L.C.G. ROGERS

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by

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1. Introduction.

Let \((S,d)\) be a complete metric space with Borel \(\sigma\)-field \(S\), and let \((X_t)_{t \geq 0}\) be an \(S\)-valued strong Markov process whose paths are right continuous with left limits. We ask

(Q) Is \(P(X_{t_1} = \cdots = X_{t_k} \text{ for some } 0 < t_1 < \cdots < t_k) > 0\) ?

This is equivalent to the question

(Q') Is \(P(X(I_1) \cap \cdots \cap X(I_k) \neq \emptyset) > 0\) for some disjoint compact intervals \(I_1, \ldots, I_k\) ?

We shall find conditions sufficient to ensure that \(X\) has \(k\)-multiple points with positive probability, and we will apply this to Lévy processes, providing another proof of a result of Le Gall, Rosen and Shieh [6], and its improvement due to Evans [3]. However, it is advantageous to begin with the easier question

(Q) Is \(P(X(I_1) \cap \cdots \cap X(I_k) \neq \emptyset) > 0\) for some disjoint compact intervals \(I_1, \ldots, I_k\) ?

Here, \(X(I_j) \equiv \text{closure } \{(X_s : s \in I_j)\}\), a compact subset of \(S\). In recent years, much effort has been devoted to a study of (Q), usually in the form of constructing some non-trivial random measure on the set \(\{(t_1, \ldots, t_k) : X_{t_1} = \cdots = X_{t_k}\}\) from which the existence of common points in the ranges \(X(I_j)\) follows immediately. We mention only the work of Dynkin [1] and Evans [2] on symmetric Markov processes, of Rosen [8], [9], Geman, Horowitz and Rosen [4], Le Gall, Rosen and Shieh [6] and Evans [3] on more concrete Markov processes in \(\mathbb{R}^d\), as a sample of recent activity. Typically, one studies the random variables

(1) \[ Z_\varepsilon \equiv \int_C I_U(X_{t_1}) F_\varepsilon(X_t) \, dt, \]

where \(C = I_1 \times \cdots \times I_k\), with the \(I_j\) disjoint compact intervals in \(\mathbb{R}^+\), \(U \in S\), and

(2) \[ F_\varepsilon(x_1, \ldots, x_k) \equiv \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}), \]
(where $f_\varepsilon$ is some 'spike' function such that $f_\varepsilon(x,y) = 0$ if $d(x,y) > \varepsilon$), and proves $L^2$-convergence of the $Z_\varepsilon$ to some non-trivial limit as $\varepsilon \downarrow 0$.

This will be the approach used here, but, since we are concerned only with an answer to (Q), and not with the (more refined) $L^2$-convergence of the $Z_\varepsilon$, we can weaken the assumptions somewhat. In particular, we give sufficient conditions to ensure the existence of points of intersection for general (i.e. non-symmetric) Markov processes.

**If we could prove that**

(i) for some $\eta > 0$, \[ (Z_\varepsilon : 0 < \varepsilon < \eta/k) \text{ is bounded in } L^2; \]

(ii) \[ \limsup_{\varepsilon \downarrow 0} E Z_\varepsilon > 0, \]

then the answer to (Q) is, "Yes". The point is that $(Z_\varepsilon)_0 < \varepsilon < \eta/k$ is then uniformly integrable; if there were no common points in the closed ranges $\bar{X}(I_j)$, then the $Z_\varepsilon$ would (almost surely) be zero for all small enough $\varepsilon > 0$, and hence the $Z_\varepsilon \to 0$ in $L^1$, contradicting (3.ii).

2. **The main result.** We suppose that there is a $\sigma$-finite measure $\mu$ on $S$ such that for all $x \in S$

(4) \[ \mu(B_\varepsilon(x)) > 0 \quad \forall \ \varepsilon > 0. \]

Here, $B_\varepsilon(x) = \{ y : d(x,y) \leq \varepsilon \}$. (The assumption (4) is no great restriction, since we could always confine ourselves to the closed set of $x$ for which it is true.)

