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PLANAR SEMIMARTINGALES OBTAINED BY TRANSFORMATIONS OF TWO-PARAMETER MARTINGALES

By

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Abstract. In this paper we study the weak local submartingale property and the quasimartingale property of processes obtained by the composition of a two-parameter continuous martingale by $\mathcal{C}^2$ -functions whose second derivative is convex.

0. INTRODUCTION

In this paper we are concerned with some properties of the process obtained by the composition of a two-parameter martingale with a $\mathcal{C}^2$ -class function.

In the one-parameter case the composition of a martingale with a convex function gives a local submartingale. On the other hand, the semimartingale property is preserved under transformation by convex functions, as can be proved by means of a general version of Tanaka's formula ([5]).

Some results in this direction have been given for two-parameter processes. In a previous work ([6], [7]) it has been established that the convexity of f'' implies the weak submartingale property of $f(M)$ under some hypotheses on f and the martingale M .

The notion of semimartingale does not possess up to date an equivalent in the two-parameter theory. For this reason it seems more convenient to deal with the quasi-martingale property (see [1], [4]), which can be expressed in terms of the total variation of the Doléans-Föllmer measure associated to the process. In [7] an explicit expression for the variation of $f(W)$ has been obtained, where f is a $\mathcal{C}^2$ -class function with some requirements, and W the Brownian sheet. In particular, we have sufficient conditions for $f(W)$ to possess the quasi-martingale property.

The aim of this paper is to generalize the above mentioned results. One part, section 2, deals with the weak submartingale property of $f(M)$ in the local sense, when f is a $\mathcal{C}^2$ -function whose second derivative is convex and positive. This property is obtained using two different methods. The first one consists in proving a suitable two-parameter Itô formula for $\mathcal{C}^2$ -functions and continuous martingales bounded in L^2 , with path independent variation. The second one is based on the compact Itô formula of [13] and uses a regularization procedure. It requires M to be bounded in L^4 , but the martingale may belong to a strictly larger class than before.

In Section 3 we study the quasi-martingale property of $f(M)$, when f is a $\mathcal{C}^2$ -function such that f'' is the difference of two convex functions. We prove a formula for the variation of $f(M)$ (see Theorem 3.1) involving the local time, and this allow us to state a necessary and sufficient condition to ensure the quasi-martingale property.

1. PRELIMINARIES AND NOTATION

The parameter space is $T = [0, 1]^2$ endowed with the partial ordering $(s_1, t_1) \leq (s_2, t_2)$ if and only if $s_1 \leq s_2$, $t_1 \leq t_2$; $(s_1, t_1) < (s_2, t_2)$ means $s_1 < s_2$ and $t_1 < t_2$. If f is a map from T to $\mathbb{R}$, the increment of f on a rectangle $(z_1, z_2] = \{z \in T, z_1 < z \leq z_2\}$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ is $f((z_1, z_2]) = f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$. For any $z \in T$ we define $R_z = [0, z]$.

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ together with an increasing family of sub- σ -fields of $\mathcal{F}$, $(\mathcal{F}_z)_{z \in T}$ satisfying the usual conditions (F 1) to (F 4) of [2].

Besides the classical notion of martingale, in the two-parameter case, other related definitions can be given. If $M = \{M_z, z \in T\}$ is a real valued, integrable and $\mathcal{F}_z$ -adapted process, M is a *i-martingale* ($i = 1, 2$) if for any $z \leq z'$, $z = (s, t)$, $E\{M((z, z'])/\mathcal{F}_z^i\} = 0$, where $\mathcal{F}_z^1$ (resp. $\mathcal{F}_z^2$) is the σ -field $\mathcal{F}_{s|}$ (resp. $\mathcal{F}_{t|}$). M is said to be a *weak martingale* (resp. *weak submartingale*) if $E\{M(z, z')/\mathcal{F}_z\} = 0$ (resp. ≥ 0).

Let $\mathfrak{m}^p$ (resp. $\mathfrak{m}_c^p$) be the class of two-parameter martingales (resp. continuous martingales) bounded in L^p , $p \geq 1$. It is well known (see [2]) that for $M \in \mathfrak{m}^2$, there exists an increasing, predictable process $\langle M \rangle$, called the *quadratic variation* of M , such that $M^2 - \langle M \rangle$ is a weak martingale. Moreover, it has been proved in [8] that if $M \in \mathfrak{m}_c^2$, $\langle M \rangle$ has a continuous modification.

If $M \in \mathfrak{m}_c^2$ and vanishes on the axes we say that it has *path-independent variation* (p.i.v.) if $\langle M_{\cdot, t} \rangle_s = \langle M_{s, \cdot} \rangle_t = \langle M \rangle_{s, t}$.

