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JIA-AN YAN

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GENERALIZATIONS OF GROSS' AND MINLOS' THEOREMS

by Jia An YAN
Institute of Applied Mathematics
Academia Sinica, Beijing

The purpose of this note is to give simple proofs, with some extensions, of the well known theorems of Gross, Dudley–Feldman–LeCam and Minlos, and also of the general version of Gross' theorem given by Lindstrøm.

## 1. Introduction

Let $X$ be a Banach space (or more generally any locally convex space) and $X'$ be its dual. Denote by $<x,y>$ the natural pairing between $X$ and $X'$. Let $\mathcal{K}(X')$, or simply $\mathcal{K}$ if the meaning is clear, be the collection of all finite dimensional subspaces of $X'$. Given $K \in \mathcal{K}$ we denote by $\mathcal{S}(K)$ the $\sigma$-algebra of all cylinder sets based on $K$, *i.e.* of all sets of the following form

$$C = \{x \in X \, : \, (<x,y_1>, \ldots, <x,y_n>) \in E\} \, ,$$

where $y_1, \ldots, y_n$ belong to $K$ and $E$ is a Borel set in $\mathbb{R}^n$. Let $\mathcal{R}(X)$ denote the algebra $\bigcup_{K \in \mathcal{K}} \mathcal{S}(K)$. A non-negative set function $\mu$ defined on $\mathcal{R}(X)$ is called a *cylinder (probability) measure* if $\mu(X) = 1$ and $\mu$ is $\sigma$-additive on each $\sigma$-algebra $\mathcal{S}(K)$. A function $f$ defined on $X$ is called a *cylinder function* if there exists some $K \in \mathcal{K}$ such that $f$ is $\mathcal{S}(K)$-measurable. The value of a cylinder measure on a bounded cylinder function is well defined, and denoted $\int_X f(x)\mu(dx)$. In particular, the *characteristic functional* of the cylinder measure $\mu$ on $X$ is the function defined as follows for $y \in X'$

$$\widehat{\mu}(y) = \int_X e^{i<x,y>} \mu(dx) \, .$$

In this paper we consider a *basic triple* $(H, B, \mu)$, where $H$ is a real separable Hilbert space with inner product $<\cdot,\cdot>$ and norm $|\cdot|$, $\mu$ is a cylinder measure on $H$, and $B$ is the completion of $H$ under a norm $\|\cdot\|$ which is weaker than the norm $|\cdot|$. Thus $H$ is identified with a subset of $B$. We shall always identify $H'$ with $H$ so that $B'$ in turn can be identified to a subset of $H$, *i.e.*

$$B' = \{y \in H \, : \, \sup_{x \in H, \|x\| \le 1} |<x,y>| < \infty\} \, .$$

In this way the same notation $<x,y>$ can denote without ambiguity the inner product in $H$ and (for $x \in B$, $y \in B'$) the natural pairing between $B$ and $B'$. Since we have $\mathcal{K}(B') \subset \mathcal{K}(H)$, for every cylinder set $C \in \mathcal{S}(B)$ the intersection $C \cap H$ belongs to $\mathcal{S}(H)$, and therefore we can define a cylinder measure $\mu^*$ on $B$ by the relation $\mu^*(C) = \mu(C \cap H)$. We call $\mu^*$ the *lifting* of $\mu$ on $B$.

A natural question is the following : under which conditions on $\mu$ and on the norm $\|\cdot\|$ is $\mu^*$ $\sigma$-additive on $\mathcal{R}(B)$ ? (It can then be extended as an ordinary probability measure on the $\sigma$-field generated by $\mathcal{R}(B)$, which is also the Borel $\sigma$-field of $B$ since $B$ is separable). Lindstrøm's extension of Gross' theorem asserts that it suffices that the norm $\|\cdot\|$ be *$\mu$-measurable* in the sense given below. In the Gaussian case the theorem of Dudley-Feldman-LeCam asserts that this condition is also necessary. Finally, the theorem of Minlos asserts that if the characteristic functional $\widehat{\mu}$ of $\mu$ is continuous on $H$, then we can take for $\|\cdot\|$ any norm on $H$ of the form $\|x\| = |Ax|$ where $A$ is a Hilbert-Schmidt operator on $H$.

