SHENG-WU HE
JIA-GANG WANG

Remarks on absolute continuity, contiguity and convergence in variation of probability measures

Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 260-270

<http://www.numdam.org/item?id=SPS_1988__22__260_0>
Remarks on Absolute Continuity, Contiguity and Convergence
in Variation of Probability Measures*

by

S. W. He J. G. Wang

Let \((\Omega^n, \mathcal{F}^n), n \geq 1\) be measurable spaces with right-continuous filtrations \(\mathcal{F}^n_t = (\mathcal{F}^n_{t+})_{t \geq 0}\), and \(\mathcal{F}^n = \bigvee_{t \geq 0} \mathcal{F}^n_t\). Let \(P^n\) and \(\mathbb{P}^n\) be probability measures defined on \(\mathcal{F}^n\).

From [2]-[4], it is known that Hellinger processes are the main tools for the study of absolute continuity, contiguity and convergence in variation of probability measures. In §1, by using the results about the convergence of submartingales at infinity, we give the Lebesgue's decomposition between measures. Then the conditions for absolute continuity and singularity can be deduced immediately. These facts are easy, but they supplement the known results completely. In §2 and §3, we give new proofs of the conditions for contiguity and convergence in variation respectively. These proofs start directly from derivative processes, don't need the deeper properties of Hellinger processes. Hence, they are straightforward and can be easily followed. All results are applied to semimartingale cases.

1. Absolute Continuity

1.1 Preliminaries. We'll adopt all denotations of [1] without specification. For the sake of convenience, we always omit the index \(n\). It appears only in the case, where it is indispensable.

Set \(Q = \frac{1}{2}(P+\mathbb{P})\). Suppose that \((\Omega, \mathcal{F}, Q)\) is a complete probability space, and under \(Q\) \(\mathcal{F} = (\mathcal{F}_t)\) satisfies the usual conditions. Let \(Z\) and \(\overline{Z}\) be the derivative processes of \(P\) and \(\mathbb{P}\) with respect to \(Q\) respectively,

\[
Z = \left( E^Q \frac{dP}{dQ} | \mathcal{F}_t \right), \quad \overline{Z} = \left( E^Q \frac{d\mathbb{P}}{dQ} | \mathcal{F}_t \right),
\]

\[
T_k = \inf \{ t : Z_t \leq 1/k \}, \quad \overline{T}_k = \inf \{ t : \overline{Z}_t \leq 1/k \},
\]

\[
T = \sup_k T_k = \inf \{ t : Z_t = 0 \}, \quad \overline{T} = \sup_k \overline{T}_k = \inf \{ t : \overline{Z}_t = 0 \},
\]

* Research supported by National Natural Science Foundation of China.
Denote by \( \mu \) the jump measure of \( Z \), by \( \nu \) the compensator of \( \mu \) under \( Q \):
\[
\nu = \mu_{P,Q}.
\]
Set
\[
\lambda = 1 + \frac{x}{Z}, \quad \tilde{\lambda} = 1 - \frac{x}{\tilde{Z}},
\]
then \( \lambda, \tilde{\lambda} \in [0, \infty) \), and (see [3])
\[
\mu_{P,P} = \lambda \nu, \quad \mu_{P,\tilde{P}} = \tilde{\lambda} \nu.
\]

(Oviously, \( 1_{x<0} \nu = 1_{\tilde{x}<0} \nu = 0 \).

From now on all discussions proceed under the probability measure \( Q \), unless otherwise specified. We have
\[
Z = Z_c + x(\mu - \nu), \quad \tilde{Z} = \tilde{Z}_c - x(\mu - \nu),
\]
\[
Z + \tilde{Z} = 2, \quad Z_c + \tilde{Z}_c = 2, \quad \Delta Z + \Delta \tilde{Z} = 0.
\]
The Hellinger process (with index 1/2) of \( P \) and \( P \) is
\[
H = \frac{1}{8} (1/Z_c + 1/\tilde{Z}_c)^2 Z_c + \frac{1}{2} (\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 \nu
\]
and
\[
1 (\Gamma \cap \tilde{\Gamma})^c \cdot H = 0.
\]

1.2. Theorem. Set \( N = \{ S < \infty \} \) or \( H_\infty = \infty \). Then
(1) \( \tilde{\mathcal{F}} \perp P \) on \( N \),
(2) \( \tilde{\mathcal{F}} \sim P \) on \( N^c \).