Consider a grid S of T given by

$$S = \{(s_i, t_j) \in T, i = 0, \dots, p; j = 0, \dots, q; 0 = s_0 < s_1 < \dots < s_p < 1, 0 = t_0 < t_1 < \dots < t_q < 1\}.$$

For any $(s_i, t_j) = z_{ij} \in S$ we define

$$\Delta_{ij} = (s_i, s_{i+1}] \times (t_j, t_{j+1}], \Delta_{ij}^1 = (s_i, s_{i+1}] \times (0, t_j], \Delta_{ij}^2 = (0, s_i] \times (t_j, t_{j+1}],$$

with the convention $s_{p+1} = t_{q+1} = 1$.

Throughout this paper we will deal with an increasing sequence of grids $\{S^n, n \geq 1\}$ of T , whose norm tends to zero (i.e. $|S^n| = \max\{|s_{i+1} - s_i| + |t_{j+1} - t_j|, (s_i, t_j) \in S^n\} \xrightarrow{n \rightarrow \infty} 0$), and for any $z \in T$ we define $I_z^n = \{(i, j) \in \mathbb{N}^2, (s_i, t_j) \in S^n, (s_i, t_j) < z\}$. I^n will denote the set $\{(i, j) \in \mathbb{N}^2, (s_i, t_j) \in S^n\}$. I_z and I are defined in an analogous way, but referred to S .

In the Doob-Meyer decomposition of a martingale $M \in \mathfrak{m}_c^2$ (and therefore in the Itô formulas for $f(M)$) we encounter a martingale $\tilde{M}$, obtained as the L^1 -limit of the sequence

$$\sum_{(i,j) \in I_z^n} M(\Delta_{ij}^1) M(\Delta_{ij}^2)$$

(cf. e.g. [8] and [9]).

For martingales M of $\mathfrak{m}_c^4$, null on the axes, it has been proved in [10] that the property p.i.v. implies $\langle M, \tilde{M} \rangle = 0$. The reciprocal is not true in general.

For any integrable, adapted process $X = \{X_z, z \in T\}$ and any rectangle $(z_1, z_2]$, $z_1 \leq z_2$, we define

$$\text{Var}_{(z_1, z_2]}^S X = \sum_{(i,j) \in I} E \left| E \{ X(\Delta_{ij} \cap (z_1, z_2]) / \mathcal{F}_{z_{ij}} \} \right|.$$

When $(z_1, z_2] = (0, 1]^2$ we simply write $\text{Var}^S X$. We also define

$$\text{Var}_{(z_1, z_2]} X = \sup_S \text{Var}_{(z_1, z_2]}^S X,$$

and $\text{Var} X = \sup_S \text{Var}^S X$.

The total variation of the Doléans-Föllmer measure of the process X on $(z_1, z_2] \times \Omega$ coincides with $\text{Var}_{(z_1, z_2]} X$. Then, following [4] the process $X = \{X_z, z \in T\}$ is said to be a *planar quasimartingale* if $\text{Var} X < \infty$.

Sometimes it is convenient to localize the above definitions. This can be done by means of the notion of *stopping domain*. A set $D \subset T \times \Omega$ is a stopping domain if 1_D is a progressively measurable process, and $R_z \subset D(\omega)$, whenever $z \in D(\omega)$. For any class $\mathcal{C}$ of processes, $\mathcal{C}_{\text{loc}}$ will denote the class of processes X such that there exists a sequence $\{X^n, n \geq 1\}$, $X^n \in \mathcal{C}$, and an increasing sequence $\{D_n, n \geq 1\}$ of stopping domains, with $\bigcup_n D_n = T$, such that for any n , $X_z^n = X_z$, if $z \in D_n$.

Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$ we introduce a sequence of smooth functions $\{f_m, m \geq 1\}$ by means of a sequence of regularization kernels of the form

$$\alpha_m(x) = m \alpha(mx), \quad (1.1)$$

where $\alpha \in \mathcal{C}_0^\infty(\mathbb{R})$ is a nonnegative function whose compact support is contained in the interval $(-\infty, 0]$ and such that $\int_{\mathbb{R}} \alpha(x) dx = 1$. We take

$$f_m(x) = (f * \alpha_m)(x) = \int_{\mathbb{R}} f(x+y) \alpha_m(y) dy. \quad (1.2)$$

2. WEAK SUBMARTINGALE PROPERTY OF $F(M)$

In this section we study the weak submartingale property of $f(M)$, f being a real function and M a two-parameter martingale. Two slightly different results in this direction are presented. The first one requires M to be a martingale belonging to $\mathfrak{M}_c^2$ and of path independent variation; its proof is based on a special two-parameter Itô's formula for

$\mathbb{C}^2$ -functions. The second one assumes that M is in $\mathfrak{m}_c^4$ and $\langle M, \tilde{M} \rangle = 0$. The proof uses the compact Itô formula of [13] and a regularization procedure.

We first state a deterministic result.