In the last example, to have a true norm we must assume the injectivity of $A$, an unnatural condition. In reality, we might deal in most cases with a seminorm instead of a norm. We leave this easy extension to the reader. One may always assume that the linear support of $\mu$ is $H$, which is equivalent to saying that there is no $K \in \mathcal{K}$ except $\{0\}$ such that $\mu$ coincides on $\mathcal{S}(K)$ with the unit mass at $0$. However, this hypothesis is used *only* in the proof of Theorem 3.2.

We recall the definition of a measurable norm. Let $\mathcal{P}$ denote the collection of all orthogonal projections in $H$ with finite dimensional ranges. It is obvious that for each $P \in \mathcal{P}$ the function $f(x) = \|Px\|$, defined on $H$, is a cylinder function. The notation $P \perp Q$ between projections means that their ranges are orthogonal.

DEFINITION 1.1. *Let $(H, B, \mu)$ be a basic triple. The norm $\|\cdot\|$ on $H$ is said to be* measurable *w.r.t. $\mu$ if, for every $\varepsilon > 0$, there exists a $P_\varepsilon \in \mathcal{P}$ such that*

$$\mu\{x \in H : \|Px\| > \varepsilon\} < \varepsilon \quad \text{for every} \quad P \perp P_\varepsilon .$$

## 2. Some lemmas

Let $(H, B, \mu)$ be a basic triple. Since $\mu$ is a cylinder measure, it is well known from Kolmogorov's theorem that there exists a probability space $(\Omega, \mathcal{F}, \mathrm{m})$ and a linear mapping $F$ from $H$ to the space $L(\Omega)$ of all real random variables on $\Omega$ such that for any $n \geq 1$, $y_1, \ldots, y_n \in H$, $E \in \mathcal{B}(\mathbb{R}^n)$ we have

$$\mu\{x \in X : (<x, y_1>, \ldots, <x, y_n>) \in E\} = \mathrm{m}\{\omega : (F(y_1)(\omega), \ldots, F(y_n)(\omega)) \in E\} . \tag{2.1}$$

We call $(\Omega, \mathcal{F}, \mathrm{m}, F)$ a *representation* of $\mu$. If $\mu$ were a true measure on $H$, we might construct a representation as follows : on some probability space $\Omega$ choose an $H$-valued r.v. $\xi$ with law $\mu$, and then define $F(y) = <\xi, y>$ for $y \in H$, so that $\xi = \sum_j F(e_j) e_j$ a.s. for every ONB (orthonormal basis) of $H$. The main idea of our proof is to start from an arbitrary representation (which we keep fixed in the sequel), and study the convergence of this sum to a $B$-valued r.v..

The following lemma is well known (see [1], p. 51), but we include a proof for the reader's convenience.

**LEMMA 2.1.** *The following conditions are equivalent*

*(a)* $\hat{\mu}$ *is continuous at* $0$ ;

*(b)* $\hat{\mu}$ *is continuous on* $H$ ;

*(c)* $F$ *is continuous in probability from* $H$ *to* $L(\Omega)$ ;

*(d)* $(h_n \to 0 \text{ in } H) \Rightarrow (\forall \epsilon > 0,\ \mu\{x \in H : |<x, h_n>| > \epsilon\} \to 0)$.