Proof. We have (see [1])
\[
\{ Z_{\infty} > 0 \} = \{ \tilde{Z}_{\infty} > 0 \} = \{ \tilde{T} = \infty, \tilde{\lambda} \}, \quad \frac{1}{2} Z_c^2 + (1 - \sqrt{\tilde{\lambda}})^2 \nu \}
\]
(1.2)
\[
\{ \tilde{Z}_{\infty} > 0 \} = \{ \tilde{T} = \infty, \tilde{\lambda} \}, \quad \frac{1}{2} \tilde{Z}_c^2 + (1 - \sqrt{\lambda})^2 \nu \}
\]
(1.3)
Since 1 is between \( \sqrt{\lambda} \) and \( \sqrt{\tilde{\lambda}} \),
\[
(1 - \sqrt{\lambda})^2 \leq (\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2, \quad (1 - \sqrt{\tilde{\lambda}})^2 \leq (\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2
\]
Comparing (1.1) with (1.2) and (1.3), we get
\[
\{ S = \infty, H_{\infty} < \infty \} = \{ Z_{\infty} > 0, \tilde{Z}_{\infty} > 0 \}
\]
(1.4)
\[
\{ S < \infty \text{ or } H_{\infty} = \infty \} = \{ Z_{\infty} = 0 \} \cup \{ \tilde{Z}_{\infty} = 0 \}
\]
(1.5)
But \( P(Z_{\infty} = 0) = \tilde{\mathcal{F}}(\tilde{Z}_{\infty} = 0) = 0 \). The conclusions follow from (1.4) and (1.5).

1.3. Corollary. ([2],[3]) \( \tilde{\mathcal{F}} \ll P \) iff
(i) \( \tilde{\mathcal{F}} \ll P_0 \),
(ii) \( \tilde{\mathcal{F}}(H_{\infty} < \infty) = 1 \),
(iii) \( \tilde{\mathcal{F}}(1_{\lambda = 0} \nu = 0) = 1 \).
Proof. Since $\widetilde{P} \ll P$ iff $P(N) = 0$, but

$$P(N) = \widetilde{P}( T < \infty \text{ or } H_{\infty} = \infty)$$

and

$$\{T < \infty \} \cup \{ H_{\infty} = \infty \} = \{ T = 0 \} \cup \{ 0 < T < \infty, H_{T} < \infty \} \cup \{ H_{\infty} = \infty \}.$$  

Also notice that

$$\{ T = 0 \} = \{ Z_{0} = 0 \},$$

$$\{ 0 < T < \infty, H_{T} < \infty \} = \{ 0 < T < \infty, Z_{T} > 0 \} = \{ \lambda_{0} = 0 * \mu_{\infty} > 0 \}.$$ 

Hence

$$\{ T < \infty \} \cup \{ H_{\infty} = \infty \} = \{ Z_{0} = 0 \} \cup \{ \lambda_{0} = 0 * \mu_{\infty} > 0 \} \cup \{ H_{\infty} = \infty \},$$

but

$$\widetilde{P}( \lambda_{0} = 0 * \mu_{\infty} > 0 ) = 0 \iff \widetilde{E}(1_{\lambda_{0} = 0} * \mu_{\infty}) = 0 \iff \widetilde{E}(1_{\lambda_{0} = 0} * \mu_{\infty}) = 0 \iff \widetilde{E}(1_{\lambda_{0} = 0} * \mu_{\infty} > 0 ) = 0,$$

therefore the Corollary holds.

