $f \in C^2(\mathbb{R}^n, \mathbb{R})$, it is possible to approximate it by a sequence of polynomials f_p such that $f_p - f$, $\nabla(f_p - f)$ and $\text{Hess}(f_p - f)$ tend to zero uniformly on compacts. Let K be a strictly positive, predictable process such that K , KU^f and KV^f are bounded. On the predictable set

$$A_q = \{t : ||M_{t-}|| + ||M_{t-} + H_t \Phi_t|| \leq q\},$$

U^{f_p} and V^{f_p} converge uniformly to U^f and V^f . So $I_{A_q} K U^{f_p}$ and $I_{A_q} K V^{f_p}$ are bounded, and, in the equality

$$\int_{A_q} K d(f_p \circ M) = \int_{A_q} K U^{f_p} dX + \int_{A_q} K V^{f_p} ds,$$

the integrals are true semimartingales. The index p can be dropped by taking

limits (the left-hand side converges since f_p approximates f in the C^2 -topology); then one gets rid of I_{A_q} by letting $q \rightarrow \infty$ and using the boundedness of K , KU^f and KV^f . So the formula is true for f .

Last, if $f \circ M$ is a special semimartingale, it can be decomposed into a local martingale N and a predictable process A with finite variation. Since, for every bounded predictable K , $\int K dN + \int K dA$ is the decomposition of $\int K d(f \circ M)$, making K small enough gives $\int K U dX = \int K dN$ and $\int K V ds = \int K dA$, hence $N = \int U dX$ and $A = \int V ds$. So these integrals exist in the usual sense. ■

REMARK. A more general formula, that we will not use, can be shown: for a function $f(M_t, t)$ (or $f(M_t, A_t)$ with $A = A_0 + \int G_s ds$, G predictable and $\mathbb{R}^m$ -valued) a third term must be added, namely $\int \frac{\partial f}{\partial t}(M_{s-}, s) ds$ (or $\int G_s \cdot \nabla_A f(M_{s-}, A_s) ds$).

PROPOSITION 3. (i) On $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$, let Φ be a predictable process. A solution X of (SE) has the predictable representation property iff P is an extreme point of the set of all probabilities on $\mathcal{F}$ for which X is a martingale and verifies (SE).

(ii) If Φ is a functional of the form

$$\Phi_t = \Psi(t, (X_s, 0 \leq s < t))$$

and if uniqueness in law holds for (SE) with initial condition x_0 , every solution X has the predictable representation property with respect to its natural filtration.

PROOF. (i) Owing to the result of Jacod-Yor [4], we just have to verify that P is extremal in this set S iff it is extremal in the larger set M of all probabilities such that X is a martingale. The "if" part is obvious. For the necessary condition, suppose $P = \lambda Q_1 + (1 - \lambda)Q_2$ with Q_1 and Q_2 in M and $0 < \lambda < 1$. Both Q_1 and Q_2 are absolutely continuous with respect to P , so $[X, X]$ and $\int \Phi dX$, when computed with P , are also valid for Q_1 and Q_2 . Since (SE) holds for P , it also holds for Q_1 and Q_2 , and they belong to S . So if P is not extremal in M , it is not in S either.

(ii) Taking for Ω the canonical space of càdlàg paths and for X the canonical process, (i) shows that, among the solutions of (SE), the predictable representation property is possessed by those with an extremal law. In particular, if uniqueness holds, the set S has only one point, and extremality is trivial. ■

(c) THE CASE WITH INDEPENDENT INCREMENTS. An important generalization of (a) and (b) is obtained when the process Φ in (SE) is a deterministic function of time.

PROPOSITION 4. ⁷⁾ Let ϕ be a Borel function on $[0, \infty)$; consider the structure equation

$$d[X, X]_t = dt + \phi(t) dX_t, \quad X_0 = x_0.$$

(i) EXISTENCE. If B is a Brownian motion and P an independent Poisson point process on $[0, \infty)$ with intensity $I_{\{\phi(t) \neq 0\}} \frac{dt}{\phi^2(t)}$, the martingale

$$X_t = x_0 + \int_0^t I_{\{\phi(s)=0\}} dB_s + M_t$$

is a solution, where M is the purely discontinuous martingale with jump times the points of P and $\Delta M_T = \phi(T)$ if T is a jump time.

(ii) UNIQUENESS. If, on some $(\Omega, \mathcal{F}, P, (F_t)_{t \geq 0})$, X is a solution, then, for $s \geq 0$, the process $(X_{s+t} - X_s)_{t \geq 0}$ is independent of $\mathcal{F}_s$ and its law depends upon the function $\phi|_{[s, \infty)}$ only.

(iii) CHAOTIC REPRESENTATION PROPERTY. If $\mathcal{G}$ is the σ -field generated by X , the chaoses are total in $L^2(\mathcal{G})$.