We shall suppose that the Green’s functions of $X$ have densities with respect to $\mu$: for $0 \leq a < b < \infty$, there exists $g_{a,b}(\cdot,\cdot)$ such that

(5) \[ G_{a,b}(x,A) \equiv E_x \left[ \int_a^b I_A(X_s) ds \right] = \int_A g_{a,b}(x,y) \mu(dy) \quad (\forall x \in S, A \in \mathcal{S}). \]

We suppose also that there are open $U \subset V \subset S$ such that for some $\eta > 0$ the $\eta$-neighbourhood of $U$ is contained in $V$, and that there are positive finite $K, T$ such that
(A) \( \mu(B_{2\varepsilon}(x)) \leq K \mu(B_{\varepsilon}(x)) \), \( \forall \varepsilon \in (0,\eta], \forall x \in V; \)

(B) \( \int_{V \times V} g_{0,T}(x,y)^k \mu(dx) \mu(dy) < \infty; \)

(C) for each \( \delta \in (0,2T), \)
\[
\sup_{x,y \in V} g_{\delta,2T}(x,y) < \infty;
\]

(D) for each \( 0 < a < b < \infty, \ g_{a,b}(\cdot,\cdot) \) is lower semicontinuous on \( V \times V; \)

(E) for some \( \xi \in U \) and \( \tau \in (0,T), \)
\[
g_{0,\tau}(\xi,\xi) > 0.\]

Remarks on conditions (A)-(E). Condition (A) seems fairly mild; it is trivially satisfied for Lebesgue measure on Euclidean space. The purpose of (A) is to let us take

\[ f_\varepsilon(x,y) = \mu(B_\varepsilon(x))^{-1} I_{d(x,y) \leq \varepsilon}, \]

and estimate

\[ f_\varepsilon(x,y) \leq K \mu(B_{2\varepsilon}(x))^{-1} I_{d(x,y) \leq \varepsilon} \]
\[ \leq K \mu(B_{\varepsilon}(y))^{-1} I_{d(x,y) \leq \varepsilon} \]
\[ = K f_\varepsilon(y,x). \]

Condition (B) is the 'folklore' condition for \( k \)-multiple points. Condition (C) may appear severe, but is frequently satisfied. Conditions (A)-(C) will give us (3.i), and conditions (D) and (E) will give us (3.ii). We may (and shall) suppose that the \( \xi \) appearing in (E) is a point of increase of \( g_{0,\tau}(\xi,\xi). \)

THEOREM 1. Assuming conditions (A), (B), and (C), the family \( \{Z_\varepsilon : 0 < \varepsilon < \eta/k\} \) is bounded in \( L^2. \) Assuming also conditions (D) and (E), there exist initial distributions such that for some disjoint compact intervals \( I_1, \ldots, I_k \)
\[ P(\{X(I_1) \cap \ldots \cap X(I_k) \neq \emptyset\}) > 0. \]

Proof. (i) Let \( m \) be the law of \( X_0. \) For ease of exposition, we shall suppose that \( X \) has a transition density \( p_\varepsilon(\cdot,\cdot) \) with respect to \( \mu; \) the result remains true without this assumption though.
The time-parameter set \( C = I_1 \times \cdots \times I_k \) used in the definition of \( Z \) is chosen so that \( \gamma \tau \) is in the interior of \( I_j \) for each \( j \), so that \( 0 < \delta \leq t - s \leq 2T \) if \( t \in I_j, s \in I_{j-1} \) \((j = 2, \ldots, k)\), and so that \(|I_j| < T\) for all \( j \). Then

\[
E Z^2_e = E \int_{C \times C} ds \, dt \, I_U(X_s, I_U(X_{t_1}) \cdot F_\epsilon(X_s) \cdot F_\epsilon(X_t)
\]

\[
= \sum_R \int_{C_2^R} ds \, dt \int m(dy_0)I_U(x_1)I_U(y_1)F_\epsilon(x')F_\epsilon(y') \prod_{j=1}^k p_{s_j-t_{j+1}}(y_{j-1},x_j)p_{t_j-s_j}(x_j,y_j)\mu(dx_j)\mu(dy_j),
\]

where \( C_2^R = \{(s,t) \in C^2 : s_i \leq t_i \text{ for } i = 1, \ldots, k\} \), \( t_0 = 0 \), the sum is taken over all subsets \( R \) of \( \{1, \ldots, k\} \), and

\[
x'_i = x_i, \quad y'_i = y_i \quad \text{if } i \in R
\]

\[
x'_i = y_i, \quad y'_i = x_i \quad \text{if } i \notin R.
\]

