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RIESZ TRANSFORMS : A SIMPLER ANALYTIC PROOF OF P.A. MEYER'S INEQUALITY

by Gilles PISIER

INTRODUCTION.

The aim of this note is to give a rather simple analytic proof of the following inequality due to P.A. Meyer [M1]. For $1 < p < \infty$, there are positive constants K_p, K'_p such that for all n and all polynomials f on $\mathbb{R}^n$

$$\frac{1}{K'_p} \| L^{1/2} f \|_p \leq \| \text{grad } f \|_p \leq K_p \| L^{1/2} f \|_p$$

where L is the generator of the Ornstein-Uhlenbeck semi-group and the L_p-norm is with respect to the canonical Gaussian measure γ_n on $\mathbb{R}^n$. A different proof has already been given by Gundy [G]. Our proof is different. We simply use a transference argument to show that the boundedness of the "Riesz transforms" $\frac{\partial}{\partial x_i} L^{-1/2}$ can be deduced from that of the one dimensional Hilbert transform.

This approach also gives a different proof of some classical (one dimensional) results of Muckenhoupt [Mu] but (as usual in this context) the case $p = 1$ cannot be handled efficiently although in [Mu] it is proved that (in dimension one) the operator $\frac{\partial}{\partial x} L^{-1/2}$ is of weak type (1,1).

In § 1, we quickly give a proof of the classical case ($\mathbb{R}^n$ here is equipped with Lebesgue measure and L is replaced by the Laplacian).

In § 2, we prove the inequality in the Gaussian case as stated above. We note in passing that since K_p, K'_p are independent of the dimension and since γ_n makes sense also for $n = \infty$, we might as well state it as an infinite dimensional inequality on $\mathbb{R}^{\mathbb{N}}$ equipped with its canonical Gaussian measure.

In fact this infinite dimensional formulation is the main motivation for the study of such inequalities in the context of the "Malliavin calculus".

Notation : Let μ be a positive measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. We will always write $\| \text{grad } f \|_{L^p(\mu)}$ instead of $\left(\int \left(\Sigma \left| \frac{\partial f}{\partial x_i} \right|^2 \right)^{p/2} d\mu \right)^{1/p}$.

§ 1. THE CLASSICAL CASE.

The classical Riesz transforms are defined as follows for a smooth function f (say) in $\mathcal{S}(\mathbb{R}^n)$.

$$(1.1)\qquad (R_k f)^\wedge(\xi) = \frac{i\,\xi_k}{\left(\sum_1^n \xi_k^2\right)^{1/2}}\hat{f}(\xi) \qquad (k = 1,2,\dots,n)\ .$$

Equivalently we have "symbolically"

$$R_k f = \frac{\partial}{\partial x_k}\left(\frac{1}{\sqrt{\Delta}} f\right)$$

where we have denoted by $x_1,\dots,x_n$ the coordinates of a point x in $\mathbb{R}^n$ and where

$$\Delta = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\ ,$$

clearly we have $\int \Sigma |R_k f|^2\,dx = \int |f|^2\,dx$.

Similarly, by Parseval's identity, one checks easily

$$\int |\sqrt{\Delta}\, f|^2\,dx = \langle \Delta f, f\rangle_{L^2} = \int \|\operatorname{grad} f\|_2^2\,dx$$

or equivalently after polarization

$$(1.2)\qquad \langle \sqrt{\Delta} f, \sqrt{\Delta} g\rangle_{L^2} = \int \langle \operatorname{grad} f, \operatorname{grad} g\rangle\,dx$$

for all f and g in $\mathcal{S}(\mathbb{R}^n)$.

The classical Riesz transforms inequalities in $L_p(\mathbb{R}^n)$ are the following.

THEOREM 1.1. If $1 < p < \infty$, there are constants C_p, C'_p such that for all f in $\mathcal{S}(\mathbb{R}^n)$ we have

$$(1.3)\qquad \frac{1}{C'_p}\|f\|_p \le \left\|\left(\sum_1^n |R_k f|^2\right)^{1/2}\right\|_p \le C_p\,\|f\|_p\ .$$

Of course, this extends to f in $L_p(\mathbb{R}^n)$ by density, we will usually ignore such matters in the sequel.

Recently, E. Stein [S] discovered that the best constants C_p, C'_p can be bounded above independently of the dimension n . Actually, this phenomenon can be viewed also as an immediate consequence of an earlier probabilistic proof of (1.3) due to R. Gundy an N. Varopoulos [GV].

Alternate proofs of (1.3) have been given in [DR] and [Ba] (with constants independent of the dimension).

Although this is not the object of the present note we will give below a proof of (1.3), to be compared with that of § 2. We use a simple "transference argument" in the sense of [CW] and Gaussian measures.

In the case $n = 1$, the Riesz transforms reduce to the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\, \text{p.v.} \int f(x - t)\, \frac{dt}{t} .$$

By "transference", we will show that the boundedness of H on $L_p(\mathbb{R})$ implies (1.3) for any n .