PROOF. (i) Let $A_n = \{t: 2^n \leq |\phi(t)| < 2^{n+1}\} \subset [0, \infty)$; the family $(A_n)_{n \in \mathbb{Z}}$ is a partition of $\{\phi \neq 0\}$. The restriction P_n of P to A_n is locally finite. The process with independent increments

$$M_t^n = \sum_{s \in P_n \cap [0, t]} \phi(s) - \int_0^t I_{A_n}(s) \frac{ds}{\phi(s)}$$

has mean zero, so it is a martingale. It verifies

$$[M^n, M^n]_t = \sum_{s \in P_n \cap [0, t]} \phi^2(s) = \int_0^t I_{A_n}(s) ds + \int_0^t \phi(s) dM_s^n$$

and

$$E[M^n, M^n]_t = \int_0^t I_{A_n}(s) ds.$$

For fixed $\tau > 0$, each $(M_t^n)_{0 \leq t \leq \tau}$ is a square integrable martingale, with norm $\|M_t^n\|_2^2 = \int_0^t I_{A_n}(s) ds$. Depending upon P_n only, the M^n are mutually independent,

hence orthogonal in L^2 . Since $\sum_n \|M_t^n\|_2^2 < \infty$, they add up to a square integrable martingale $(M_t)_{0 \leq t \leq \tau}$; letting $\tau \rightarrow \infty$, we get a martingale M verifying

$$d[M, M]_t = I_A(t) (dt + \phi(t) dM_t)$$

with $A = \bigcup_{n \in \mathbb{Z}} A_n = \{\phi \neq 0\}$. Now, adding the independent martingale

$N_t = \int_0^t I_{A^c}(s) dB_s$ yields a solution to the structure equation.

(ii) Let X be a solution. If u is a bounded real Borel function with compact

support on $[0, \infty)$, since the functions $\frac{e^{ix} - 1}{x}$ and $\frac{e^{ix} - 1 - ix}{x^2}$ are bounded

for x real, the functions

$$\begin{aligned} h(t) &= \begin{cases} \frac{e^{iu(t)\phi(t)} - 1}{\phi(t)} & \text{if } \phi(t) \neq 0 \\ iu(t) & \text{if } \phi(t) = 0 \end{cases} \\ k(t) &= \begin{cases} \frac{e^{iu(t)\phi(t)} - 1 - iu(t)\phi(t)}{\phi^2(t)} & \text{if } \phi(t) \neq 0 \\ -\frac{1}{2} u^2(t) & \text{if } \phi(t) = 0 \end{cases} \end{aligned}$$

are also bounded and compactly supported. The change of variable formula

(Proposition 2) applied to $Y_t = \exp[i \int_0^t u(s) dX_s]$ shows that

$$Y_t = 1 + \int_0^t Y_{s-} (h(s) dX_s + k(s) ds);$$

hence the bounded process $Z_t = \exp[i \int_0^t u(s) dX_s - \int_0^t k(s) ds]$ verifies the Doléans exponential equation

$$Z_t = 1 + \int_0^t Z_{s-} h(s) dX_s.$$

So Z is a martingale, and $E[Z_\infty Z_s^{-1} | \mathcal{F}_s] = 1$:

$$E[\exp i \int_s^\infty u(t) dX_t | \mathcal{F}_s] = \exp [-\int_s^\infty k(t) dt].$$

Taking for u a linear combination $\sum_{j=1}^n \alpha_j I_{[0, t_j]}$ gives the conditional characteristic function of the process $(X_{t+s} - X_s)_{t \geq 0}$ given F_s . As it is deterministic, this process is independent of F_s ; taking $s = 0$ gives the law of $X - x_0$, whence uniqueness in law.

(iii) The notations are the same as in (ii). For $0 \leq t \leq \infty$, the random variables

$$Z'_t = 1 + \sum_{n \geq 1} \int_{0 < t_1 < \dots < t_n < t} h(t_1) \dots h(t_n) dx_{t_1} \dots dx_{t_n}$$

are well defined in L^2 because

$$\int_{0 < t_1 < \dots < t_n < t} h^2(t_1) \dots h^2(t_n) dt_1 \dots dt_n = \frac{1}{n!} \|hI_{[0, t]}\|_2^{2n}$$

is the general term of a summable series; moreover $Z'_t = E[Z'_\infty | F_t]$ and Z' is a martingale. Taking the right-continuous version shows that Z' solves the same Doléans equation

$$Z'_t = 1 + \int_0^t Z'_s h(s) dx_s$$

as Z ; so $Z' = Z$. Hence the random variable

$$Z_\infty = \exp[i \int_0^\infty u(t) dx_t - \int_0^\infty k(t) dt]$$

is chaos-decomposable, and so is also its multiple $\exp[i \int_0^\infty u(t) dx_t]$: the chaotic representation property will be established if we prove that the random variables

$\exp[i \sum_{j=1}^n \alpha_j X_{t_j}]$ are total in $L^2(G)$.