The typical term in the sum is bounded above by some constant times

\[
\int m(dy_0) I_U(x_1)I_U(y_1)F_\epsilon(x')F_\epsilon(y') \prod_{j=1}^k q(y_{j-1},x_j) g(x_j,y_j) \mu(dx_j) \mu(dy_j),
\]

where we have made the abbreviations

\[
q(x,y) \equiv g_{8,2T}(x,y),
\]

\[
g(x,y) \equiv g_{0,T}(x,y).
\]

By assumption (C), the factors \( q(y_{j-1},x_j) \) are globally bounded, because \( x_1,y_1 \in U \), and \( d(x'_i,x'_{i+1}) \leq \epsilon < \eta/k \) for each \( i \), and therefore by assumption \( x_i \in V \) for all \( i = 1, \ldots, k \). Thus we have an upper bound in terms of

\[
\int I_U(x_1)I_U(y_1)F_\epsilon(x')F_\epsilon(y') \prod_{j=1}^k g(x_j,y_j) \mu(dx_j) \mu(dy_j)
\]

\[
\leq \prod_{j=1}^k \left( \int I_U(x_1)I_U(y_1)F_\epsilon(x')F_\epsilon(y') g(x_j,y_j)^k \mu(dx_j) \mu(dy_j) \right)^{1/k},
\]

by Hölder's inequality, where, of course \( \mu(dx) \equiv \prod_{1}^k \mu(dx_j) \). The \( j^{th} \) term in this product, raised to the power \( k \), is bounded by

\[
\int I_V(x_j)I_V(y_j) g(x_j,y_j)^k \prod_{i=1}^{k-1} f_\epsilon(x'_i, x'_{i+1}) f_\epsilon(y'_i, y'_{i+1}) \mu(dx_j) \mu(dy_j),
\]

which we deal with by integrating out successively \( x_k,y_k,x_{k-1}, \ldots, x_{j+1},y_{j+1} \), and then,
exploiting (6), integrating out $x_1, y_1, ..., x_{j-1}, y_{j-1}$ to leave as an upper bound

$$K^{2j-2} \int I_V(x_j) I_V(y_j) g(x_j, y_j)^k \mu(dx_j) \mu(dy_j)$$

which is finite, by assumption (B). Hence for $0 < \epsilon < \eta/k$, $E(Z^2_\epsilon)$ is bounded above by a finite constant independent of $\epsilon$, which proves the first statement.

(ii) We next exploit (D) and (E) to give us (3.ii). By the choice of the set $C$, we have that for some small enough $\theta > 0$,

$$C \supseteq C_0 = \{ (t_1, ..., t_k) : |t_i - t_{i-1} - \tau| < \theta \quad \text{for} \quad i = 1, ..., k \},$$

where $t_0 = 0$. Hence

$$EZ_\epsilon \geq E \left[ \int_{\mathbb{R}^k} dt U(X_{t_1}) F_\epsilon(X_{t_1}) \right]$$

$$= \int m(dx_0) I_U(x_1) \prod_{i=1}^k g(x_{i-1}, x_i) \prod_{i=1}^{k-1} f_\epsilon(x_i, x_{i+1}) \mu(dx),$$

where we write $g$ as an abbreviation for $g_{\tau, \tau, \tau}$. Since $\tau$ is a point of increase of $g_{x_0, (\xi, \xi)}$, we know that $g(x_0, (\xi, \xi)) > 0$. Thus

$$EZ_\epsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) g_\epsilon(x_1)^{k-1} \prod_{i=1}^{k-1} f_\epsilon(x_i, x_{i+1}) \mu(dx),$$

where

$$g_\epsilon(x_1) = \inf \{ g(x, y) : d(x, x_1) \leq k\epsilon, d(y, x_1) \leq k\epsilon \},$$

which, in view of (D), increases as $\epsilon \downarrow 0$ to $g(x_1, x_1)$. By integrating out the variables $x_k, x_{k-1}, ..., x_2$ in (8), we obtain the lower bound

$$EZ_\epsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) g_\epsilon(x_1)^{k-1} \mu(dx_1),$$

and hence the estimate

$$\liminf_{\epsilon \downarrow 0} EZ_\epsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) g(x_1, x_1)^{k-1} \mu(dx_1).$$