PROOF. Since $\hat{\mu}(y) = E_{\mathrm{m}}[e^{iF(y)}]$ for $y \in H$, (a) means that $F(h_n)$ converges in law to $0$ as $h_n \to 0$ in $H$. This is well known to be equivalent to convergence in probability to $0$. Thus (a) $\Longleftrightarrow$ (c) from the linearity of $F$. On the other hand (c) $\Longleftrightarrow$ (d) from (2.1), and (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is trivial. □

**LEMMA 2.2.** *Let* $P \in \mathcal{P}$ *and* $(e_1, \ldots, e_n)$ *be an ONB of the space* $P(H)$. *Then the following random element of* $P(H)$

$$\ell(P) = \sum_{j=1}^{n} F(e_j) e_j \tag{2.2}$$

*doesn't depend on the choice of the ONB* $(e_j)$. *Moreover, we have*

$$<\ell(P), h> = F(Ph) \quad \text{for every} \quad h \in H \tag{2.3}$$

$$\mathrm{m}\{\omega : \ell(P)(\omega) \in C\} = \mu(C) \quad \text{for every} \quad C \in \mathcal{S}(P(H)) \tag{2.4}$$

PROOF. We need only prove (2.4), the other properties being obvious. Given $C \in \mathcal{S}(P(H))$ there exists some $E \in \mathcal{B}(\mathbb{R}^n)$ such that

$$C = \{x \in H : (<x, e_1>, \ldots, <x, e_n>) \in E\}.$$

We have by (2.2)

$$\{\omega : \ell(P)(\omega) \in C\} = \{\omega : (F(e_1)(\omega), \ldots, F(e_n)(\omega)) \in E\}.$$

Thus (2.4) follows from (2.1). □

We denote by $\prec$ the partial ordering of $\mathcal{P}$ defined by $(P \prec Q) \Longleftrightarrow (P(H) \subset Q(H))$. Then we consider the mapping $\ell(\cdot)$ as defining a net of $B$-valued (rather than $H$-valued) random variables, indexed by the directed set $(\mathcal{P}, \prec)$. Note that, to prove the convergence in probability of this net to a $B$-valued r.v., it is sufficient to prove that $\ell(P_n)$ is a Cauchy sequence in probability for every sequence $(P_n)$ of projectors which increases to $I$, the identity operator on $H$.

**LEMMA 2.3.** *The norm* $\|\cdot\|$ *is measurable w.r.t.* $\mu$ *iff the net* $\ell(P), P \in \mathcal{P}$ *of* $B$*-valued r.v.'s converges in probability in* $B$.

PROOF. The order relation $P \prec Q$ between projections can be written as $Q = P + R$, where $R$ is a projection orthogonal to $P$, and then we have $\ell(Q) - \ell(P) = \ell(R)$. Using this remark and the relation (2.4)

$$\mathrm{m}\{\omega : \|\ell(R)(\omega)\| > \epsilon\} = \mu\{x : \|Rx\| > \epsilon\}$$

the lemma becomes obvious.

The following result is Minlos' lemma (see lemma 3.1, p. 119 of Hida [4]; the same proof works though here the probabilities of two ellipsoids are compared instead of an ellipsoid and a sphere. This will be necessary to our generalization of Minlos' theorem). See also Bourbaki *Intégration, Chap IX*, n° 6.9, Prop. 10.

LEMMA 2.4. *Let $\mu$ be a probability measure on $\mathbb{R}^n$ and $\varphi$ be its characteristic function. Set*

$$S_1 = \{z = (z_1, \dots, z_n) \; : \; \sum_{j=1}^{n} \beta_j z_j^2 \leq s^2\} \quad ; \quad S_2 = \{z = (z_1, \dots, z_n) \; : \; \sum_{j=1}^{n} \gamma_j z_j^2 \leq t^2\}$$

*where $s$, $t$, $\beta_j$, $\gamma_j$ $(1 \leq j \leq n)$ are non negative numbers. Then the property $|\varphi(z) - 1| \leq \varepsilon$ on $S_1$ implies*

$$\mu(S_2^c) \leq C\,(\varepsilon + \frac{2}{s^2 t^2} \sum_{j=1}^{n} \beta_j \gamma_j) \quad \text{with} \quad C = (1 - e^{-1/2})^{-1} \,. \tag{2.5}$$

## 3. Generalizations of Gross' and Minlos' theorems

Our first result will be the following extension of Gross' theorem, inspired from Kallianpur [5] (the cylinder measure $\mu$ isn't assumed to be Gaussian, however).