This is a classical consequence of the Fourier transform being injective:

Take a $U \in L^2(F)$ orthogonal to every $\exp[i \sum_{j=1}^n \alpha_j X_{t_j}]$. The Fourier transform of

the finite measure μ on $\mathbb{R}^n$ defined, for f Borel and bounded, by

$$\int f d\mu = E[U f(X_{t_1}, \dots, X_{t_n})]$$

is $\int e^{i\alpha \cdot y} \mu(dy) = E[U \exp[i \sum_j \alpha_j X_{t_j}]] = 0$, so $\mu = 0$ and U is orthogonal to every

$f(X_{t_1}, \dots, X_{t_n})$, whence to all $L^2(G)$. ■

(d) THE "MARKOV" CASE.

In discrete time, it is clear that the solution of a structure equation is

a homogeneous Markov process iff the predictable process Φ_n is a function of X_{n-1} only. This is why the special instance when the structure equation has the form

$$d[X, X]_t = dt + f(X_{t-})dX_t$$

will be referred to as the Markov case. But we shall see that uniqueness does not always hold, allowing non-Markovian solutions (whence the quotation marks above).

It is possible to depict heuristically the solutions:

when $f(X_{t-}) = 0$, X behaves as a Brownian motion, when $f(X_{t-}) \neq 0$, X jumps, with intensity $\frac{dt}{f^2(X_{t-})}$ and amplitude $f(X_{t-})$, and, between jumps, drifts with speed $-1/f(X_{t-})$ (as $[X, X]$ is constant between jumps, X must obey the ordinary differential equation $dt + f(x)dx = 0$). So pathological behaviours, or phenomena such as non-uniqueness will be observed for X if they already occur in this deterministic equation.

Taking $M = X$ in Proposition 2 shows that, intuitively, the generator L of the Markov process X should be, for a C^2 function g ,

$$\begin{aligned} Lg(x) &= \int_0^1 g''(x + \theta f(x)) (1 - \theta) d\theta \\ &= \begin{cases} \frac{g(x + f(x)) - g(x) - f(x)g'(x)}{f^2(x)} & \text{if } f(x) \neq 0 \\ \frac{1}{2} g''(x) & \text{if } f(x) = 0. \end{cases} \end{aligned}$$

The following existence result is due to Meyer [7].

PROPOSITION 5. If f is a continuous function on the real line, for every $x \in \mathbb{R}$ the structure equation

$$d[X, X]_t = dt + f(X_{t-})dX_t$$

has a solution with $X_0 = x$, defined on some $(\Omega, \mathcal{F}, P, (F_t)_{t \geq 0})$.

[Meyer's proof [7] consists in discretizing time, and solving step by step the corresponding equation. By Rebolledo's criterion [9], the set of laws of martingales X verifying $\langle X, X \rangle_t = t$ and $X_0 = x$ is tight, so passing to the limit when the mesh size tends to zero is possible using the Skorohod theorem or a nonstandard hyperfinite setting.]

When f is not continuous, Markovian structure equations may have zero, one, or infinitely many solutions. Consider for instance the equation

$$d[X, X]_t = dt - \text{sign}(X_{t-})dX_t,$$

where $\text{sign}(0) = 0$. If the initial value X_0 is not zero, there is a unique

solution: between the jumps, X drifts away from zero with speed 1; the jumps are given by a Poisson process with intensity 1 and have amplitude $-\text{sign}(X_{t-}) : X$ jumps towards zero and may overpass it, but not hit it (the jumps occur at totally inaccessible times and the set $\{t : |X_{t-}| = 1\}$ is a countable union of graphs of predictable times). But if $X_0 = 0$, it is possible to choose the initial value of $\frac{dX}{dt}$ as 1, -1, or any random choice between these: uniqueness does not hold.

Now look at equation $d[X, X]_t = dt + \text{sign}(X_{t-})dX_t$. Whatever the sign of X_0 , $X \text{sign} X_0$ behaves as a compensated Poisson process until it hits zero. Then it may neither leave zero (when $X_- = 0$, $\Delta X = 0$; and even if it succeeded in reaching a small non-zero value, the drift would immediately bring it back to zero) nor stay there (for on $\{X = 0\}$, $d[X, X]_t = dt$ and X must be a Brownian motion). So this equation has no solution, mainly because the deterministic equation $0 = dt + \text{sign}(x)dx$ has no solution $x(t)$ starting from 0.

(e) THE AZÉMA MARTINGALES.

The rest of this article is devoted to studying structure equations of the form

$$d[X, X]_t = dt + (\alpha + \beta X_{t-})dX_t,$$

where α and β are two constants. This is of course a sub-case of the Markov case (with f affine), but also a particular instance of (SE'), with $f_1 = \alpha$, $f_2 = \beta$ and $f_n = 0$ for $n \geq 3$.