1.4. Remark. The condition (iii) in Corollary 1.3 is equivalent to the following

$$(iii') \forall A \in \mathcal{F}, 1_{A} * \mu_{\infty} = 0 \text{ a.s. } \widetilde{P} \Rightarrow 1_{A} * \mu_{\infty} = 0 \text{ a.s. } \widetilde{P}.$$ 

Proof. (iii) $\Rightarrow$ (iii'). If $A \in \mathcal{F}$ and $1_{A} * \mu_{\infty} = 1_{A} * \mu_{\infty} = 0 \text{ a.s. } \widetilde{P}$, then

$$1_{A}(\lambda > 0) * \nu_{\infty} = 0 \text{ a.s. } \widetilde{P}.$$ 

By (iii)

$$1_{A} * \mu_{\infty} = 1_{A} * \mu_{\infty} = 1_{A}(\lambda > 0) * \mu_{\infty} = [\lambda * (1_{A}(\lambda > 0) * \mu_{\infty})]_{\infty} = 0 \text{ a.s. } \widetilde{P}.$$ 

(iii') $\Rightarrow$ (iii). Obviously, we have $\lambda_{0} * \mu_{\infty} = 0$. By (iii')

$$\lambda_{0} * \mu_{\infty} = 0 \text{ a.s. } \widetilde{P}.$$ 

Since $Z_{-} + \lambda_{0} * \mu_{\infty} = 2$, from (1.6) we get

$$\lambda_{0} * \mu_{\infty} = 0 \text{ a.s. } \widetilde{P}.$$ 

1.5. Corollary. $\widetilde{P}$ is $\mathcal{F}$-martingale iff

$$\widetilde{P}( Z_{0} = 0 \text{ or } H_{\infty} = \infty \text{ or } 1_{\lambda_{0} = 0} * \mu_{\infty} > 0 ) = 1.$$ 

Proof. It is sufficient to notice that $\widetilde{P}$ is $\mathcal{F}$-martingale if $\widetilde{P}(N) = 1$ and similarly to the proof of Corollary 1.3. we have $\widetilde{P}(N) = \widetilde{P}( Z_{0} = 0 \text{ or } H_{\infty} = \infty \text{ or } 1_{\lambda_{0} = 0} * \mu_{\infty} > 0 )$, hence the Corollary holds.

1.6. Application to semimartingales. Suppose that $\widetilde{Q}$ is a probability measures on $\mathcal{F}$.
such that $P \ll \tilde{Q}$ and $\tilde{P} \ll \tilde{Q}$. ($\tilde{Q}$ is not necessarily $Q$, but $Q \ll \tilde{Q}$. This is the difference from the assumption in [5].) Suppose $X = (X_t)_{t \geq 0}$ is a semimartingale under $\tilde{Q}$ (and so under $P$ and $\tilde{P}$). The predictable characteristics of $X$ under $P$, $\tilde{P}$ and $\tilde{Q}$ are $(B,C,\nu)$, $(\tilde{B},\tilde{C},\tilde{\nu})$ and $(\tilde{B},\tilde{C},\tilde{\nu})$ respectively, and

$$\nu = \tilde{\nu}, \quad \tilde{C} = \tilde{\nu}, \quad \tilde{P}, \tilde{\nu} \in P^e$$

Set $a = (a_t, \tilde{a}_t) = (\tilde{a}_t)$ and $\tilde{a} = (a_t):$

$$a_t = \nu([t]_{\tilde{Q}}), \quad \tilde{a}_t = \tilde{\nu}([t]_{\tilde{Q}}), \quad \tilde{a}_t = \tilde{\nu}([t]_{\tilde{Q}}).$$

Define

$$\tau = \inf \{ t: \text{Var}_t(B-B_t + |x|_{[t]_{\tilde{Q}}})^{1/2} \text{Var} (\nu - \tilde{\nu})_t = \infty \}$$

$$A_t = \begin{cases} 0, & \tau = 0, \\ \tilde{B}_t - B_t - x_1 |x|_{[t]_{\tilde{Q}}} \text{Var} (\nu - \tilde{\nu})_t, & t < \tau, \tau > 0, \\ +\infty, & t > \tau, \tau > 0. \end{cases}$$

$$K = \frac{dA}{d\tilde{C}} \in P.$$  

(If on $[0, t]$ $A$ is not absolutely continuous with respect to $\tilde{C}$, $K_t = +\infty$.)

$$N = \{ Z_0 \tilde{Z}_0 = 0 \} \cup \{ C = \tilde{C} \} \cup \{ \tau = 0 \} \cup \{ K^2 C_{\infty} + (L - L)2* \tilde{\nu}_\infty + S_{\infty}(\sqrt{1 - a_1} - \sqrt{1 - \tilde{a}_1})^2 = \infty \} \cup \{ (I_{\tilde{P} = 0} \mu_X)_{\infty} > 0 \} \cup \{ S_{\infty}(\Delta X = 0, a_1 = 1 \text{ or } \tilde{a}_1 = 1) > 0 \},$$

where $\mu_X$ is the jump measure of $X$.