By lower semi-continuity and the fact that $g(\xi, \xi) > 0$, we know that $g(x, y)$ is positive in a neighbourhood of $(\xi, \xi)$ and so taking $m = \delta_{\xi, \xi}$, for example, yields

$$\liminf_{\epsilon \downarrow 0} EZ_\epsilon > 0.$$
We now turn to the more difficult question Q. Let us suppose further that every singleton is polar:

(F) \( P^x(X_t = y \text{ for some } t > 0) = 0 \quad \forall \ x, y \in S \),

and that

(G) for each \( \mu \in Pr(S) \), for each previsible stopping time \( \tau > 0 \) we have

\[ X_\tau = X_{\tau^-} \quad P^\mu \quad \text{a.s. on } \{ \tau < \infty \} . \]

For example, if \( S \) is locally compact and separable, and the process \( X \) is Feller-Dynkin, then (G) holds; see Rogers and Williams [7], Theorem VI.15.1.

THEOREM 2. Assuming conditions (A)-(G), there exist initial distributions such that for some disjoint compact intervals \( I_1, \ldots, I_k \)

\[ P(X(I_1) \cap \cdots \cap X(I_k) = \emptyset) > 0 . \]

Proof. The proof uses Theorem 1, and proceeds by induction on \( k \). For \( k = 1 \), the result is trivial. We suppose the result is true for \( k \leq K \), and, using Theorem 1, take some initial distribution, and disjoint compact intervals \( I_1, \ldots, I_{K+1} \) such that \( I_{j+1} \) is to the right of \( I_j \) for each \( j \), and

\[ P(\overline{R}_K \cap \overline{X}(I_{K+1}) \neq \emptyset) > 0 , \tag{9} \]

where \( \overline{R}_K = \overline{X}(I_1) \cap \cdots \cap \overline{X}(I_K) \). Let \( R_K = X(I_1) \cap \cdots \cap X(I_K) \). Then

\[ P(\overline{R}_K \cap X(I_{K+1}) \neq \emptyset) > 0 , \]

because, if not, from (9), the previsible time set

\( \{ t \in I_{K+1} : X_{t^-} \in \overline{R}_K \} \)

is non-empty with positive probability and can therefore be sectioned by a previsible time \( \tau \); but, by (G), \( X_\tau = X_{\tau^-} \in \overline{R}_K \).

Finally we deduce that

\[ P(R_K \cap X(I_{K+1}) \neq \emptyset) > 0 , \]

for if not, we would have to have

\[ P((\overline{R}_K \setminus R_K) \cap X(I_{K+1}) \neq \emptyset) > 0 ; \tag{10} \]
since $\overline{R}_K \setminus R_K \subset \bigcup_{j=1}^{K} (\overline{X}(I_j) \setminus X(I_j))$, and $\overline{X}(I_j) \setminus X(I_j)$ is contained in the (countable) set of left endpoints of jumps of $X$ during time interval $I_j$, it follows from (F) that the set $\overline{R}_K \setminus R_K$ is polar, contradicting (10).

3. Multiple points of Lévy processes. Let $X$ be a Lévy process in $\mathbb{R}^n$, with resolvent $(U_\lambda)_{\lambda > 0}$. We shall assume that the resolvent is strong Feller (equivalently, that each $U_\lambda(x, \cdot)$ has a density with respect to Lebesgue measure - see Hawkes [5]), in which case there is for each $\lambda > 0$ a $\lambda$-excessive lower semi-continuous function $u_\lambda$ such that

$$U_\lambda f(x) = \int u_\lambda(y) f(y + x) \, dy.$$ 

To establish sufficient conditions for $k$-multiple points, we shall need three lemmas on Lévy processes of interest in their own right.

**LEMMA 1.** The resolvent $(U_\lambda)_{\lambda > 0}$ is strong Feller if and only if for every $0 \leq a < b < \infty$ the kernel $G_{a,b}$ has a density $g_{a,b}$.