We will use $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the product $dx\, d\gamma_n(y)$ of the Lebesgue measure $dx_1\, dx_2 \ldots dx_n$ and the canonical Gaussian measure $\exp - \frac{1}{2}(\sum_1^n y_k^2)\, dy_1 \ldots dy_n (2\pi)^{-n/2}$.

The transformation $R_t : L_p(dx) \to L_p(dx\, d\gamma_n(y))$

defined by $(R_t f)(x,y) = f(x + ty)$

is clearly an isometric embedding.

By a simple transference argument (cf. [CW]), the transformation

$$\tilde{H} = \text{p.v.} \int_{-\infty}^{\infty} R_t \frac{dt}{t}$$

is also bounded from $L_p(dx)$ into $L_p(dx\, d\gamma_n(y))$ with norm

$$\| \tilde{H} \| \le \tilde{K}_p$$

where $\tilde{K}_p$ is a constant bounding the norm of H (or rather the truncated versions of H) on L_p . In particular, we have

$$\tilde{K}_p \in O(p) \quad \text{when} \quad p \to \infty \quad \text{and} \quad \tilde{K}_p \in O\left(\frac{1}{p-1}\right) \quad \text{when} \quad p \to 1 .$$

We will also use the orthogonal projection from $L_2(\gamma_n)$ onto the span of $y_1,\ldots,y_n$. We denote this projection by Q .

Let $\gamma(p) = \left(\int |t|^p e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}\right)^{1/p}$.

We claim that (this is a well known fact)

(1.4) $\quad$ if $2 \le p < \infty$ $\quad \forall f \in L_p(\gamma_n(dy)) \quad \| Qf \|_p \le \gamma(p)\, \| f \|_p$,

and consequently by duality

(1.4)' $\quad$ if $1 < p \le 2$ $\quad \| Qf \|_p \le \gamma(p')\, \| f \|_p$.

Indeed, since any linear combination of independent Gaussian variables is again Gaussian, we have (if $2 \le p < \infty$)

$$\| Qf \|_p = \gamma(p) \| Qf \|_2$$
$$\leq \gamma(p) \| f \|_2$$
$$\leq \gamma(p) \| f \|_p ,$$

therefore (1.4) is immediate.

Proof of theorem 1.1. :

We can view theorem 1.1 as a consequence of the following formula, valid for any f (say) in $L_2(dx)$.

$$(1.5) \qquad Q_y \left[\frac{1}{\pi} \text{ p.v. } \int f(x+ty) \frac{dt}{t}\right] = (2/\pi)^{1/2} \sum_1^n y_k R_k f(x) ,$$

where we have denoted by Q_y the operator $I \otimes Q$ acting on $L_2(dx\, d\gamma_n(y))$.

This formula immediately implies (1.3). Indeed we deduce from (1.5),(1.4) and (1.4)'

$$\| \Sigma y_k R_k f(x) \|_{L_p(dx\, d\gamma_n(y)} \leq (\pi/2)^{1/2} \| Q \|_{L_p \to L_p} \| \tilde{H} \|_{p\to p} \| f \|_p$$
$$\leq (\pi/2)^{1/2} \tilde{K}_p \max(\gamma(p),\gamma(p')) \| f \|_p$$

On the other hand, we have clearly for any (λ_k) in $\mathbb{R}^n$

$$\| \Sigma \lambda_k y_k \|_{L_p(d\gamma_n(y))} = (\Sigma |\lambda_k|^2)^{1/2} \gamma(p)$$

Hence

$$\| (\Sigma |R_k f(x)|^2)^{1/2} \|_{L_p(dx)} = \gamma(p)^{-1} \| \Sigma y_k R_k f(x) \|_{L_p(dx\, d\gamma_n(y))}$$
$$\leq (\pi/2)^{1/2} \tilde{K}_p \max\left(1, \frac{\gamma(p')}{\gamma(p)}\right) \| f \|_p .$$

Thus we obtain the right side of (1.3) with $C_p \in O(p)$ when $p \to \infty$ and $C_p \in O((p-1)^{-3/2})$ when $p \to 1$.

The left side of (1.2) follows from the right side by a well known duality argument based on (1.2).

Finally, the identity (1.5) is easy to check using Fourier transforms in the x variable.

Indeed, let $F = \tilde{H}f$. The Fourier transform (with respect to x) of $F(x,y)$ computed at $\xi \in \mathbb{R}^n$ is equal to $i\, \text{sign}(\xi(y))\, \hat{f}(\xi)$. (This follows using (1.1) for $n=1$) .

On the other hand, by symmetry $\text{sign}(\xi(y))$ is clearly orthogonal to $(\xi(y))^\perp$, hence

$$Q(\text{sign}(\xi(y)) = \alpha\, \xi(y)$$

with $\alpha = \left\langle \frac{\xi(y)}{\| \xi(y) \|_2^2} , \text{sign}(\xi(y)) \right\rangle = \frac{\| \xi(y) \|_1}{\| \xi(y) \|_2^2} = \frac{\gamma(1)}{\| \xi \|_2}$.

Since $\gamma(1) = (2/\pi)^{1/2}$, we find the Fourier transform (in x) of $Q\tilde{H}f$ equal to $i(2/\pi)^{1/2} \frac{\xi(y)}{\|\xi\|_2} \hat{f}(\xi)$. Since by (1.1) this is clearly the Fourier transform in x of the right side of (1.5), this proves (1.5).