When $\beta = 0$, the right-hand side is $dt + \alpha dX_t$, and we are back to the Poisson case (b) if $\alpha \neq 0$, and to the Brownian one (a) if $\alpha = 0$ too. So we may suppose $\beta \neq 0$; and now X is a solution of this equation (with initial condition x_0) iff $X + \frac{\alpha}{\beta}$ is a solution to

$$(*) \quad d[X, X]_t = dt + \beta X_{t-} dX_t$$

(with initial value $x_0 + \frac{\alpha}{\beta}$). So, at the cost of changing the initial condition, we do not lose any generality by assuming that $\alpha = 0$.

For $\beta = 0$, equation (*) is just the Brownian case; but for arbitrary β , as observed by Parthasarathy, it has the same scaling property as Brownian motion: if X_t is a solution to (*), then so is also $\frac{1}{\lambda} X_{\lambda^2 t}$ for every $\lambda \neq 0$; in

particular, if uniqueness holds (we shall establish it for $\beta \leq 0$), the solution X_t starting from 0 and $\frac{1}{\lambda} X_{\lambda^2 t}$ have the same law. Remark that this scaling property remains true (for $\lambda > 0$ only) for the more general Markov equation

$$d[X, X]_t = dt + (\beta^+ X_{t-}^+ + \beta^- X_{t-}^-) dX_t$$

with two coefficients β^+ and β^- .

Besides Brownian motion, obtained for $\beta = 0$, two interesting processes can be found among the solutions of (*).

First, the process X such that $X_t^2 = t$ and

$$P[X_t = \sqrt{t}] = P[X_t = -\sqrt{t}] = \frac{1}{2},$$

with jumps (that is, changes of sign) occurring according to a Poisson point process with intensity $dt/4t$. This process is a martingale since, for $0 < s < t$,

$$X_t X_s^{-1} = \sqrt{t/s} (-1)^N$$

where N is independent of F_s and has a Poisson law with parameter $\frac{1}{4} \log \frac{t}{s}$. It verifies (*) with $\beta = -2$ since

$$[X, X]_t = X_t^2 - 2 \int_0^t X_s^- dX_s = t - 2 \int_0^t X_s^- dX_s.$$

Using Proposition 1, one can show that this solution is the only one (another proof of uniqueness will be given below). See Parthasarathy [8] for a decomposition of Fermionic Brownian motion as the product of this X and another commutative process, with values ± 1 , that does not commute with X .

Second, the martingale obtained by Azéma ([1] p.464; see also Azéma-Yor [0]) when projecting a Brownian motion B starting from 0 on the filtration of sign (B) : if $G_t = \sup\{s : s \leq t, B_s = 0\}$, let

$$X_t = (\text{sign } B_t) \sqrt{2(t - G_t)}$$

(the above mentioned projection gives a multiple of this X ; the constant $\sqrt{2}$ featured here is chosen so that $\langle X, X \rangle_t = t$). Assuming that X is a martingale (this can also be shown directly, without using Azéma's projection), we shall show that it verifies (*), with $\beta = -1$. First, $\int I_{\{|X_-| > \epsilon\}} dX$ clearly has finite variation for every $\epsilon > 0$; so $X^C = \int I_{\{X_- = 0\}} dX^C = 0$ (the latter equality holds

for all semimartingales), and $[X, X]_t = \sum_{0 < s \leq t} \Delta X_s^2$. But ΔX_s^2 is zero unless s is the end of some excursion of B , in which case ΔX_s^2 is twice the length of that excursion; so $[X, X]_t = 2G_t$. And

$$\int_0^t X_- dx = \frac{1}{2} X_t^2 - \frac{1}{2} [X, X]_t = (t - G_t) - G_t = t - [X, X]_t$$

establishes the claim.

We know by Proposition 5 that solutions to (*) can be found for every value of β . These processes will be called the Azéma martingales; formally¹ they should be Markov, with generator

$$Lg(x) = \begin{cases} \frac{g(x + \beta x) - g(x) - \beta x g'(x)}{\beta^2 x^2} & \text{if } \beta x \neq 0 \\ \frac{1}{2} g''(x) & \text{if } \beta x = 0. \end{cases}$$

An amusing consequence of this formula is that, when $e^\beta + \beta + 1 = 0$ (this happens for some β between -2 and -1) and $x_0 \neq 0$, not only the process X (we shall see that it is unique) but also $L = \log|X|$ is a martingale -- and X is the stochastic exponential of L . This is easy to verify rigorously by Proposition 2, using Proposition 1 to see that the first time when X or X_- hits zero, having the same law as $\int_0^\infty \exp[-2\beta(N_t - t)] dt$, must be infinite.

PROPOSITION 6. Consider the structure equation

$$d[X, X]_t = dt + \beta X_{t-} dX_t, \quad X_0 = x_0.$$

- (i) If $\beta \leq 0$, the solution is unique in law and is a strong Markov process.
 (ii) If $-2 \leq \beta \leq 0$, the solution has the chaotic representation property.