THEOREM 3.1. *Assume that the characteristic function $\widehat{\mu}$ is continuous on $H$. If there exists an increasing sequence $P_n \subset \mathcal{P}$ such that $P_n \uparrow I$ and for every $\varepsilon > 0$*

$$\lim_{n,m \to \infty} \mu\{x \in H \; : \; \|P_n x - P_m x\| > \varepsilon\} = 0 \,, \tag{3.1}$$

*then the lifting $\mu^*$ of $\mu$ is $\sigma$-additive on $\mathcal{R}(B)$.*

According to lemma 2.3, the condition (3.1) is satisfied whenever the norm $\|\cdot\|$ is measurable w.r.t. $\mu$. We will see in Theorem 4.1 that these two properties are equivalent under some conditions on the cylinder measure $\mu$. Also, if $B$ is reflexive we give in Theorem 3.3 a condition which is easier to verify.

PROOF. Condition (3.1) implies that $\ell(P_n)$ converges in probability in $B$ to a r.v. $\xi$. According to (c) in lemma 2.1, for every $y \in H$ the real valued r.v. $F(P_n y) = \langle \ell(P_n), y \rangle$ converges in probability to $F(y)$. On the other hand for $y \in B'$ it converges to $\langle \xi, y \rangle$. Therefore $F(y) = \langle \xi, y \rangle$ m-a.s. for $y \in B'$, implying that the law $\nu$ of the r.v. $\xi$ is a countably additive extension of $\mu^*$. □

There is a slightly different version of the preceding proof, which implies the version of Gross' theorem given by Lindstrøm in [7], with a proof based on non-standard analysis.

THEOREM 3.1'. *Let $(P_n) \subset \mathcal{P}$ be such that $P_n \uparrow I$ and (3.1) holds, and let $L = B' \cap (\bigcup_n P_n(H))$. Let $\mathcal{R}(L)$ denote the collection of all cylinder sets in $B$ based on subspaces of $L$. Then $\mu^*$ is $\sigma$-additive on $\mathcal{R}(L)$.*

We remark that if the norm is measurable w.r.t. $\mu$, the net $(\ell(P), P \in \mathcal{P})$ converges in probability to a $B$-valued r.v. $\xi$. Let $\nu$ denote the law of $\xi$. We can find for any finite dimensional subspace $K$ of $B'$ a sequence $P_n \uparrow I$ such that $\ell(P_n)$ converges in where $y_1,\dots,y_p \in L$ and $E \in \mathcal{B}(\mathbb{R}^p)$. We define the image measure $\nu\circ\varphi^{-1}$ on $\mathbb{R}^p$ and (though $\mu$ is just a cylinder measure on $H$) we define the image measure $\mu\circ\varphi^{-1}$ by the formula

$$\mu\circ\varphi^{-1}(F) = \mu\{x\in H\ :\ (<x,y_1>,\dots,<x,y_p>)\in F\}\ ,\ (F\in\mathcal{B}(\mathbb{R}^p))\ .$$

The result we have to prove amounts to the equality of these two measures on $\mathbb{R}^p$, and to this order we need only show that they have the same characteristic function. Again this amounts to showing that $\widehat{\mu}(y) = E_{\mathbf{m}}[e^{i<\xi,y>}]$ for every $y\in B'$ which is a linear combination of $y_1,\dots,y_p$. Now by the definition of $L$ all the $y_j$ belong to $P_m(H)$ for some $m$, and for all $n\geq m$ we have $\widehat{\mu}(y) = E_{\mathbf{m}}[e^{i<\ell(P_n),y>}]$. Since $\ell(P_n)\to\xi$ in probability the result is obvious.