Remark : The preceding proof is close in spirit to the proof of [DR].

§ 2. THE GAUSSIAN (OR ORNSTEIN-UHLENBECK) CASE.

Motivated by the "Malliavin calculus" and related estimates of the Ornstein-Uhlenbeck semi-group, P.A. Meyer proved the following result. Here again γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$.

THEOREM 2.1. : [M1]

If $1 < p < \infty$ there are constants K_p and K'_p such that for all n for all f in $L_p(\mathbb{R}^n, \gamma_n)$ we have

$$(2.1) \quad \frac{1}{K'_p} \| \sqrt{L} f \|_{L_p(\gamma_n)} \leq \| grad\ f \|_{L_p(\gamma_n)} \leq K_p \| \sqrt{L} f \|_{L_p(\gamma_n)},$$

where L is the generator of the Ornstein-Uhlenbeck semi-group

$$L = \Delta - \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

Note that if $h_\alpha(x) = h_{\alpha_1}(x_1)\, h_{\alpha_2}(x_2) \ldots h_{\alpha_n}(x_n)$ $(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n)$ is a Hermite polynomial in n variables, then we have

$$Lh_\alpha = -|\alpha|\, h_\alpha \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n .$$

Indeed, this is easy to check using the classical identity

$$\forall\ \xi \in \mathbb{R}^n \qquad \exp\{\xi(x) - \tfrac{1}{2} \|\xi\|_2^2\} = \sum_\alpha \frac{h_\alpha(x)}{\alpha!} \xi^\alpha .$$

We refer the reader e.g. to [N] for examples of the use of this formula.

It is well known that $(e^{-Lt})_{t>0}$ is a semi-group of positive contractions on $L_p(\gamma_n)$, $1 \leq p \leq \infty$.

The analogue of (1.2) in the Gaussian case is the following :

For all f in $L_2(\gamma_n)$ we have

$$< Lf, f >_{L_2(\gamma_n)} = \int < grad\ f, grad\ f > \gamma_n(dx)$$

hence after polarization for all g in $L_2(\gamma_n)$

$$(2.2) \quad < \sqrt{L} f, \sqrt{L} g >_{L_2(\gamma_n)} = \int < grad\ f(x), grad\ g(x) > \gamma_n(dx) .$$

This again shows that the right side of (2.1) (for p) implies the left side of (2.1) (for p') . Hence it is enough to prove the right side of (2.1).

Recently, R. Gundy gave a simpler proof of P.A. Meyer's inequality [G] based on previous work of Gundy-Varopoulos [GV]. Both Meyer's and Gundy's proofs are probabilistic.

We will give below a rather simple analytic proof based on transference of a perturbation of the one dimensional classical Hilbert transform to the n dimensional (or infinite dimensional) Gaussian case. We will again work in $\mathbb{R}^n \times \mathbb{R}^n$ but this time equipped with the measure $\gamma_n(dx) \times \gamma_n(dy)$.

This measure is invariant by the "rotations"

$$R_t(x,y) = (x \cos t + y \sin t, -x \sin t + y \cos t) .$$

Therefore $f \to f \circ R_t$ is a measure preserving group of isometries of $L_p(\gamma_n \times \gamma_n)$.

Our proof will be very similar to that given for the classical case in § 1 .

In the Gaussian case, we will replace formula (1.5) by the following formula which is the crucial point for the proof of (2.1). For any f in $L_2(\gamma_n)$ with mean zero we have :

$$(2.3)\quad Q_y \left[\text{p.v.} \int_{-\pi}^{\pi} f(x \cos t + y \sin t)\, \varphi_1(t) \frac{dt}{2\pi}\right] = 2(2\pi)^{-1/2} \sum_{i=1}^{n} y_i\, D_i\left(\frac{1}{\sqrt{L}}\, f\right)$$

where p.v. means "principal value", and where φ_1 is a function on the circle group Π which does not depend on n and satisfies

$$(2.4)\qquad \varphi_1(t) - \cot(t/2) \in L_\infty(\Pi) .$$

A fortiori $\varphi_1(t) - \cot(t/2) \in L_1(dt)$ therefore the operator T_{φ_1} (convolution by φ_1) differs from the Hilbert transform $\mathscr{H}$ (on the circle group) by an operator which is bounded on L_p for all $1 \leq p \leq \infty$. (More generally, for any Banach space B , the operator $T_{\varphi_1} - \mathscr{H}$ is bounded on $L_p(B)$ for all $1 \leq p \leq \infty$.) Therefore, the boundedness of T_{φ_1} will be an easy consequence of the boundedness of $\mathscr{H}$.

To prove the boundedness of the operation between brackets in (2.3) we will use an elementary transference argument, as follows (cf. [CW], cf. also [M2]).