LEMMA 7. Let X be a solution to the structure equation

$$d[X, X]_t = dt + f(t) dX_t + \left(\int_0^t g(s, t) dX_s \right) dX_t$$

with $X_0 = x_0$, where f and g are locally bounded, Borel functions. Every random variable of the form $Z = Q(X_{t_1}, \dots, X_{t_n})$ with Q a polynomial is in L^2

¹ This means: informally!

and chaos-representable. More precisely, if Q has degree k , Z is in $\bigoplus_{j \leq k} \chi_j$.

Naturally, in this structure equation, one has to take a predictable version of the integral $\int_0^t \dots$; so the equation actually means

$$[X, X]_t = t + \int_0^t f(s) dX_s + \int_{0 < r < s < t} g(r, s) dX_r dX_s$$

and is of the type (SE').

PROOF OF LEMMA 7. As the conclusion involves only finitely many instants t_i , we shall restrict ourselves to some fixed compact $[0, T]$.

Define $\chi_n^T(a)$ as the set of all elements of χ_n of the form

$$\int_{0 < t_1 < \dots < t_n < T} \phi(t_1, \dots, t_n) dX_{t_1} \dots dX_{t_n}$$

for some ϕ with $\|\phi\|_\infty \leq a$.

The conclusion clearly holds for $k \leq 1$; to establish it for all k , it suffices to prove the following claim, where $k \geq 1$: if $M = \int_0^T u_t dX_t$ for some

Borel, bounded function u , and if Z is in $\chi_k^T(a)$, then ZM is

in $\bigoplus_{j=0}^{k+1} \chi_j^T(c_k a \|u\|_\infty)$, where c_k depends on f, g, T but not M nor Z .

The important point is that this implies by induction on n that every product of the form $X_{t_1} \dots X_{t_n}$ is chaos-representable. The estimate involving a and u is not particularly interesting, but it will be useful when proving the claim (the reason why the proof may be easier with a stronger conclusion is that it goes by induction).

To prove the claim, suppose that it holds when k is replaced by any smaller value, and write

$$Z = \int_{0 < t_1 < \dots < t_{k-1} < t_k} \phi(t_1, \dots, t_k) dX_{t_1} \dots dX_{t_k}$$

with $|\phi| \leq a$; so the martingale $Z_t = E[Z|F_t]$ can be written $\int_0^t H_s dX_s$, with

$$H_t = \int_{0 < t_1 < \dots < t_{k-1} < t} \phi(t_1, \dots, t_{k-1}, t) dX_{t_1} \dots dX_{t_{k-1}}$$

(if $k = 1$, $Z = \int_0^T \phi(t) dX_t$ and $H_t = \phi(t)$). Similarly, the martingale

$M_t = E[M|F_t]$ is $\int_0^t u_s dX_s$. The integration by parts formula gives

$$\begin{aligned} ZM &= \int_0^T Z_t^- dM_t + \int_0^T M_t^- dZ_t + [Z, M]_T \\ &= \int_0^T (Z_t^- u_t + M_t^- H_t + u_t H_t \Phi_t) dX_t + \int_0^T u_t H_t dt \end{aligned}$$

(the given structure equation has been abbreviated as $d[X, X]_t = dt + \Phi_t dX_t$). The conclusion will be checked separately for each of those four integrals. Clearly,

$$\int_0^T u_t H_t dt = \int_{0 < t_1 < \dots < t_{k-1}} \left[\int_{t_{k-1}}^T \phi(t_1, \dots, t_{k-1}, t) u(t) dt \right] dX_{t_1} \dots dX_{t_{k-1}}$$

is in $\chi_{k-1}^T(Ta||u||_\infty)$, and

$$\int_0^T Z_t^- u_t dX_t = \int_{0 < t_1 < \dots < t_k < t} \phi(t_1, \dots, t_k) u(t) dX_{t_1} \dots dX_{t_k} dX_t$$

in $\chi_{k+1}^T(a||u||_\infty)$; breaking Φ_t into $f(t) + \int_0^{t-} g(s, t) dX_s$, the first part of $\int_0^T u_t H_t \Phi_t dX_t$,

$$\int_0^T u_t f(t) H_t dX_t = \int_{0 < t_1 < \dots < t_{k-1} < t} \phi(t_1, \dots, t_{k-1}, t) u_t f(t) dX_{t_1} \dots dX_{t_{k-1}} dX_t,$$

is in $\chi_k^T(||f||_\infty a||u||_\infty)$.

Both remaining terms will be dealt with using the induction hypothesis.