Lemma 2.1. Let X and Y be two functions in $C(T)$ and assume that Y is increasing, in the measure sense. Consider the sequences defined by

$$\varphi_n(z) = \sum_{(i,j) \in I_z^n} (X_{s_i, t_{j+1} \wedge t} - X_{s_i, t_j}) (Y_{s_{i+1} \wedge s, t_j} - Y_{s_i, t_j}), \quad (2.1)$$

$$\begin{aligned} \psi_n(z) &= \sum_{(i,j) \in I_z^n} \int_{s_i}^{s_{i+1} \wedge s} (X_{\sigma, t_{j+1} \wedge t} - X_{\sigma, t_j}) d_{\sigma} Y_{\sigma, t_j} \\ &= \sum_{\{j, t_j < t\}} \int_0^s (X_{\sigma, t_{j+1} \wedge t} - X_{\sigma, t_j}) d_{\sigma} Y_{\sigma, t_j}. \end{aligned} \quad (2.2)$$

Then $\{\varphi_n, n \geq 1\}$ and $\{\psi_n, n \geq 1\}$ converge to the same function $X * Y \in C(T)$.

Proof: We first remark that $\varphi_n(z)$ and $\psi_n(z)$ can be written in the following alternative way

$$\begin{aligned} \varphi_n(z) &= \sum_{\{i, 0 \leq s_i < s\}} X_{s_i, t} (X_{s_{i+1} \wedge s, t} - Y_{s_i, t}) - \sum_{\{i, 0 \leq s_i < s\}} X_{s_i, 0} (Y_{s_{i+1} \wedge s, 0} - Y_{s_i, 0}) \\ &\quad - \sum_{\{j, 0 < t_j \leq t\}} \sum_{\{i, s_i < s\}} X_{s_i, t_j} Y(\Delta_{ij-1}). \end{aligned} \quad (2.3)$$

$$\begin{aligned} \psi_n(z) &= \int_0^s X_{\sigma, t} d_{\sigma} Y_{\sigma, t} - \int_0^s X_{\sigma, 0} d_{\sigma} Y_{\sigma, 0} \\ &\quad - \sum_{\{j, 0 < t_j \leq t\}} \int_0^s X_{\sigma, t_j} d_{\sigma} (Y_{\sigma, t_j} - Y_{\sigma, t_{j-1}}). \end{aligned} \quad (2.4)$$

Therefore, we obtain

$$\sup_{z \in T} |\varphi_n(z) - \psi_n(z)| \leq C \sup_{|u-v| \leq |S|} |X_u - X_v| Y_{t_1}. \quad (2.5)$$

On the other hand $\{\psi_n, n \geq 1\}$ is a Cauchy sequence. Indeed, take $m > n$ and suppose

that $S^m = \{(\sigma_{i'}, \tau_{j'}) \in T, i' = 0, \dots, p_m, j' = 0, \dots, q_m\}$. For every $j = 0, \dots, q_n$ define $J_j = \{j', \tau_{j'} \in (t_j, t_{j+1}]\}$, then, using (2.4) we obtain

$$\begin{aligned} & \sup_{z \in T} |\psi_n(z) - \psi_m(z)| \\ &= \sup_{z \in T} \left| \sum_{\{j, 0 < t_j \leq t\}} \sum_{j' \in J_j} \int_0^s [X_{\sigma, t_j} - X_{\sigma, \tau_{j'}}] d_{\sigma} (Y_{\sigma, \tau_{j'}} - Y_{\sigma, \tau_{j'-1}}) \right| \\ &\leq C \sup_{|u-v| \leq \frac{1}{n}} |X_u - X_v| Y_{\frac{1}{n}}, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$, uniformly in m .

This fact together with inequality (2.5) proves the Lemma. $\square$

Let us now prove a particular Itô formula.

Theorem 2.2. Let M be a martingale belonging to $\mathcal{M}_c^2$ such that $M^2 - \langle M \rangle$ is a martingale. For any real function f of class $\mathcal{C}^2$, and every $z = (s, t) \in T$ we have

$$\begin{aligned} f(M_z) &= f(M_{0,t}) + \int_0^s f'(M_{\sigma,t}) d_{\sigma} M_{\sigma,t} + \frac{1}{2} \int_0^s f''(M_{\sigma,0}) d_{\sigma} \langle M \rangle_{\sigma,0} \\ &\quad + \frac{1}{2} \int_{R_z} f''(M_u) d \langle M \rangle_u + \frac{1}{2} [f''(M) * \langle M \rangle]_z. \end{aligned} \quad (2.6)$$

Remark: The assumptions on M are equivalent to the property of p.i.v. if M vanishes on the axes (see [10]).