LEMMA 2.2. Let $1 < p < \infty$ and let k be in $L_1(\Pi)$.
Let $T_k : L_p(\Pi) \to L_p(\Pi)$ be the operator of convolution by k , i.e. $T_k(f) = f * k$. We denote its norm on $L_p(\Pi)$ simply by $\| T_k \|_p$. Then for any g in $L_p(\gamma_n \times \gamma_n)$ we have

(2.5) $\| \int g(R_t(x,y))k(-t)dt \|_{L^p(\gamma_n(dx)\gamma_n(dy))} \leq \| T_k \|_p \| g \|_{L^p(\gamma_n \times \gamma_n)}$

In particular, for any f *in* $L_p(\gamma_n)$ *we have*

(2.6) $\| \int f(x \cos t + y \sin t)k(-t)dt \|_{L_p(\gamma_n(dx)\gamma_n(dy))} \leq \| T_k \|_p \| f \|_{L_p(\gamma_n)}$.

Proof : By successive integrations and Fubini's theorem, we have obviously

$$(2.7) \qquad \| \int g(R_{s-t})\ k(t)\ dt \|_{L_p(ds\ d\gamma_n\ d\gamma_n)} \leq \| T_k \|_p \| g \|_{L^p(\gamma_n \times \gamma_n)} .$$

but since R_s preserves the measure $\gamma_n \times \gamma_n$.

$$\| \int g(R_{s-t})\ k(t)\ dt \|_{L_p(\gamma_n \times \gamma_n)} = \| \int g(R_{-t})\ k(t)\ dt \|_{L_p(\gamma_n \times \gamma_n)}$$

hence (2.7) implies (2.5).

Moreover, (2.5) implies (2.6) by setting

$$g(x,y) = f(x) . \qquad \text{q.e.d.}$$

It is well known that Lemma 2.2 extends to singular integrals (i.e. $k \notin L_1$) by a suitable approximation. Let us consider in particular the Hilbert transform on the circle

$$\forall\ \varphi \in L_p(\Pi) \qquad \mathscr{H}\varphi(s) = \lim_{\varepsilon \to 0} \int_{|t| \geq \varepsilon} \varphi(s-t)\ \cot(\tfrac{t}{2})\ \frac{dt}{2\pi}$$

or in short

$$= \text{p.v.} \int \varphi(s-t)\ \cot(\tfrac{t}{2})\ \frac{dt}{2\pi} .$$

Actually, in our specific situation, we do not need to invoke the classical approximation of a general multiplier by a compactly supported one ([CW], see also [M2]). We use instead the following result.

LEMMA 2.3. For all f *in* $L_p(\gamma_n)$ *, the integral*

p.v. $\int f(x \cos t + y \sin t) \cot(t/2)\ dt$ *exists for a.e.* (x,y) *and we have :*

(2.8) $\| \text{p.v.} \int_{-\pi}^{\pi} f(x \cos t + y \sin t) \cot(t/2) \frac{dt}{2\pi} \|_{L_p(\gamma_n \times \gamma_n)} \leq \| \mathscr{H} \|_p \| f \|_{L_p(\gamma_n)}$.

Proof. Let $\omega(t) = x \cos t + y \sin t$. By Fubini's theorem $\int |f(\omega(t))|^p\, dt < \infty$ for a.e. (x,y) , hence the integral

$$\text{p.v.} \int_{-\pi}^{\pi} f(\omega(s-t))\ \cot(t/2)\ dt$$

exists for a.e.s.

Note that for each s $(\omega(-t))_{t\in\mathbb{R}}$ and $(\omega(s-t))_{t\in\mathbb{R}}$ have the same distribution. Therefore if we let

$$p(s) = \gamma_n \times \gamma_n \{(x,y) \mid \text{p.v.} \int f(\omega(s-t)) \cot(t/2)\, dt \text{ exists}\} ,$$

we must have $p(s)$ independent of s. But by Fubini's theorem $\int p(s)\, ds = 1$.

This implies that $p(0) = 1$, and (2.8) follows exactly as in Lemma 2.2. q.e.d.

Actually, we will work with a convolution operator T_{φ_1} which is a "perturbation" of the Hilbert transform on the circle, given by the next lemma.

<u>*LEMMA 2.4*</u>. *(i) There is an odd function* φ_1 *on* $[-\pi,\pi]$ *such that* $\varphi_1(\theta) - \cot(\theta/2) \in L_\infty(d\theta)$ *and such that*

$$(2.9) \quad \forall\, m \geq 0 \quad \int_{-\pi}^{\pi} (\cos\theta)^m\, (\sin\theta)\, \varphi_1(\theta)\, \frac{d\theta}{2\pi} = 2(2\pi)^{-1/2}\, (m+1)^{-1/2}$$

(ii) More generally, for every odd $k \geq 1$, *there is an odd function* φ_k *on* $[-\pi,\pi]$ *such that* $\varphi_k(\theta) - \cot(\theta/2) \in L_\infty(d\theta)$ *and such that*

$$(2.10) \quad \forall\, m \geq 0 \quad \int_{-\pi}^{\pi} (\cos\theta)^m\, (\sin\theta)^k\, \varphi_k(\theta)\frac{d\theta}{2\pi} = A_k(2\pi)^{-1/2}\, (m+k)^{-k/2}$$

where $A_k = 2 \int_{-\infty}^{\infty} x^{k-1} \exp - \frac{x^2}{2}\, dx/(2\pi)^{-1/2}$.