Since, for every t , H_t is in $\chi_{k-1}^T(a)$, the product

$M_t^- H_t = H_t \int_0^T I_{[0, t)}(s) u_s dX_s$ is in $\bigoplus_{j \leq k} \chi_j^T(c_{k-1} a||u||_\infty)$; hence the integral

$\int_0^T M_t^- H_t dX_t$ is in $\bigoplus_{1 \leq j \leq k+1} \chi_j^T(c_{k-1} a||u||_\infty)$. Similarly, the product $H_t \int_0^{t-} g(s, t) dX_s$

is in $\bigoplus_{j \leq k} \chi_j^T(c_{k-1} a||g||_\infty)$, and the last term $\int_0^T u_t H_t \int_0^{t-} g(s, t) dX_s dX_t$ is in

$\bigoplus_{1 \leq j \leq k+1} \chi_j^T(c_{k-1} ||g||_\infty a||u||_\infty)$. So the result holds, with

$$c_k = 1 + T + ||f||_\infty + c_{k-1}(1 + ||g||_\infty). \quad \blacksquare$$

PROOF OF PROPOSITION 6. (ii) We know by lemma 7 that for an Azéma martingale X ,

each polynomial $Q(X_{t_1}, \dots, X_{t_n})$ is chaos representable. The chaotic

representation property follows if these polynomials are dense in $L^2(G)$ (where G

denotes the σ -field generated by X). For $-2 \leq \beta < 0$, this is true simply

because X is bounded on each compact interval $[0, T]$. Indeed, from the structure equation

$$d[X, X]_t = dt + \beta X_t^- dX_t$$

and the integration by parts formula

$$d(X_t^2) = 2X_t dX_t + d[X, X]_t,$$

one deduces easily

$$(\beta + 2)d[X, X]_t + (-\beta)d(X_t^2) = 2dt,$$

whence $X_t^2 \leq x_0^2 + \frac{2}{-\beta}t$.

(i) The proof of Lemma 7 is constructive: at each step, the expansion of ZM as a sum of iterated integrals is obtained in terms of u, f, g and the coefficients in the expansion of Z . So the chaotic expansion of $Q(X_{t_1}, \dots, X_{t_n})$ involves

$Q, t_1, \dots, t_n, f$ and g only. This applies in particular to the first term $E[Q(t_1, \dots, t_n)]$. So, for the structure equation of Lemma 7, uniqueness in law holds as soon as X is bounded on compacts, for in this case approximating the function $\exp[i(\alpha_1 x_1 + \dots + \alpha_n x_n)]$ uniformly on compacts by polynomials gives the characteristic function of $(X_{t_1}, \dots, X_{t_n})$. As we have seen, such a boundedness

holds for $-2 \leq \beta < 0$, whence the uniqueness in that case. The strong Markov property follows by using the uniqueness of the conditional law of $(X_{T+t})_{t \geq 0}$ given F_T , for a stopping time T .

Now for $\beta < -2$. In that case, the above equality

$$(-\beta)d(X_t^2) = 2dt + (-\beta - 2)d[X, X]_t$$

gives $X_t^2 \geq x_0^2 + \frac{2}{-\beta}t$. We shall first study the case when $x_0 \neq 0$. The preceding equation shows that X is bounded away from zero. Proposition 1 applies, and, for some standard Poisson process N ,

$$\begin{cases} X_t = x_0 (1 + \beta)^{N_{A_t}} e^{-\beta A_t} \\ A_t = \int_0^t \frac{ds}{\beta^2 X_s^2} \end{cases}$$

Considering N as given, this system has a unique solution, that can be constructed pathwise from N : replacing X_s^2 in the second equation by its value from the first one yields an equation of the form $dA_t = C \exp(2\beta A_t)dt$ between the jump times. So the law of $(X_t, A_t)_{t \geq 0}$ must be the image of the law of N by this deterministic operation on paths, whence uniqueness.

Observe that the change of time A transforming the filtration of N into

that of X is continuous, strictly increasing and a.s. unbounded (else, X would have a limit X_∞ , contradicting $X_t^2 \geq x_0^2 + \frac{2}{-\beta}t$), realizing an isomorphism of filtered probability spaces. If T is a stopping time for X , A_T is one for N , and the conditional law on F_T of the process

$$X_{T+t} = X_T(1 + \beta)^{N_{A_T+t} - N_{A_t}} e^{-\beta(A_{T+t} - A_t)}$$

is the law of X , with initial condition X_0 replaced by X_T ; so X has the strong Markov property.

When $x_0 = 0$, uniqueness is a little more difficult. The estimate $X_t^2 \geq \frac{2}{-\beta}t$ shows that X is bounded away from zero on every interval $[T_\epsilon, \infty)$, where $T_\epsilon = \inf\{t: |X_t| \geq \epsilon\}$. On this interval, the above applies, and the law of $(X_{T_\epsilon+t})_{t \geq 0}$ can be obtained by replacing x_0 by X_{T_ϵ} in the preceding formulae (with X_{T_ϵ} and N independent). As $|X_{T_\epsilon}| \leq \epsilon|1 + \beta|$ we just have to show that the law of X starting from $x_0 \neq 0$ has a limit when $x_0 \rightarrow 0$. Repeating the same argument, we just have to show that for each $\epsilon > 0$ the law of X_{T_ϵ} has a limit when the initial condition x_0 tends to zero (for then the law of $(X_{T_\epsilon+t})_{t \geq 0}$ will be determined, and, since $T_\epsilon \leq \frac{1}{2}\beta(1 + \beta)^2\epsilon^2$, the law of X will necessarily be its limit for $\epsilon \rightarrow 0$).