Proof: To simplify the notation we take $z = (1, 1)$. By Itô's formula we can write

$$f(M_{1,1}) = f(M_{0,1}) + \int_0^1 f'(M_{\sigma,1}) d_{\sigma} M_{\sigma,1} + \frac{1}{2} \int_0^1 f''(M_{\sigma,1}) d_{\sigma} \langle M \rangle_{\sigma,1}.$$

We have

$$\int_0^1 f''(M_{\sigma,1}) d_{\sigma} \langle M \rangle_{\sigma,1} - \int_0^1 f''(M_{\sigma,0}) d_{\sigma} \langle M \rangle_{\sigma,0}$$

$$\begin{aligned}
&= \sum_{(i,j) \in I^n} \left(\int_{s_i}^{s_{i+1}} f''(M_{\sigma, t_{j+1}}) d_{\sigma} \langle M \rangle_{\sigma, t_{j+1}} - \int_{s_i}^{s_{i+1}} f''(M_{\sigma, t_j}) d_{\sigma} \langle M \rangle_{\sigma, t_j} \right) \\
&= a_n + b_n + c_n,
\end{aligned}$$

where:

$$\begin{aligned}
a_n &= \sum_{(i,j) \in I^n} f''(M_{z_{ij}}) \langle M \rangle(\Delta_{ij}), \\
b_n &= \sum_{(i,j) \in I^n} \int_{s_i}^{s_{i+1}} [f''(M_{\sigma, t_{j+1}}) - f''(M_{\sigma, t_j})] d_{\sigma} \langle M \rangle_{\sigma, t_j},
\end{aligned}$$

and

$$c_n = \sum_{(i,j) \in I^n} \int_{s_i}^{s_{i+1}} [f''(M_{\sigma, t_{j+1}}) - f''(M_{z_{ij}})] d_{\sigma} (\langle M \rangle_{\sigma, t_{j+1}} - \langle M \rangle_{\sigma, t_j}).$$

The sequence a_n converges a.s. to $\int_T f''(M_u) d \langle M \rangle_u$, as $n \rightarrow \infty$.

By Lemma 2.1 applied to the paths of $X = f''(M)$ and $Y = \langle M \rangle$, we obtain $b_n \rightarrow (f''(M) * \langle M \rangle)_{1,1}$ as $n \rightarrow \infty$, a.s. Finally, since

$$|c_n| \leq \sup_{|u-v| \leq |S|} |f''(M_u) - f''(M_v)| \langle M \rangle(R_{1,1}),$$

$\lim_{n \rightarrow \infty} c_n = 0$ a.s., and the theorem is established. $\square$

In order to study the weak submartingale property of $f(M)$ we need the following lemma.

Lemma 2.3. Let φ be a real convex function and M a martingale in $\mathfrak{M}_c^2$. The continuous process $\varphi(M) * \langle M \rangle$ is a local weak submartingale.

Proof: For any positive integer k , define

$$D_k = \left\{ (z, \omega), \sup_{u \in R_z} |M_u(\omega)| \leq k \right\}, \quad (2.7)$$

and put $N = \varphi(M) * \langle M \rangle$. We have to show the existence of a sequence of weak submartingales $\{N^k, k \geq 1\}$ such that, for any $k \geq 1$,

$$1_{D_k}(z) N_z = 1_{D_k}(z) N_z^k. \quad (2.8)$$

Let $z = (s, t)$. For each $i \in \mathbb{N}$ such that, $s_i < s$, define

$$T_i^k = \inf \{ \tau, M_{s_i, \tau} \notin D_k \},$$

this is a stopping time with respect to the filtration $\{\mathcal{F}_{t_j}, t \in [0, 1]\}$, and we have

$$\begin{aligned} & 1_{D_k}(z) \sum_{(i,j) \in I_z^n} \left[\varphi(M_{s_i, t_{j+1} \wedge t}) - \varphi(M_{s_i, t_j}) \right] \left[\langle M \rangle_{s_{i+1} \wedge s, t_j} - \langle M \rangle_{s_i, t_j} \right] \\ &= 1_{D_k}(z) \sum_{(i,j) \in I_z^n} \left[\varphi(M_{s_i, t_{j+1} \wedge t}^{T_i^k}) - \varphi(M_{s_i, t_j}^{T_i^k}) \right] \left[\langle M \rangle_{s_{i+1} \wedge s, t_j} - \langle M \rangle_{s_i, t_j} \right], \end{aligned}$$

where $\{M_{s_i, t}^{T_i^k}, t \geq 0\}$ denotes the martingale, $M_{s_i, t}$ stopped at T_i^k .

Using the convexity of φ we obtain

$$\begin{aligned} & E \left\{ \left[\varphi(M_{s_i, t_{j+1} \wedge t}^{T_i^k}) - \varphi(M_{s_i, t_j}^{T_i^k}) \right] \left[\langle M \rangle_{s_{i+1} \wedge s, t_j} - \langle M \rangle_{s_i, t_j} \right] / \mathcal{F}_{z_{ij}} \right\} \\ &= E \left\{ \left[\langle M \rangle_{s_{i+1} \wedge s, t_j} - \langle M \rangle_{s_i, t_j} \right] E \left\{ \varphi(M_{s_i, t_{j+1} \wedge t}^{T_i^k}) - \varphi(M_{s_i, t_j}^{T_i^k}) / \mathcal{F}_{1, t_j} \right\} / \mathcal{F}_{z_{ij}} \right\} \\ &\geq 0. \end{aligned}$$

Hence $J_z^{n,k} = \sum_{(i,j) \in I_z^n} \left[\varphi(M_{s_i, t_{j+1} \wedge t}^{T_i^k}) - \varphi(M_{s_i, t_j}^{T_i^k}) \right] \left[\langle M \rangle_{s_{i+1} \wedge s, t_j} - \langle M \rangle_{s_i, t_j} \right]$ is a weak submartingale.