In particular, if the norm is measurable w.r.t. $\mu$, the whole net $(\ell(P),\ P\in\mathcal{P})$ converges in probability to a $B$–valued r.v. $\xi$. Therefore the measure $\nu$ of the preceding proof doesn't depend on the approximating sequence, and we can find for any finite dimensional subspace $K$ of $B'$ a sequence $P_n\uparrow I$ such that $\ell(P_n)$ converges in probability to $\xi$ and $K\subset P_1(H)$. Then $\mu^*$ coincides with $\nu$ on $\mathcal{S}(K)$ so it is $\sigma$–additive on $\mathcal{R}(B)$ (Lindstrøm's result), without any assumption on the continuity of $\widehat{\mu}$ as in Theorem 3.1.

As an application of Theorem 3.1, we prove the following theorem which generalizes Minlos' theorem. The two classical cases correspond to $A_1 = I$, $A_2$ being of Hilbert–Schmidt type (Sazonov's theorem) and to $A_2 = I$, $A_1$ Hilbert–Schmidt. The meaning here is that trace class operators are "radonifying", *i.e.* map cylinder measures with a continuous characteristic functional into Radon measures. However, our hypotheses add an unessential injectivity restriction, which could have been avoided if we had used seminorms from the beginning.

THEOREM 3.2. *Let $A_1$ and $A_2$ be two bounded operators operators on $H$ such that $A_1$ commutes with $A_2$ and $A_2^*$ (hence also $A_2$ commutes with $A_1$ and $A_1^*$) and $A_1A_2$ is of Hilbert–Schmidt type. We set for $x\in H$*

$$||x||_1 = |A_1x|\ ,\qquad ||x||_2 = |A_2x|$$

*and assume that $||x||\leq||x||_1$. Then if $\widehat{\mu}$ is continuous on $H$ w.r.t. the seminorm $||\cdot||_2$, the lifting $\mu^*$ of $\mu$ on $B$ is $\sigma$–additive on $\mathcal{R}(B)$.*

We might replace $||\cdot||$ by $||\cdot||_1$ from start, and thus get a measure carried by the completion of $H$ w.r.t. $||\cdot||_1$, a smaller space than $B$. Thus the weaker is $||\cdot||_2$ (*i.e.* the stronger is the continuity assumption on the characteristic functional), the smaller is the support of the measure.

PROOF. Let $B_1$ and $B_2$ denote the bounded self-adjoint operators $A_1^*A_1$ and $A_2^*A_2$. Our hypotheses imply that $B_1$ and $B_2$ commute, and their product is a compact operator $B$. Using a joint spectral representation of the Hilbert space $H$ as a space $L^2(\nu)$ and of these operators as multiplication operators by bounded functions $b_1$, $b_2$, we can see that the product function $b = b_1 b_2$ only takes a countable set of values $\lambda_n$ (the eigenvalues of the compact operator $B$) and each set $\{b = \lambda_n\}$ is a finite union of atoms of $\nu$, except possibly for the eigenvalue $0$. Our hypotheses imply that $B_1$ is injective. On the other hand the hypothesis that the linear support of $\mu$ is $H$ prevents $\widehat{\mu}$ from being equal to $1$ on a non trivial subspace, and therefore $A_2$ from having a non trivial kernel, and $B_2$ from having the eigenvalue $0$. Thus we may assume that $\nu$ is purely atomic. This implies the existence of an ONB $(e_j)$ of $H$ in which the operators $B_1 = A_1^* A_1$ and $B_2 = A_2^* A_2$ are diagonal, with non-negative eigenvalues $\gamma_j$ and $\beta_j$. Explicitly

$$A_1^* A_1 x = \sum_{j=1}^{\infty} \gamma_j <x, e_j> e_j \qquad ; \qquad A_2^* A_2 x = \sum_{j=1}^{\infty} \beta_j <x, e_j> e_j .$$