By scaling, X is a solution of our equation with initial condition x_0 iff $\frac{1}{\epsilon}X_{\epsilon^2 t}$ is a solution starting from x_0/ϵ . So it suffices to check it for $\epsilon = 1$: we have to show that the law of X_{T_1} has a limit when $x_0 \rightarrow 0$.

Using the time change A , this amounts to saying that the first value exceeding 1 or -1 of $x_0(1 + \beta)^{N_t} e^{-\beta t}$ has a limit law. Taking logarithms, letting $a = \frac{-\beta}{\log|1 + \beta|} > 1$, $x = -\frac{\log|x_0|}{\log|1 + \beta|}$ and $T = \inf\{t: N_t + at \geq x\}$, we have to check that the law of $((-1)^{N_T}, N_T + aT - x)$, carried by $\{-1, 1\} \times [0, 1]$, has a limit when $x \rightarrow +\infty$, invariant by the symmetry exchanging -1 and 1. This law can be expressed in terms of the two functions

$$u(x) = P[N_T \text{ even}, N_T + aT = x]$$

$$v(x) = P[N_T \text{ odd}, N_T + aT = x]$$

Indeed, for $x \geq 1$, $0 \leq y < y + dy \leq 1$ and dy infinitesimal,

$$\begin{aligned}
 & P[N_T \text{ even, } x + y < N_T + aT \leq x + y + dy] \\
 &= \sum_{n \text{ even}} P[N_T = n, \frac{x + y - n}{a} < T \leq \frac{x + y + dy - n}{a}] \\
 &= \sum_{n \text{ even}} P[N_{\frac{x + y - n}{a}} = n - 1, N_{\frac{x + y + dy - n}{a}} = n] \\
 &= \sum_{n \text{ even}} P[N_{\frac{x + y - n}{a}} = n - 1] \frac{dy}{a} \\
 &= \sum_{n \text{ odd}} P[N_{\frac{x + y - n - 1}{a}} = n] \frac{dy}{a} \\
 &= v(x + y - 1) \frac{dy}{a}
 \end{aligned}$$

(the probability that more than one jump of N occurs during an interval $\frac{dy}{a}$ has been neglected as an infinitesimal of higher order).

So on $\{1\} \times [0, 1]$, the probability is $u(x)\epsilon_0(dy) + \frac{1}{a} v(x + y - 1)dy$; similarly, on $\{-1\} \times [0, 1]$, the probability is $v(x)\epsilon_0(dy) + \frac{1}{a} u(x + y + 1)dy$. All we have to do is prove that, when $x \rightarrow \infty$, $u(x)$ and $v(x)$ both have the same limit α ; and the limit law will be

$$(\epsilon_1 + \epsilon_{-1}) \otimes \alpha[\epsilon_0(dy) + \frac{1}{a} I_{[0, 1]}(y)dy].$$

This could be done by using an explicit expression of $u(x)$ and $v(x)$: they are respectively the contribution of the even and odd terms in the series

$$\sum_{n \geq 0} \frac{1}{n!} I_{\{x \geq n\}} e^{-\frac{x-n}{a}} \left(\frac{x-n}{a}\right)^n.$$

But it is quicker and simpler to use some probabilistic information on u and v .

First, $u(x)$, $v(x)$ and $u(x) + v(x)$ are probabilities, so they are in $[0, 1]$.

Second, the law of $((-1)^{N_T}, N_T + aT - x)$ has a total mass equal to one, whence, for $x \geq 1$,

$$u(x) + v(x) + \frac{1}{a} \int_{x-1}^x [v(t) + u(t)] dt = 1.$$

So $\sigma(x) = u(x) + v(x) - \frac{a}{a+1}$ verifies $\sigma(x) = -\frac{1}{a} \int_{x-1}^x \sigma(t) dt$ for $x > 1$. As $|\sigma(x)| \leq 1$ for $x > 0$, this implies by induction that $|\sigma(x)| \leq a^{-n}$ for $x > n$. Since $a > 1$, this gives $\sigma(x) \rightarrow 0$ for $x \rightarrow \infty$, and $(u + v)(x) \rightarrow \frac{a}{a+1}$.