Notice that $\{J_z^{n,k}, n \geq 1\}$ is uniformly integrable, therefore by Lemma 2.1 $L^1\text{-}\lim_{n \rightarrow \infty} J_z^{n,k}$ exists and defines a weak submartingale. Finally since

$$1_{D_k}(z) N_z = 1_{D_k}(z) L^1\text{-}\lim_{n \rightarrow \infty} J_z^{n,k},$$

(2.8) holds with $N_z^k = L^1\text{-}\lim_{n \rightarrow \infty} J_z^{n,k}$, and the proof of the Lemma is complete. $\square$

We can now state the main result of this section.

Theorem 2.4. Let f be a $\mathcal{C}^2$ function such that f'' is convex and positive. Then, if M is a martingale in $\mathfrak{M}_c^2$ such that $M^2 - \langle M \rangle$ is a martingale, the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: It is an immediate consequence of Theorem 2.2 and lemma 2.3 applied to $\phi = f''$. $\square$

The conclusion of Theorem 2.4 can also be obtained by a more direct approach, using the compact Itô formula of [13]. Here the martingale M is supposed to belong to $\mathfrak{M}_c^4$, but the path independent variation property can be replaced by a weaker one. More precisely we can prove the following result:

Theorem 2.5. Let f be a $\mathcal{C}^2$ function whose second derivative is convex and positive. Let $M \in \mathfrak{M}_c^4$, null on the axes and such that $\langle M, \tilde{M} \rangle = 0$. Then the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: (a) Let us first prove the theorem under stronger hypotheses on f : Assume that $f \in \mathcal{C}^4(\mathbb{R})$, f'' convex and positive. By the Itô formula proved in [13] we have

$$\begin{aligned} f(M_{s,t}) &= f(0) + \int_{R_{st}} f'(M_z) dM_z \\ &+ \int_{R_{st}} \left[f''(M_z) d\tilde{M}_z + f''(M_z) dS_z^{(1)} + f''(M_z) dS_z^{(2)} + \frac{1}{2} f''(M_z) d\langle M \rangle_z \right] \\ &+ \int_{R_{st}} \left[\frac{1}{2} f'''(M_z) dW_z^{(1)} + \frac{1}{2} f'''(M_z) dW_z^{(2)} \right] + \frac{1}{4} \int_{R_{st}} f^{(4)}(M_z) d\langle \tilde{M} \rangle_z. \quad (2.9) \end{aligned}$$

We recall that $\{S_z^{(1)}, z \in T\}$ (resp. $\{S_z^{(2)}, z \in T\}$) is a continuous 1-martingale (resp. 2-martingale) obtained as the L^2 -limit of the sequence $\left\{ \sum_{(i,j) \in I_z^n} M(\Delta_{ij}^2) M(\Delta_{ij}), z \in T; n \geq 1 \right\}$ (resp. $\sum_{(i,j) \in I_z^n} M(\Delta_{ij}^1) M(\Delta_{ij}), z \in T; n \geq 1$), and $\{W_z^{(1)}, z \in T\}$, (resp. $\{W_z^{(2)}, z \in T\}$) is the continuous 1-martingale (resp. 2-martingale) obtained by the L^1 -limit of the sequence $\left\{ \sum_{(i,j) \in I_z^n} M(\Delta_{ij}^2)^2 M(\Delta_{ij}^1), z \in T; n \geq 1 \right\}$ (resp. $\left\{ \sum_{(i,j) \in I_z^n} M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2), z \in T; n \geq 1 \right\}$). These processes are two-parameter stochastic integrators.

The properties on f ensure that the process defined by

$$Z_{s,t} = \int_{R_{st}} \left[\frac{1}{2} f'(M_z) d\langle M \rangle_z + \frac{1}{4} f''(M_z) d\langle \tilde{M} \rangle_z \right],$$

is increasing in the measure sense.

Consider the increasing sequence $\{D_k, k \geq 1\}$ of stopping domains defined by 2.7. For $(s, t) \in D_k$, $f(M_{s,t})$ coincides with the weak submartingale given by

$$\begin{aligned} & \int_{R_{st}} 1_{D_k}(z) \left\{ f'(M_z) dM_z + f''(M_z) d\tilde{M}_z + f''(M_z) dS_z^{(1)} + f''(M_z) dS_z^{(2)} + \frac{1}{2} f''(M_z) d\langle M \rangle_z \right. \\ & \left. + \frac{1}{2} f'''(M_z) dW_z^{(1)} + \frac{1}{2} f'''(M_z) dW_z^{(2)} + \frac{1}{4} f''(M_z) d\langle \tilde{M} \rangle_z \right\}, \end{aligned}$$

and consequently the assertion is proved.

(b) In the general case, we consider a sequence $\{\alpha_n(x), n \geq 1\}$ of regularization kernels of the form (1.1), and the corresponding $\{f_n, n \geq 1\}$.