*If this happens, the densities $g_{a,b}(\cdot)$ may be chosen so that*

(i) $g_{a,b}(\cdot)$ is lower semicontinuous for each $0 \leq a < b < \infty$;

(ii) $(a,b) \to g_{a,b}(x)$ is left-continuous increasing in $b$ and right-continuous decreasing in $a$ for each $x$;

(iii) for all $0 \leq a < b < \infty$ and all $x \in \mathbb{R}^n$

$$g_{a,b}(x) = \lim_{\delta \downarrow 0} \delta^{-1} \int g_{0,\delta}(y) g_{a,b-\delta}(x-y) \, dy.$$ 

**LEMMA 2.** For a Lévy process with a strong Feller resolvent, the following are equivalent:

(i) for some $\varepsilon, T > 0$,

$$\int_{|x| \leq \varepsilon} g_{0,T}(x)^k \, dx < \infty;$$
(ii) for some $T > 0$, $g_{0,T} \in L^k$;
(iii) for some $\lambda > 0$, $u_\lambda \in L^k$;
(iv) for some $\varepsilon, \lambda > 0$,
$$\int_{|x| \leq \varepsilon} u_\lambda(x)^k \, dx < \infty.$$  

**LEMMA 3.** Let $X$ be a Lévy process with a strong Feller resolvent such that $g_{0,T}(0) > 0$ for some $T$, and $\{\xi\}$ is non-polar for some $\xi \in \mathbb{R}^n$. Then $\{x\}$ is non-polar for every $x \in \mathbb{R}^n$.

We defer the proofs of these lemmas so as to show how to deduce the following result from them and Theorem 2. Fix some integer $k > 1$.

**THEOREM 3** (LeGall-Rosen-Shieh; Evans). Assuming that the Lévy process $X$ has a strong Feller resolvent, the conditions

(11.i) for some $\varepsilon, T > 0$
$$\int_{|x| \leq \varepsilon} g_{0,T}(x)^k \, dx < \infty;$$

(11.ii) for some $T > 0$, $g_{0,T}(0) > 0$

are sufficient to ensure that the paths of $X$ have points of multiplicity $k$ almost surely.

**Proof.** In view of Lemma 3, we may assume that every singleton is polar, for, if not, every singleton is non-polar, and the existence of multiple points is trivial! To apply Theorem 2, we must check conditions (A)-(G); (A) is immediate, (B) is guaranteed by (11.i), (D) follows from Lemma 1, (E) comes from (11.ii), (F) is by assumption, and (G) is valid because the Lévy process is a Feller-Dynkin process. Finally, to check (C), (11.i) implies that $g_{0,T}$ is square-integrable in a neighbourhood of $0$, so, by Lemma 2, $g_{0,T} \in L^2$. Hence $g_{0,T}^* \, g_{0,T}$ is bounded and continuous. But for $f \geq 0$ measurable, of compact support, and $0 < \delta < T$

$$\int g_{0,T}^* \, g_{0,T}(x) \, f(x) \, dx = \int_0^T dt \int_0^T ds \, P_{t+s} f(0)$$
$$\geq \delta \sqrt{2} \int_0^{2T-\delta} P_t f(0) \, dt$$
$$= \delta \sqrt{2} \int g_{\delta, 2T-\delta}(x) \, f(x) \, dx,$$
whence $g_{s,T}(.)$ is bounded globally (exploiting lower semi-continuity).

This completes the proof that (11.i-ii) implies that $X$ has $k$-multiple points with positive probability, and hence, by Borel-Cantelli, there are almost surely $k$-multiple points.

Proof of Lemma 1. The arguments used are similar to those of Hawkes [5], so we will just give an outline. The first statement of the lemma is immediate. To get good versions of the densities $g_{a,b}$, firstly take any densities $g'_{p,q}(.)$ for $G_{p,q}$, $0 \leq p < q < \infty$ rational, then define

$$g''_{a,b}(x) \equiv \sup \{g'_{p,q}(x) : a < p < q < b \},$$

which have property (ii) (which remains preserved under the subsequent modifications). Next, for $n > (b - a)^{-1}$ define