Since $C = A_1 A_2$ is a Hilbert–Schmidt operator and $C^* C = B_1 B_2$, $\mathrm{Tr}(C^* C) = \sum_j \beta_j \gamma_j$ is finite. We set for $n \geq 1$

$$P_n x = \sum_{j=1}^{n} <x, e_j> e_j .$$

Then $P_n \in \mathcal{P}$ and $P_n \uparrow I$. For $n > m$ we have

$$||P_n x - P_m x||_1^2 = \sum_{j=m+1}^{n} \gamma_j <x, e_j>^2 = \sum_{j=m+1}^{n} \gamma_j <P_n x - P_m x, e_j>^2 \tag{3.2}$$

By the continuity of $\widehat{\mu}$ w.r.t. $||\cdot||_2$ we see that for every $\varepsilon > 0$ there is a $s > 0$ such that, for all $n$, $m$ with $n > m$ we have

$$\sum_{j=m+1}^{n} \beta_j <P_n x - P_m x, e_j>^2 \leq s^2 \Rightarrow |\widehat{\mu}(P_n x - P_m x) - 1| < \varepsilon .$$

Consequently, by Lemma 2.4 we have for $n > m$

$$\begin{aligned} \mu\{x \in H : ||P_n x - P_m x|| > t\} &\leq \mu\{x \in H : ||P_n x - P_m x||_1 > t\} \\ &= \mu\{x \in H : \sum_{j=m+1}^{n} \gamma_j <P_n x - P_m x, e_j>^2 > t^2\} \\ &\leq C\Big(\varepsilon + \frac{2}{s^2 t^2} \sum_{j=m+1}^{\infty} \beta_j \gamma_j\Big) \end{aligned}$$

from which (3.1) follows. On the other hand, the semi-norm $||\cdot||_2$ being weaker than the norm $||\cdot||$, the continuity of $\widehat{\mu}$ w.r.t. $||\cdot||_2$ implies its continuity w.r.t. $||\cdot||$, thus by Theorem 3.1 $\mu^*$ is $\sigma$-additive on $\mathcal{R}(B)$. $\square$

The following theorem can be considered as another generalization of Minlos' theorem, since condition (3.4) below is weaker than condition (3.1).

THEOREM 3.3. *Assume that $B$ is reflexive. If $\widehat{\mu}$ is continuous on $H$ and there exists a sequence $(P_n) \subset \mathcal{P}$ with $P_n \uparrow I$ such that*

$$\lim_{N\to\infty} \lim_{n\to\infty} \mu\{x \in H : \sup_{1\le k\le n} \|P_k x\| > N\} = 0 ,$$

*then $\mu^*$ is $\sigma$-additive on $\mathcal{R}(B)$.*

PROOF. Let $(e_j)$ be an ONB of $H$ such that $P_k$ is the projection on the subspace generated by $(e_1, \ldots, e_{n_k})$. We then have

$$\ell(P_k) = \sum_{j=1}^{n_k} F(e_j) e_j .$$

Since by (2.1) and (2.2) we have

$$\mathrm{m}\{\omega : \sup_{1\le k\le n} \|\ell(P_k)(\omega)\| > N\} = \mu\{x \in H : \sup_{1\le k\le n} \|P_k x\| > N\} \tag{3.5}$$

it follows from this and (3.4) that

$$\mathrm{m}\{\omega : \sup_k \|\ell(P_k)(\omega)\| = \infty\} = \lim_{N\to\infty} \lim_{n\to\infty} \mathrm{m}\{\omega : \sup_{1\le k\le n} \|\ell(P_k)(\omega)\| > N\} = 0 . \tag{3.6}$$

On the other hand by Lemma 2.1 $F$ is continuous in probability on $H$. Consequently, for any $y \in B'$ we have m-a.s.

$$|F(y)| \le \limsup_{n\to\infty} |F(P_n y)| = \limsup_{n\to\infty} |\langle \ell(P_n), y\rangle| \le \sup_n \|\ell(P_n)\| \, \|y\|_{B'} . \tag{3.7}$$