Each f_n satisfies the hypotheses of part (a), thus, $\{f_n(M_z), z \in T\}$ is a local weak submartingale, for any $n \geq 1$.

Consider the martingales $m^k \in \mathfrak{M}_c^4$ and $\tilde{m}^k \in \mathfrak{M}_c^2$ given by

$$m_{st}^k = \int_{R_{st}} 1_{D_k}(z) dM_z, \quad \tilde{m}_{st}^k = \int_{R_{st}} 1_{D_k}(z) d\tilde{M}_z.$$

Notice that

$$\langle m^k, \tilde{m}^k \rangle = \int_R 1_{D_k}(z) d\langle M, \tilde{M} \rangle_z = 0,$$

so, (2.9) still holds with f replaced by f_n and M by m^k , and it follows that $\{f_n(m_z^k), z \in T\}$ is, for any $n \geq 1$, a weak submartingale.

Then, since $1_{D_k}(z) f_n(M_z) = 1_{D_k}(z) f_n(m_z^k)$, taking account of the convergences

$$1_{D_k}(z) f_n(M_z) \xrightarrow[n \rightarrow \infty]{L^1} 1_{D_k}(z) f(M_z),$$

and

$$1_{D_k}(z) f_n(m_z^k) \xrightarrow[n \rightarrow \infty]{L^1} 1_{D_k}(z) f(m_z^k),$$

we obtain

$$1_{D_k}(z) f(M_z) = 1_{D_k}(z) f(m_z^k),$$

where $f(m_z^k)$ is a weak submartingale. This finishes the proof of the Theorem. $\square$

3. QUASIMARTINGALE PROPERTY OF $f(M)$

The second part of this paper is devoted to study the quasimartingale property of the transformation of a two-parameter martingale by a $\mathcal{C}^2$ -function.

In order to carry out our program we will state a formula for the total variation of the Doléans-Föllmer measure (or conditional variation) of $f(M)$. This formula involves the local time of the martingale M and is obtained by means of the compact Itô's formula proved in [13]. At the same time this provides a necessary and sufficient condition for $f(M)$ to be a quasimartingale. We refer the reader to [14] for analogue results in the one-parameter setting.

In the sequel the martingale M is supposed to be null on the axes.

Assume that $M \in \mathcal{M}_c^4$ satisfies the following hypothesis:

(H 1) The measure $\langle M \rangle$ is absolutely continuous with respect to the product of its marginals.

It has been proved in [11] (see Corollary 4.2) that, under (H 1), $\langle M \rangle$ is absolutely continuous with respect to $\langle \tilde{M} \rangle$, a.s. Furthermore, since the local time L of the martingale M with respect to the measure $\langle \tilde{M} \rangle$ always exist (Theorem 3.1, [9]), so does the local time L of M with respect to $\langle M \rangle$ (cf. Lemma 5.1, [11]), and we have

$$L(x, A) = \int_A \phi(u) \tilde{L}(x, du), \quad \text{a.s.}, \quad (3.1)$$

ϕ being the Radon-Nikodym derivative $\frac{d\langle M \rangle}{d\langle \tilde{M} \rangle}$.

Following [14] a function $g: \mathbb{R} \longrightarrow \mathbb{R}$ is said to be of class (β) if its second derivative in the distributional sense is a bounded measure ν (i.e. $\int_{\mathbb{R}} |\nu(dx)| < \infty$).

We can now state our main result:

Theorem 3.1. Let M be a martingale belonging to $\mathcal{M}_c^p$, for some $p > 4$. Assume that M satisfies (H 1), and that $\langle M, \tilde{M} \rangle = 0$. Let $f \in \mathcal{C}^2$, such that f'' is of class (β) . Then, if we denote by ν the second derivative of f'' in the distributional sense, we have, for any $z_1, z_2 \in T$, $z_1 < z_2$,

$$\text{Var}_{(z_1, z_2]} f(M) = E \int_{(z_1, z_2] \times \mathbb{R}} \left\{ \left| \frac{1}{2} f''(a) \phi(z) + \frac{1}{4} \phi(a) \right| \tilde{L}(a, dz) da + \frac{1}{4} \tilde{L}(a, dz) |v^s|(da) \right\} \quad (3.2)$$

where $v(da) = \phi(a) da + v^s(da)$ is the Lebesgue decomposition of v in its absolutely continuous and singular part, with respect to the Lebesgue measure on $\mathbb{R}$.

Hence, $f(M)$ is a quasimartingale if and only if

$$E \int_T |dA_z| < \infty,$$

where

$$A_z = \int_{\mathbb{R}_z \times \mathbb{R}} \left(\frac{1}{2} f''(a) \phi(u) da + \frac{1}{4} v(da) \right) \tilde{L}(a, du), \quad (3.3)$$

$z \in T$.