$$\tilde{g}^n_{a,b}(x) = n \int g_{0,\delta}(y) g_{a,b-\delta}(x - y) \, dy,$$

which is lower semicontinuous in $x$ (it is the increasing limit as $M \uparrow \infty$ of

$$n \int g_{0,\delta}(y) (M \wedge g_{a,b-\delta}(x - y)) \, dy,$$

which are continuous by the strong Feller property of $G_{0,\delta}$). Finally, we take

$$g_{a,b}(.) \equiv \sup \{\tilde{g}^n_{a,b}(.) : n > (b - a)^{-1}\}.$$

Since, for fixed $a < b$, $\tilde{g}^n_{a,b}$ is increasing almost everywhere to a version of the density of $G_{a,b}$, this provides a version with the desirable properties (i) - (iii). \qed

Proof of Lemma 2. The implications (iii) => (iv) => (i) are trivial. The implication (ii) => (iii) follows easily from the estimate

$$\int g_{a,a+T}(x)^k \, dx = \left( \int P_a(dy) g_{0,T}(x-y) \right)^k \, dx \leq \int dx \int P_a(dy) g_{0,T}(x-y)^k = \int g_{0,T}(x)^k \, dz.$$

So, finally, we assume (i) and prove (ii). Specifically, let $K$ denote the cube

$$K \equiv \{x \in \mathbb{R}^n : |x_i| \leq \frac{1}{2} \quad \text{for} \quad i = 1, \ldots, n\},$$

and assume without loss of generality that
\[ \int_{\mathbb{Z}^n} g(x)^k dx < \infty, \]

where we have abbreviated \( g_{0,T} \) to \( g \). For \( j \in \mathbb{Z}^n \), let

\[ \tau_j = \inf \{ t > 0 : X_t \in j + K \}. \]

Then for \( x \in j + K \), we have from the strong Markov property at \( \tau_j \) that

\[ g(x) \leq \int_{j + K} P(\tau_j < T, X(\tau_j) \in dy) g(x - y), \]

from which

\[ g(x)^k \leq P(\tau_j < T)^{k-1} \int_{j + K} P(\tau_j < T, X(\tau_j) \in dy) g(x - y)^k, \]

and, integrating,

\[ \int_{j + K} g(x)^k dx \leq P(\tau_j < T)^k \int_{j + K} g(z)^k dz. \]

The proof is finished if we can show that

\[ (\zeta - T) \leq T \]

Since \( \phi(T) \) is evidently increasing, it is enough to prove that

\[ \int_0^\infty \lambda e^{-\lambda T} \phi(T) dT = \sum_j P(\tau_j < \zeta) < \infty, \]

where \( \zeta \) is an \( \text{exp}(\lambda) \) random variable independent of \( X \). But we have the lower bound

\[ \int_{j + K} u_\lambda(x) dx \geq P(\tau_j < \zeta) \int_K u_\lambda(x) dx. \]

The sum over \( j \in \mathbb{Z}^n \) of the left-hand sides of (12) is clearly finite, and \( \int_K u_\lambda(x) dx > 0 \), so the proof is finished.

**Proof of Lemma 3.** If \( \{\xi\} \) is non-polar, the resolvent density \( u_\lambda(\cdot) \) must be bounded, since

\[ E^x \exp(-\lambda H_\xi) = c_\lambda u_\lambda(\xi - x) \]

for some constant \( c_\lambda \). (Here, \( H_\xi = \inf\{t > 0 : X_t = \xi\} \).) By lower semicontinuity, \( u_\lambda(0) > 0 \) implies that \( u_\lambda > 0 \) in some neighbourhood of zero and hence, by the resolvent equation, \( u_\lambda > 0 \) everywhere. Thus \( P^x(H_\xi < \infty) > 0 \) for every \( x \), and translation invariance implies that every point is non-polar.
Remarks. (i) It is evident that (11.ii) is equivalent to the condition

(9.ii) for some \( \lambda > 0 \), \( u_\lambda(0) > 0 \).

Hence, in view of Lemma 2, the conditions (11) are equivalent to those imposed by Evans [3].

(ii) Similar techniques can be used to study the problem of the existence of common points in the ranges of \( k \) independent Markov processes, a technically easier problem.

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Statistical Laboratory
16 Mill Lane
Cambridge CB2 1SB
Great Britain