Now we use a theorem from Badrikian [1] p.42, according to which our representation of $\mu$ may be chosen so that, for every $\omega \in \Omega$, $F(\cdot)(\omega)$ is a linear function on $H$ (in this representation, $\Omega$ is taken to be the algebraic dual of $H$). Let $\Gamma$ be a countable dense subset of $B'$. According to (3.6) the r.v. $\sup_k \|\ell(P_k)(\omega)\| = C(\omega)$ is m-a.s. finite, and by (3.7) we a.s. have $F(y)(\omega) \le C(\omega)\|y\|$ for all $y \in \Gamma$. Deleting a set of measure 0 we may assume that these properties hold everywhere on $\Omega$. Since $F$ is linear and $\Gamma$ is dense in $B'$, we may extend the mapping $F(\cdot)(\omega)$ to a bounded linear functional $\tilde{F}(\cdot)(\omega)$ on $B'$, and the reflexivity of $B$ implies the existence of a unique $\xi(\omega) \in B$ such that

$$\tilde{F}(y)(\omega) = \langle \xi(\omega), y\rangle \quad \text{for} \quad y \in B' .$$

For every $y \in \Gamma$ we have m-a.s. $F(y) = \tilde{F}(y) = \langle \xi, y\rangle$, and therefore the right side is a random variable. Therefore $\xi$ itself is a $B$-valued r.v., and it only remains to prove that its law $\nu$ is an extension of $\mu^*$. Now the characteristic functions of $\nu$ and $\mu^*$ are equal on the dense set $\Gamma \subset B'$, and since $\widehat{\mu}$ is continuous on $H$ its restriction to $B'$ is also continuous in the stronger topology of $B'$. □

## 4. A simple proof of the Dudley–Feldman–LeCam Theorem

In this section we assume that $\mu$ is Gaussian and the lifting to $B$ of the cylinder measure $\mu$ is $\sigma$–additive on $\mathcal{R}(B)$, and we prove that the norm $||\cdot||$ is measurable. The symmetry of $\mu$ will be used in the proof, as well as the following characteristic property of Gaussian measures : for any orthogonal system $(h_1, \ldots, h_n)$ in $H$ one has $\hat{\mu}(\sum_i h_i) = \prod_i \hat{\mu}(h_i)$.

We begin with a few remarks. Since $\mu^*$ is $\sigma$–additive on $\mathcal{R}(B)$ we extend it to a probability measure on the Borel $\sigma$–field $\mathcal{B}(B)$ and define a representation of $\mu$ as follows : we take $(\Omega, \mathcal{F}, \mathrm{m})$ to be $(B, \mathcal{B}(B), \mu^*)$ and the random variable $\xi : \Omega \longmapsto B$ to be the identity mapping. Then $F(y) = <\xi, y>$ is well defined for $y \in B'$. Since $B'$ is dense in $H$ and $\hat{\mu}$ is continuous on $H$, Lemma 2.1 shows that $F$ can be extended as a linear mapping from $H$ to $L(\Omega)$ which is continuous in probability. Thus we have defined a representation of $\mu$.

We also observe that, for every sequence $(P_n) \subset \mathcal{P}$ which increases to $I$ and satisfies (3.1) ( *i.e.* the random variables (from $\Omega = B$ to $B$) $\ell(P_n)$ converge in probability) $\ell(P_n)$ converges to the identity mapping on $B$. Indeed, denoting the limit by $\eta$ we have $<\eta(x), y> = <x, y>$ for $y \in B'$.

The next result is the Dudley–Feldman–LeCam theorem for centered Gaussian cylinder measures.