Proof: (1) The hypotheses on f imply the existence of some constants α, β such that

$$f''(x) = \alpha x + \beta + \int_0^x dy \int_{(-\infty, y]} v(dz).$$

Consequently f'' is the difference of two convex functions of class (β) , ϕ^1 and ϕ^2 , and the following conditions hold:

$$\begin{aligned} |f'(x)| &\leq a_1 |x|^2 + b_1 |x| + c_1, \\ |f''(x)| &\leq a_2 |x| + b_2, \\ |f'''(x)| &\leq a_3. \end{aligned} \quad (3.4)$$

Consider the sequence $\{f_m, m \geq 0\}$ defined by (1.2). We remark that f_m'' is the difference of two convex functions (ϕ_m^1 and ϕ_m^2), and that $(\phi_m^i)' \uparrow (\phi^i)'$, $i = 1, 2$. (Here we are dealing with the left derivatives). Furthermore

$$\begin{aligned} |f_m^{(k)}(x)| &\leq \sup_{|y| \leq 1} |f^{(k)}(x+y)|, & \text{for } k = 1, 2 \\ |f_m^{(k)}(x)| &\leq C, & \text{for } k = 3, 4 \end{aligned} \quad (3.5)$$

(2) We apply the compact Itô's formula of [13] to $f_m(M)$. Then, we obtain

$$\begin{aligned}
f_m(M_{s,t}) &= f_m(0) + \int_{R_{st}} f'_m(M_z) dM_z \\
&+ \int_{R_{st}} \left\{ f''_m(M_z) d\tilde{M}_z + f''_m(M_z) dS_z^{(1)} + f''_m(M_z) dS_z^{(2)} + \frac{1}{2} f''_m(M_z) d\langle M \rangle_z \right\} \\
&+ \int_{R_{st}} \frac{1}{2} f'''_m(M_z) dW_z^{(1)} + \frac{1}{2} f'''_m(M_z) dW_z^{(2)} \\
&+ \int_{R_{st}} \frac{1}{4} f^{iv}_m(M_z) d\langle \tilde{M} \rangle_z. \quad (3.6)
\end{aligned}$$

Since $M \in \mathbb{M}_c^p$ for $p > 4$, and due to conditions (3.5), every term in the right-hand side of (3.6) belongs to L^1 . This is obvious for $\int_{R_{st}} f''_m(M_z) d\langle M \rangle_z$ and $\int_{R_{st}} f^{iv}_m(M_z) d\langle \tilde{M} \rangle_z$. In order to check this property for the stochastic integrals it suffices to prove that the corresponding sequences of Riemann sums are bounded in $L^{1+\varepsilon}$, for some $\varepsilon > 0$. Consider, for example, the sequence $\left\{ \sum_{(i,j) \in I_{st}^n} f''_m(M_{z_{ij}}) M(\Delta_{ij}^1) M(\Delta_{ij}^2), n \geq 1 \right\}$. By Burkholder's inequality we have

$$\begin{aligned}
&E \left| \sum_{(i,j) \in I_{st}^n} f''_m(M_{z_{ij}}) M(\Delta_{ij}^1) M(\Delta_{ij}^2) \right|^{1+\varepsilon} \\
&\leq C E \left(\sum_{(i,j) \in I_{st}^n} f''_m(M_{z_{ij}})^2 M(\Delta_{ij}^1)^2 M(\Delta_{ij}^2)^2 \right)^{\frac{1+\varepsilon}{2}} \\
&\leq C \left\{ E \left\{ \sup_{z \in T} |f''_m(M_z)|^{2(1+\varepsilon)} \right\}^{1/2} \left\{ E \left\{ \sup_i \left(\sum_j M(\Delta_{ij}^1)^2 \right)^{2(1+\varepsilon)} \right\} \right\} \right. \\
&\quad \left. \cdot E \left\{ \sum_i \sup_j M(\Delta_{ij}^2)^2 \right\}^{2(1+\varepsilon)} \right\}^{1/4} \\
&\leq \{C_1 + C_2 E |M_{11}|^{2(1+\varepsilon)}\}^{1/2} \{E M_{11}^{4(1+\varepsilon)}\}^{1/2},
\end{aligned}$$

where the last inequality has been obtained using (3.5), Doob's maximal inequality and Lemma 2.2. of [11].

The same kind of arguments are used for the other terms.

(3) Let $\Delta = (z, z']$, $z \leq z'$. The results proved in part (2) show that

$$E [f_m(M)(\Delta)/\mathcal{F}_z] = E \left\{ \int_{\Delta} \frac{1}{2} f''_m(M_u) d\langle M \rangle_u + \frac{1}{4} f^{iv}_m(M_u) d\langle \tilde{M} \rangle_u / \mathcal{F}_z \right\}, \quad (3.7)$$

and using the density of occupation formula

$$E \left[f_m(M)(\Delta) / \mathcal{F}_z \right] = E \left\{ \int_{\mathbb{R} \times \Delta} \frac{1}{2} f_m''(a) L(a, dz) da + \frac{1}{4} f_m'(a) \tilde{L}(a, dz) da / \mathcal{F}_z \right\}. \quad (3.8)$$