(In other words, $A_k = 2 \int x^{k-1}\, \gamma_1(dx)$*)*

<u>Proof</u>. We start from the identity

$$A_k = 4 \int_0^\infty t^{k-1}\, e^{-t^2/2}\, dt\, (2\pi)^{-1/2} .$$

Changing t to $\sqrt{m+k}\,t$, we obtain

$$(m+k)^{-k/2}\, A_k = 4 \int_0^\infty \exp -[(m+k)\, t^2/2]\, |t|^{k-1}\, dt(2\pi)^{-1/2} .$$

Again, using the change of variables defined by $\cos\theta = e^{-t^2/2}$ or equivalently setting $t(\theta) = (2 \operatorname{Log} \frac{1}{\cos\theta})^{1/2}$, we find by differentiation $-\sin\theta\, d\theta = -t(\theta)\cos\theta\, dt(\theta)$.

Hence $(2\pi)^{1/2}(m+1)^{-k/2}\, A_k = 4 \int_0^{\pi/2} (\cos\theta)^{m+k}\, t(\theta)^{k-1}\, \frac{\sin\theta}{\cos\theta}\, \frac{d\theta}{t(\theta)}$.

Therefore if we define

$$\varphi_k(\theta) = 2\left(\frac{t(\theta)}{\operatorname{tg}\theta}\right)^{k-1} \frac{\operatorname{sign}(\theta)}{t(\theta)} \quad \text{for } \theta \in [-\tfrac{\pi}{2},\tfrac{\pi}{2}]$$

and $\varphi_k(\theta) = 0$ for $\theta \in [-\pi,-\pi/2] \cup [\pi/2,\pi]$,

We obtain (2.10) as announced. In particular, we have (2.9).

If $k=1$, we have $t(\theta) \to \infty$ when $\theta \nearrow \pi/2$ hence $\varphi_1(\theta) \to 0$. Therefore the only discontinuity of φ_1 is at $\theta = 0$. But in the neighbourhood of 0 , it is easy to check that $t(\theta) \sim \theta$ therefore

$$\varphi_1(\theta) = \frac{2}{\theta} + \psi_1(\theta)$$

with $\psi_1(\theta)$ bounded in the neighbourhood of zero. This shows that $\varphi_1(\theta) - \cot(\theta/2) \in L_\infty(d\theta)$. Since $tg(\theta) \sim \theta$ when $\theta \to 0$, the function $\varphi_k(\theta) - 2/\theta$ also remains bounded in the neighbourhood of 0 . Since $\varphi_k(\theta)$ is also bounded in the neighbourhoods of $\pi/2$ or $-\pi/2$, we conclude that

$$\psi_k(\theta) = \varphi_k(\theta) - \cot(\theta/2) \in L_\infty(d\theta) .$$

This concludes the proof.

Clearly we have

<u>*COROLLARY 2.5.*</u> *Let ψ_k be as above*

$$\| T_{\varphi_k} \|_p \leq \| \mathscr{H} \|_p + \| \psi_k \|_1 .$$

In particular, we have

$$\| T_{\varphi_k} \|_p \in O(\frac{p}{p-1})$$

either when $p \to 1$ or when $p \to \infty$.

We can now give the

<u>Proof of theorem 2.1.</u> : Let f be in $L_2(\gamma_n)$. We first note the elementary formal identity

$$\frac{1}{\sqrt{L+1}} D_i f = D_i \left[\frac{1}{\sqrt{L}} f\right] \tag{2.11}$$

for smooth functions with mean zero (i.e. $\int f d\gamma_n = 0$) . Indeed, this is easy to check on Hermite polynomials. Consider for example the case $i=1$, and let $\alpha \in \mathbb{N}^n$.

We have

$$h_\alpha = h_{\alpha_1} \otimes h_{\alpha_2} \otimes \dots .$$

$$D_1 h_\alpha = \alpha_1 h_{\alpha_1 - 1} \otimes h_{\alpha_2} \otimes \dots$$

$$\frac{1}{\sqrt{L+1}} D_1 h_\alpha = \frac{\alpha_1}{\sqrt{|\alpha|}} h_{\alpha_1 - 1} \otimes h_{\alpha_2} \otimes \dots .$$

while on the other hand $D_1(\frac{1}{\sqrt{L}} h_\alpha) = D_1(\frac{1}{\sqrt{|\alpha|}} h_{\alpha_1} \otimes \dots)$
$= \frac{1}{\sqrt{|\alpha|}} \alpha_1 h_{\alpha_1 - 1} \otimes h_{\alpha_2} \otimes \dots$.
More generally, the preceding well known calculation shows that symbolically we have

$$F(L+1)D_i = D_i F(L)$$

for any "function" F .

Let us check now the crucial identity announced above as (2.3).

The proof goes through an expansion in Hermite polynomials of $f(x \cos t + y \sin t)$ as a function of y . Let us recall how to compute the coefficients of such an expansion.

Consider a function g in $L_2(\gamma_n(dy))$.