Suppose that under $\tilde{Q}$ the derivative processes of $P$ and $\tilde{P}$ with respect to $\tilde{Q}$ (still denoted by $Z$ and $\tilde{Z}$) have predictable representation:

$$Z = L_{\tilde{C}}X_{\tilde{C}} + W*(\mu_{\tilde{C}} - \tilde{\nu}), \quad \tilde{Z} = \tilde{L}_X \tilde{C} + \tilde{W}*(\mu_X - \tilde{\nu}) \quad (1.7)$$

where $L, \tilde{L} \in P, W, \tilde{W} \in P^e$. Applying Theorem 1.2, we have

(1) $\tilde{P} \perp P$ on $N$,

(2) $\tilde{P} \sim P$ on $N^c$.

The conclusion (1) about singularity needn't the assumption of predictable representation (1.7). But the conclusion (2) need it in order to represent the Hellinger process as

$$H = 1_{\tilde{C} \leq 0} \frac{1}{2} (\tilde{C}^2 - \tilde{C}) \tilde{C} + \frac{1}{2} (L - \tilde{L})^2* \tilde{\nu} + \frac{1}{2} \tilde{S}(\sqrt{1 - a} - \sqrt{1 - \tilde{a}})^2.$$.  


2. Contiguity

2.1. \((\tilde{P}^n)\) is contiguous to \((P^n)\), if \(\forall A^n \in \mathcal{F}_{\tilde{P}}^n\)

\[ p^n(A^n) \to 0 \Rightarrow \tilde{P}^n(A^n) \to 0 \]

and denoted by \((\tilde{P}^n) \triangleq (P^n)\). The main result on contiguity is the following \([2],[3]\)

2.2. Theorem. \((\tilde{P}^n) \triangleq (P^n)\) iff

(i) \((\tilde{P}^n_0) \triangleq (P^n_0)\)

(ii) \(\lim_{n \to \infty} \lim_{N \to \infty} P^n(\mathcal{N}_0, N > n) = 0\),

(iii) \(\forall \theta > 0, \lim_{n \to \infty} \lim_{N \to \infty} P^n(\mathcal{N}_0, N > n) = 0\),

where \(P^n_0 (P^n_0)\) is the restriction of \(P^n (P^n)\) on \(F_0\), and \(i^n(N) = (X^n_{(N0, N2) \cap X^n, N2, N})\).

The proof of necessity, given in [2], is already very simple, needn't improving further. We'll give another proof for sufficiency. Our proof is based on the following lemma, as in [3]. But the procedure after that is greatly simpler than that in [3].

2.3. Lemma. \([3]\) \((\tilde{P}^n) \triangleq (P^n)\) iff

\[ \lim_{n \to \infty} \lim_{N \to \infty} \lim_{k \to \infty} P^n(\mathcal{N}_0, N_k > N) = 0 \]  

\[ \text{(Denote by } Z^\ast \text{ the supremum process of } Z: Z^\ast_t = \text{sup}\{Z_{s} \mid s \leq t\}.) \]

2.4. The proof of sufficiency. From exponential formula, on \[0, S_k\],

\[ \tilde{Z}/Z = \tilde{Z}^\ast_{0}/Z^\ast_0 \exp \{ (1/\tilde{Z}^\ast_-, \tilde{Z} - \frac{1}{2} (1/\tilde{Z}^\ast_-) < Z^\ast^-) + S((1 + \Delta Z/\tilde{Z}^\ast_- - \Delta Z/\tilde{Z}^\ast_-))\}

\[ - (1/\tilde{Z}^\ast_-) \tilde{Z} + \frac{1}{2} (1/\tilde{Z}^\ast_-) < Z^\ast^-) - S((1 + \Delta Z/\tilde{Z}^\ast_- - \Delta Z/\tilde{Z}^\ast_-))\}

\[ = \tilde{Z}^\ast_{0}/Z^\ast_0 \exp \{ -(1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) \tilde{Z} - \frac{1}{2} (1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) < Z^\ast^-) +