Last, using the fact that $(-1)^{N_t} |1 + \beta|^{N_t + at - x}$ is a martingale (up to the

time-change A , you have recognized our Azémartingale $x_0(1+\beta)^{N_t}e^{-\beta t}$ bounded on $[0, T]$, its expectation at time T is its starting value:

$$\begin{aligned} |1+\beta|^{-x} &= u(x) - v(x) - \frac{1}{a} \int_0^1 |1+\beta|^y [u(x+y-1) - v(x+y-1)] dy \\ &= \delta(x) - \frac{1}{a} \int_{x-1}^x |1+\beta|^{t-x+1} \delta(t) dt \end{aligned}$$

with $\delta = u - v$. Now the total mass of the measure $\frac{1}{a} |1+\beta|^{t-x-1} dt$ on the interval $[x-1, x]$ is

$$b = \frac{|1+\beta| - 1}{a \text{Log}|1+\beta|} = \frac{-\beta - 2}{-\beta} < 1,$$

so $c = \max(b, |1+\beta|^{-1})$ is also less than 1. By induction on

$n, |\delta(x)| \leq (n+1)c^n$ for $x \geq n$, so $\delta(x) \rightarrow 0$ for $x \rightarrow \infty$, and $u(x)$ and $v(x)$ have the same limit $\frac{a}{2(a+1)}$. Uniqueness is proved. ■

REMARK. What about the chaotic representation property when β is not in $[-2, 0]$? I do not know.

And uniqueness for $\beta > 0$? I do not know either; but, letting

$a = \frac{\beta}{\text{Log}(1+\beta)} > 1$ and $S = \sup_{t \geq 0} (N_t - at)$, it is possible to show that, for

$x \geq 0$

$$(1+\beta)^{-x-1} \leq P[S > x] \leq (1+\beta)^{-x},$$

and that uniqueness holds if and only if $(1+\beta)^x P[S > x]$ has a limit when

$x \rightarrow \infty$. I have not been able to decide this question, although the law of S can

be computed explicitly: it is $\frac{a-1}{a} \sum_{n \geq 0} \mu^{n*}$, where μ is the uniform measure on

$[0, 1]$ with total mass $\frac{1}{a}$ (this is the (sub-probability) law of the (not everywhere defined) random variable $N_U - aU$, with U the first time when $N_t - at > 0$); alternatively, for $x \geq 0$,

$$P[S \leq x] = \frac{a-1}{a} \sum_{n \geq 0} \frac{(-1)^n}{n!} I_{\{x \geq n\}} e^{\frac{x-n}{a}} \left(\frac{x-n}{a}\right)^n$$

(this function of x verifies $f'(x) = \frac{1}{a} [f(x) - f(x-1)]$ for $x > 1$); and the Laplace transform is

$$E[e^{-\lambda S}] = \frac{a-1}{a - \frac{1-e^{-\lambda}}{\lambda}}.$$

Letting $T = \inf\{t : N_t - at > x\}$, uniqueness is also equivalent to the conditional law

$$L[N_T - aT - x | T > \infty]$$

having a limit for $x \rightarrow \infty$.

REFERENCES.

- [0] J. Azéma et M. Yor. Étude d'une martingale remarquable. In this volume.
- [1] J. Azéma. Sur les fermés aléatoires. Séminaire de Probabilités XIX, Springer L.N. 1123, 397-495, 1985.
- [2] C. Dellacherie. Une représentation intégrale des surmartingales à temps discret. Publ. I.S.U.P. XVII, 1-19, 1968.
- [3] M. Emery. Compensation de processus v.f. non localement intégrables. Séminaire de Probabilités XIV, 152-160, Springer L.N. 784, 1980.
- [4] J. Jacod et M. Yor. Étude des solutions extrémales et représentation intégrale des solutions pour certains problèmes de martingales. Zeitschrift f. Wahrscheinlichkeitstheorie verw. Gebiete 38, 83-125, 1977.
- [5] P.A. Meyer. Notions sur les intégrales multiples. Séminaire de Probabilités X, 321-331, Springer L.N. 511, 1976.
- [6] P.A. Meyer. Éléments de probabilités quantiques. Séminaire de Probabilités XX, 186-312, Springer L.N. 1204, 1986.
- [7] P.A. Meyer. In this volume.
- [8] K.R. Parthasarathy. Remarks on the quantum stochastic differential equation $dX = (c - 1)XdA + dQ$. Preprint, Indian Statistical Institute, Delhi.
- [9] R. Rebolledo. La méthode des martingales appliquée à la convergence en loi des processus. Memoire S.M.F. 62, 1979.
- [10] L. Schwartz. Les semi-martingales formelles. Séminaire de Probabilités XV, 413-439, Springer Lecture Notes 850, 1981.

Note de la rédaction (Azéma)

L'unicité dans le cas $\beta > 0$ (page précédente du présent article) semble avoir été établie par Emery à son insu ; en effet, "l'estimation de Cramer" (Feller t.2, XI 7, p. 364) fournit précisément ce dont on a besoin : $P[S > x]$ est équivalent quand x tend vers l'infini à Ce^{-Kx} , avec $K = \text{Log}(1+\beta)$.

Note de la rédaction

La proposition 4 (p. 74) a été établie indépendamment par A. Dermoune.