Then $g(y) = \Sigma g_\alpha h_\alpha(y)$
for coefficients $g_\alpha = \langle g, h_\alpha \rangle . (\langle h_\alpha, h_\alpha \rangle)^{-1}$ for the scalar product of $L_2(\gamma_n)$. We have $\langle h_\alpha, h_\alpha \rangle = \alpha!$ where $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$.
There is a convenient formula for g_α which will be handy in the proof, namely the following

$$g_\alpha = (\alpha!)^{-1} \left(\frac{\partial^\alpha}{\partial \xi^\alpha}\right)_{\xi=0} \int g(y+\xi) \gamma_n(dy) .$$

(Here $\frac{\partial^\alpha}{\partial \xi^\alpha}$ means $\left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial \xi_2}\right)^{\alpha_2} \dots$) .

For instance, we refer the reader to [N] p. 155.

Applying this expansion to $f(x \cos t + y \sin t)$ we obtain

$$f(x \cos t + y \sin t) = \Sigma h_\alpha(y) F_\alpha(x,t) \tag{2.12}$$

where $F_\alpha(x,t) = (\alpha!)^{-1} \left(\frac{\partial^\alpha}{\partial \xi^\alpha}\right)_{\xi=0} \int f(x \cos t + (y+\xi) \sin t) \gamma_n(dy)$

hence (at least for f sufficiently smooth

$$F_\alpha(x,t) = (\alpha!)^{-1} (\sin t)^{|\alpha|} \int (D^\alpha f)(x \cos t + y \sin t) \gamma_n(dy) . \tag{2.13}$$

We now introduce the "Mehler kernel" (or equivalently the Ornstein-Uhlenbeck semi-group). Let P_k be the orthogonal projection from $L_2(\gamma_n)$ onto the span of the Hermite polynomials of degree k in $(x_1, x_2, \dots, x_n)$. For a function $g = \Sigma g_\alpha h_\alpha$ in $L_2(\gamma_n(dy))$, we define

$$Lg = \Sigma |\alpha| g_\alpha h_\alpha \qquad (\text{i.e. } L = \sum_{m>0} m P_m)$$

for all $t \geq 0$

$$e^{-Lt} g = \Sigma\, e^{-|\alpha| t} g_\alpha h_\alpha \qquad (\text{i.e. } e^{-Lt} = \sum_{n\geq 0} e^{-mt} P_m)$$

and $\quad T(\varepsilon)g = \Sigma\, \varepsilon^{|\alpha|} g_\alpha h_\alpha \quad$ for $\varepsilon \in [-1,1]$.

Hence $T(\varepsilon) = \sum_{m\geq 0} \varepsilon^m P_m$.

We also use the classical "Mehler formula"

$$[T(\varepsilon)g](x) = \int g(\varepsilon x + (1-\varepsilon^2)^{1/2} y)\ \gamma_n(dy) \ .$$

(Note : this formula is easy to check on the total set of functions $\{\varphi_\lambda\}$ defined as

$$\forall\ \lambda, x \in \mathbb{R}^n \quad \varphi_\lambda(x) = \exp\{<\lambda, x> - \tfrac{1}{2}\ \|\lambda\|_2^2\} = \sum_\alpha \frac{h_\alpha(x)}{\alpha!}\lambda^\alpha) \ .$$

Then we can rewrite (2.13) as

$$(2.14) \qquad F_\alpha(x,t) = (\alpha\,!)^{-1}\ (\text{sint})^{|\alpha|}\ [T(\text{cost})\ D^\alpha] f(x) \ .$$

We claim that if f is mean zero and if $|\alpha| = 1$

$$(2.15) \qquad \int_{-\pi}^{\pi} F_\alpha(x,t)\ \varphi_1(t)\ \frac{dt}{2\pi} = a\ (D^\alpha \frac{1}{\sqrt{L}})\ f(x)$$

with $a = 2(2\pi)^{-1/2}$.

Indeed, let $g = D^\alpha f$. We have $T(\text{cost})g = \sum_{m\geq 0} (\text{cost})^m P_m g$. Hence by (2.9)

$$\int \text{sint}\ T(\text{cost})g\ \varphi_1(\theta)\ \frac{d\theta}{2\pi} = a\ \Sigma\ (m+1)^{-1/2}\ P_m g$$
$$= a\ (L+1)^{-1/2} g \ .$$

Using (2.11), this immediately yields (2.15).

Let us define $\tilde{F}_\alpha(x) = \int_{-\pi}^{\pi} F_\alpha(x,t)\ \varphi_1(t)\ \frac{dt}{2\pi}$.

We have clearly by (2.12)

$$\text{p.v.} \int f(x\ \text{cost} + y\ \text{sint})\ \varphi_1(t)\ \frac{dt}{2\pi} = \Sigma\ h_\alpha(y)\ \tilde{F}_\alpha(x) \ ,$$

hence by definition of Q_y

$$Q_y \left[\text{p.v.} \int f(x\ \text{cost} + y\ \text{sint})\ \varphi_1(t)\ \frac{dt}{2\pi}\right] = \sum_{\substack{\alpha \\ |\alpha|=1}} h_\alpha(y)\ \tilde{F}_\alpha(x) \ ,$$

hence by (2.15)

$$= \sum_{i=1}^{i=n} y_i\ a\ D_i\ \frac{1}{\sqrt{L}}\ f(x)$$

where D_i denotes here $\frac{\partial}{\partial x_i}$.