It is simple to verify that

$$E \left[f_m(M)(\Delta) / \mathcal{F}_z \right] \xrightarrow{L^1} E \left[f(M)(\Delta) / \mathcal{F}_z \right], \quad (3.9)$$

and

$$E \left[\int_{\mathbb{R} \times \Delta} \frac{1}{2} f_m''(a) L(a, dz) da / \mathcal{F}_z \right] \xrightarrow{m \rightarrow \infty} E \left[\int_{\mathbb{R} \times \Delta} \frac{1}{2} f''(a) L(a, dz) da / \mathcal{F}_z \right].$$

Denote by ν_i the measure on $(\mathbb{R}, \mathcal{B}, (\mathbb{R}))$ whose distribution function is given by

$$\nu_i([a, b]) = (\varphi^i)'(b) - (\varphi^i)'(a), \quad \text{for all } a < b, \text{ and } i = 1, 2.$$

We have

$$(\varphi_m^i)'(b) - (\varphi_m^i)'(a) = \int_a^b (\varphi_m^i)''(x) dx \longrightarrow (\varphi^i)'(b) - (\varphi^i)'(a) = \nu_i([a, b]).$$

Therefore, on any compact set $K \subset \mathbb{R}$, the finite measures $\{\nu_m^i, m \geq 1\}$ whose distribution functions are $(\varphi_m^i)'$, $m \geq 1$, converge weakly to ν_i .

By continuity, the path $\{M_z(\omega), z \in T\}$ can only visit the points of a compact set, therefore the integral

$$\int_{\mathbb{R}} f_m'(a) \tilde{L}(a, \Delta) da \quad \left(= \int_{\mathbb{R}} (\varphi_m^1 - \varphi_m^2)'' \tilde{L}(a, \Delta) da \right)$$

is extended on a compact set K , and the weak convergence of $\{\nu_m^i, m \geq 1\}$ to ν^i entails

$$\int_{\mathbb{R}} f_m'(a) \tilde{L}(a, \Delta) da \xrightarrow{m \rightarrow \infty} \int_{\mathbb{R}} \tilde{L}(a, \Delta) \nu(da).$$

Using this fact and the convergences (3.9) we obtain from (3.8)

$$E \left[f(M)(\Delta) / \mathcal{F}_z \right] = E \left\{ \int_{\mathbb{R} \times \Delta} \frac{1}{2} f''(a) L(a, dz) da + \frac{1}{4} \tilde{L}(a, dz) \nu(da) / \mathcal{F}_z \right\}$$

$$= E \left\{ \int_{\mathbb{R} \times \Delta} \left[\frac{1}{2} f''(a) \phi(z) \tilde{L}(a, dz) da + \frac{1}{4} \tilde{L}(a, dz) v(da) / \mathfrak{F}_z \right] \right\}. \quad (3.10)$$

Thus, if $\{A_z, z \in T\}$ is the predictable, bounded variation process defined by (3.3), it follows that

$$\text{Var}_{(z_1, z_2]} f(M) = \text{Var}_{(z_1, z_2]} A. \quad (3.11)$$

(4) The equality (3.11) allow us to prove formula (3.2). Indeed, it is known (see e.g. [3]) that, if X is a predictable process of bounded variation

$$\text{Var}_{(z_1, z_2]} X = E \int_{(z_1, z_2]} |dX_z|.$$

Hence the proof of the theorem is now complete. $\square$

Remark

For the special case of the Brownian sheet, Theorem 3.1 can be paraphrased as follows:

Theorem 3.2. Let $f \in \mathcal{C}^2$ be such that f'' is of class (β) . Denote by v the second derivative of f'' in the distributional sense, and let $v(da) = \phi(a) da + v^s(da)$ the Lebesgue decomposition of v with respect to the Lebesgue measure on $\mathbb{R}$.

Then, $f(W)$ is a quasimartingale if and only if

$$E \int_T |dA_z| < \infty,$$

with

$$A_z = \int_{\mathbb{R}_z} \int_{\mathbb{R}} \left(\frac{1}{2} f''(a) da + \frac{1}{4} x \cdot y v(da) \right) L(a, dx dy),$$

L being the local time of W with respect to the Lebesgue measure. Moreover

$$\text{Var}_{(z_1, z_2]} f(W) = E \int_{(z_1, z_2] \times \mathbb{R}} \left\{ \left| \frac{1}{2} f''(a) + \frac{1}{4} \phi(a) x \cdot y \right| da + \frac{1}{4} x \cdot y |v^s|(da) \right\} L(a, dx dy),$$

for any z_1, z_2 in T , $z_1 < z_2$.

A result in the same direction has been obtained in [7], where the following formula is proved.

$$\text{Var}_{(o, z]} f(W) = \frac{1}{2} \int_0^{x,y} \frac{1}{q} \text{Var}_{(o, q]} (t f''(b_t)) dq, \quad ,$$

for $z = (x, y)$, and where $\{b_t, t \geq 0\}$ is a Brownian motion.

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