THEOREM 4.1. *Assume that $\mu$ is Gaussian. Then the following statements are equivalent*
*(i)* $\mu^*$ *is $\sigma$–additive on $\mathcal{R}(B)$.*
*(ii)* $||\cdot||$ *is a measurable norm w.r.t. $\mu$.*
*(iii) There is a sequence $P_n \uparrow I$ such that (3.1) holds.*
*(iv) For any sequence $P_n \uparrow I$ condition (3.1) holds.*

PROOF. We assume (i), and we use the representation of $\mu$ constructed above, with $\Omega = B$. Let $(P_n) \subset \mathcal{P}$ increase to $I$. Set

$$\xi_1 = \ell(P_1) \quad , \quad \xi_k = \ell(P_k) - \ell(P_{k-1}), \quad k \geq 2 .$$

Our assumption on $\mu$ and (2.1) imply that $(\xi_k)$ is a sequence of independent symmetric $B$–valued random variables. We have for $y \in B'$

$$\begin{aligned}\lim_{n\to\infty} E[\exp(i\sum_{j=1}^{n} <\xi_j, y>)] &= \lim_{n\to\infty} E[\exp(i < \ell(P_n), y >)] \\ &= \lim_{n\to\infty} E[\exp(iF(P_n y)] \\ &= \hat{\mu}^*(y) .\end{aligned}$$

Since $\mu^*$ is a measure, a theorem due to Ito–Nisio and to Buldygin (see [8], p. 271) implies that $\sum_{j=1}^{n} \xi_j = \ell(P_n)$ a.s. converges to a $B$–valued random element $\xi$. Since the sequence $P_n$ was arbitrary we have proved (iv), and the net $\ell(P)$, $P \in \mathcal{P})$ converges in probability. By lemma 2.3 the norm $||\cdot||$ is measurable. All the other implications are easy, and left to the reader.

## 5. Some remarks on the lifting of functions defined on $H$

Let $(H, B, \mu)$ be a basic triple, and $(\Omega, \mathcal{F}, \mathrm{m})$ be a representation of $\mu$. Adapting an idea of Kallianpur and Karandikar, we denote by $L(H, \mu)$ the class of all real functions $f$ defined on $H$ such that, for each $P \in \mathcal{P}$, $f(\ell(P))$ is a random variable and the net $(f(\ell(P)),\ P \in \mathcal{P})$ converges in probability to a r.v., which we call the *lifting* of $f$ and denote by $\tilde{f}$.

It is easy to see that the class $L(H, \mu)$ does not depend upon the choice of the representation. If $f$ is a cylinder function based on $P(H)$ (*i.e.* $f$ is $\mathcal{S}(P(H))$-measurable), then $\tilde{f} = f(\ell(P))$.

We shall consider the case where the norm $\|\cdot\|$ is measurable and $\hat{\mu}$ is continuous on $H$ and therefore $\mu^*$ is countably additive on $\mathcal{R}(B)$. Then we may take $\Omega = B$, and we are lifting functions from $H$ to $B$. The following theorems generalize to this situation, with simpler proofs, results of Gross concerning abstract Wiener spaces.

THEOREM 5.1. *For any continuous function $g$ defined on $B$, the restriction $f = g|_H$ belongs to $L(H, \mu)$ and the lifting of $f$ is a.s. equal to $g$.*

PROOF. Apply 2.3 (a), the remarks at the beginning of Section 4, and Lemma 2.3.

For the next result, due to Gross and proved here in a simpler way, we use the following notation. Given $P \in \mathcal{P}$ with $P(H) \subset B'$, we define for $x \in B$

$$\tilde{P}x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j\ , \tag{5.1}$$

where $(e_1, \ldots, e_n) \subset B'$ is a ONB of $P(H)$. It is easy to see that $\tilde{P}x$ doesn't depend on the choice of this ONB.

THEOREM 5.2. *Assume that $\mu$ is Gaussian and the norm $\|\ \|$ is $\mu$-measurable. Then for any continuous function $g$ defined on $B$ and any sequence $(P_n) \subset \mathcal{P}$ with $P_n \uparrow I$ and $P_n(H) \subset B'$, $g(\tilde{P}_n)$ converges in probability to $g$ as $n \to \infty$.*

PROOF. It is easy to see that $\tilde{P}_n x = \ell(P_n)(x)$, and by Theorem 4.1 (3.1) holds. Hence by the remarks at the beginning of Section 4 we have $\ell(P_n) \to I$, the identity mapping of $B$.

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