Thus we have checked the crucial identity (2.3).

It is now easy to conclude exactly as in section 1. We have

$$\gamma(p)\, a \, \| (\sum_i |D_i \frac{1}{\sqrt{L}} f(x)|^2)^{1/2} \|_{L_p(\gamma_n(dx))} = \| a \sum y_i D_i \frac{1}{\sqrt{L}} f(x) \|_{L_p(\gamma_n \times \gamma_n)}$$

hence by (2.3)

$$\leq \| Q \|_p \, \| \text{p.v.} \int f(x \cos t + y \sin t)\, \varphi_1(t) \frac{dt}{2\pi} \|_{L_p(\gamma_n \times \gamma_n)}$$

hence by lemma 2.3 and lemma 2.2 (or corollary 2.5)

$$\leq \| Q \|_p \, (\| \mathscr{H} \|_p + \| \psi_1 \|_1) \, \| f \|_{L_p(\gamma_n)} .$$

This yields the right side of (2.1) for every $1 < p < \infty$, and the left side follows by duality.

Remark : The preceding proof yields the following bounds for the constants in theorem 2.1 when $p \to 1$ or $p \to \infty$

$$K_p \in O(\frac{p}{(p-1)^{3/2}}) ,$$

exactly as in the first section.

This estimate is perhaps an improvement over the previous proofs of [M1] or [G].

By routine arguments, this yields

COROLLARY 2.6. There is a constant K such that for all n and all polynomials f on $\mathbb{R}^n$, we have

$$\| \operatorname{grad} f \|_{L^1(\gamma_n)} \leq K \, \| L^{1/2} f \|_{L(\operatorname{Log} L)^{3/2}(\gamma_n)}$$

$$\| L^{1/2} f \|_{L^1(\gamma_n)} \leq K \, \| \operatorname{grad} f \|_{L(\operatorname{Log} L)(\gamma_n)} ,$$

where we have denoted simply by $L(\operatorname{Log} L)^\alpha$ the norm in the Orlicz space associated to the function $\varphi(t) = t(\operatorname{Log}^+ t)^\alpha$.

Remark : We briefly consider here the case of the higher order Riesz transforms.

Let Q_k be the orthogonal projection on $L_2(\gamma_n(dx) \otimes \gamma_n(dy))$ defined by $Q_k = I \otimes P_k$ where I denotes the identity on $L_2(\gamma_n(dx))$ and where P_k is viewed as acting on $L_2(\gamma_n(dy))$. In other words, Q_k is the orthogonal projection from $L_2(\gamma_n \times \gamma_n)$ onto the space of all functions which are in the "Wiener chaos" of degree k with respect to the variable y.

Let φ_k be as above, with $k \geq 1$ an odd integer. By the same argument as for (2.3), we have

$$(2.16)\quad Q_k\left[\text{p.v.}\int_{-\pi}^{\pi} f(x\cos t + y\sin t)\,\varphi_k(t)\,\frac{dt}{2\pi}\right] = a_k \sum_{|\alpha|=k} (\alpha!)^{-1} h_\alpha(y)\, D^\alpha(L^{-k/2}f)$$

where $a_k = (2\pi)^{-1/2} A_k$.

We note for $p \geq 2$

$$(2.17)\quad \forall f \in L_p(\gamma_n(dy)) \qquad \| P_k f \|_2 \leq \| f \|_2 \leq \| f \|_p .$$

On the other hand, by the hypercontractive inequality (cf. [Gr]) we have

$$\| P_k f \|_p \leq (p-1)^{k/2} \| P_k f \|_2 \leq (p-1)^{k/2} \| f \|_2 .$$

By duality, this implies

$$(2.18)\quad \forall f \in L_{p'}(\gamma_n(dy)) \quad \| P_k f \|_2 \leq (p-1)^{k/2} \| f \|_{p'} = (p'-1)^{-k/2} \| f \|_{p'} .$$

Let us define, for all f in $L_p(\gamma_n)$ (assumed sufficiently smooth)

$$G_k(f) = \left(\sum_{|\alpha|=k} (\alpha!)^{-1} |D^\alpha (L^{-k/2} f)|^2 \right)^{1/2} .$$

Then, (2.16) implies with (2.17) and (2.18) if $1 < p < \infty$

$$(2.19)\qquad a_k \| G_k(f) \|_p \leq \| T_{\varphi_k} \|_p\, C(p,k)\, \| f \|_p$$

where $C(p,k) = \max\,\{1, (p-1)^{-k/2}\}$.

In [M1], Meyer considers the "iterated gradients"

$$\Gamma_k(f,f) = \sum_{1 \leq i_j \leq n} |D^{i_1} D^{i_2} \ldots D^{i_k} f|^2$$

Obviously $\Gamma_k(f,f) \leq \sum_{|\alpha|=k} |D^\alpha f|^2\, \alpha!$.

Hence

$$\Gamma_k(f,f)^{1/2} \leq (k\,!)^k\, G_k(f) .$$

Therefore, (2.19) implies

$$(2.20)\quad \| \Gamma_k(f,f)^{1/2} \|_p \leq K(p,k)\, \| f \|_p \quad \text{if } 1 < p < \infty ,$$

where (by corollary (2.5)) $K(p,k) = a_k^{-1}\, (\| \mathscr{H} \|_p + \| \psi_k \|_1)\, C(p,k)$. In particular, we note that for each fixed k (odd) we have $K(p,k) \in O(p)$ when $p \to \infty$.

This seems to be an improvement over [M1].

As an immediate consequence, if k is a fixed odd integer there is a constant δ_k depending only on k such that

$$\| f \|_\infty \le 1 \Rightarrow \int \exp \delta_k \, \Gamma_k(f,f)^{1/2} \, d\gamma_n \le 2 .$$

When $p \to 1$, we obtain $K(p,k) \in O((p-1)^{-1-k/2})$.

When k is an even integer, one can probably use a different calculation analogous to that done in [DR] to obtain similar inequalities, but with a weaker bound for the constants $K(p,k)$; we did not check it.

We should also point out that the inequality (2.20) can be reversed if f satisfies $(P_0 + P_1 + \dots + P_k)(f) = 0$. For a simple proof of this, we refer the reader to [M1].

Remark : If we use the Hilbert transform $\mathscr{H}$ instead of T_{φ_1}, we obtain the following inequality

$$\| \operatorname{grad} f \|_p \le B_p \| \Lambda f \|_p \qquad (1 < p < \infty) \quad (B_p = \| Q \|_p \, \|\mathscr{H}\|_p)$$

where $\Lambda = \sum_{m>0} \frac{m}{\lambda_m} P_m$, with λ_m defined as the sum of m i.i.d. random variables $\varepsilon_1, \dots, \varepsilon_m$ such that $\mathbf{P}(\varepsilon_i = 1) = \mathbf{P}(\varepsilon_i = -1) = 1/2$ (i.e. we define λ_m by $\lambda_m = \mathbf{E} \, |\Sigma_1^m \varepsilon_i|$.) Note that (by the Central limit theorem) $\lambda_m m^{-1/2} \to (2/\pi)^{1/2}$ when $m \to \infty$. This suggests that one can pass from Λ to $(\pi/2)^{1/2} L^{1/2}$ by a perturbation of the identity bounded on all L_p-spaces. Using explicit formulas for λ_m which we found in [H], we have indeed convinced ourselves that this is correct, but this proof is not so direct and elementary as the preceding one.

Remark : Theorem 2.1 clearly extends for functions f with values in a U.M.D. Banach space B in the sense of [Bu]. See [Bo] for closely related information.

The inequalities remain valid in that case provided we replace $\| \operatorname{grad} f \|_p$ by the following expression

$$\frac{1}{\gamma(p)} \left\| \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}(x) \right\|_{L_p(\gamma_n \times \gamma_n; B)} .$$

A similar remark applies of course also in the classical case of § 1.

Remark : The idea used in the proof of theorem 2.1 originates in [P] (chapter 2) where a weaker form of the left side of (2.1) is established for functions with values in an arbitrary Banach space. I am grateful to Michel Ledoux for stimulating conversations which made me return to the subject of this note.

Final Remark : It is natural to try to extend the above method to the case of compact Lie groups and Riemannian manifolds. It would be interesting to prove (under suitable assumptions) analogous inequalities with constants independent of the dimension. For instance, we refer the reader to the papers of Bakry [B1][B2] for various results of this kind.

The "main idea" of our proof carries over in a very general setting. We can describe it as follows :

Let M be a compact connected Riemannian manifold of dimension n. Let $T(M)$ be the tangent bundle. For each x in M, we consider the canonical Gaussian measure γ_x on $T_x(M)$, then we denote by μ the measure $dx\, d\gamma_x(y)$ on $T(M)$. (Here we denote by dx the normalized surface measure on M). By a classical result of Liouville, the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on $T(M)$ leaves the measure μ invariant. Hence, for any f in $L_p(M,dx)$ we may consider the function

$$(2.20) \qquad g(x,y) = \text{p.v.} \int \tilde{f}(\varphi_t(x,y)) \frac{dt}{t} \text{ , where } \tilde{f}(x,y) = f(x) \text{ .}$$

By transference, this function g belongs to $L_p(\mu)$ $(1 < p < \infty)$. Moreover, we can project $g(x,y)$ orthogonally onto the space of all functions of the form $\sum_1^n g_i(x)\, y_i$. This defines a projection Q which is bounded in $L_p(\mu)$ $(1 < p < \infty)$ for the same reason as above.

However, except in simple cases (such as the Euclidean sphere, which is known to be very similar to the Gaussian case) we have not been able to compute explicitly the resulting operator, or to modify the singular integral (2.20) in order to obtain something nice (after the action of Q), as in (2.3). Nevertheless, we believe that this should be possible.

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