+ (\tilde{X} - \lambda)^\ast (\mu - \nu) + (1/\log \tilde{X} - (\tilde{X} - 1)) \ast \mu - (\log \lambda - (\lambda - 1)) \ast \mu\}

\[ = \tilde{Z}^\ast_{0}/Z^\ast_0 \exp \{ -(1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) \tilde{Z} + \frac{1}{2} (1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) < Z^\ast^-) +

+ (\tilde{X} - \lambda)^\ast (\mu - \nu) + (1/\log \tilde{X} - (\tilde{X} - \lambda)) \ast \mu\}

\[ = Z^\ast_{0}/Z^\ast_0 \exp \{ A + B \} \]

where \(Z^\ast_{C, \tilde{P}} = Z^\ast - 1/\tilde{Z}^\ast_-, < Z^\ast_-, Z^\ast_+ = Z^\ast + 1/\tilde{Z}^\ast_-, < Z^\ast^-\) is the continuous local martingale part of \(Z\) under \(\tilde{P}\). Set \(\bar{\mu} = \lambda/\tilde{X}, \bar{\mu} = \mu', \bar{\mu} = \mu, \) and \(0 < \alpha < 1\) is a constant. Then

\[ A = -(1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) \tilde{Z} + \frac{1}{2} (1/\tilde{Z}^\ast_- + 1/\tilde{Z}^\ast_+) < Z^\ast^-) < Z^\ast^-\]

\[ B = (\tilde{X} - \lambda)^\ast (\mu - \nu) + (1/\log \tilde{X} - (\tilde{X} - \lambda)) \ast \mu\]

\[ = 1/|\tilde{P} - 1| > b \log \frac{1}{\tilde{P}} \ast \mu + 1/|\tilde{P} - 1| > b (\tilde{P} - 1) \ast \tilde{Z} +

+ 1/|\tilde{P} - 1| < b \log \frac{1}{\tilde{P}} \ast (\mu - \tilde{Z}) + 1/|\tilde{P} - 1| < b (\log \frac{1}{\tilde{P}} - (1 - \tilde{\mu})) \ast \tilde{Z}\]  

\[ = B^1 + B^2 + B^3 + B^4 \]
In the sequel, we'll discuss under $\mathcal{P}$, and estimate (2.2) term by term in order to get (2.1).

1° since $(\mathcal{P}_0^n) \subset (\mathcal{P}_0^n)$, we have
\[
\lim_{N \to \infty} \lim_{n \to \infty} \mathcal{P}_0^n (Z_0^n / Z_0^n \geq N) = 0 \tag{2.4}
\]

2° \(\mathcal{P}(Z^c, \mathcal{P}) = \mathcal{P}(Z^c)\), and by Lenglart's inequality
\[
\mathcal{P}(\mathcal{P}((1/Z_\cdot + 1/z_\cdot) \cdot Z^c, \mathcal{P}) \not\geq N) \leq L/N^2 + \mathcal{P}(8H_{\infty} \geq L)
\]
\[
\mathcal{P}(Z_0^c \not\geq N) \leq L/N^2 + \mathcal{P}(8H_{\infty} \geq L) + \mathcal{P}(4H_{\infty} \geq N)
\]

Let $k \to \infty$, $n \to \infty$, $N \to \infty$, $L \to \infty$ successively, we get
\[
\lim_{N \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \mathcal{P}(Z_0^c \not\geq 2N) = 0 \tag{2.5}
\]

3° Using $|\log(1+x)| < |x|/(1-|x|)$ for $|x|<1$, we have
\[
\mathcal{P}((\mathcal{B}_3)_k^c \not\geq N) \leq L/N^2 + \mathcal{P}(C_b H_{\infty} \geq L)
\]

Set $k \to \infty$, $n \to \infty$, $N \to \infty$, $L \to \infty$ successively, we get
\[
\lim_{N \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \mathcal{P}(B_3^c)_{sk}^c \not\geq N = 0 \tag{2.6}
\]

4° Using $|\log(1+x)| < x^2/(2(1-|x|))$ for $|x|<1$, we have
\[
|B^4_k| \leq \mathcal{P}(B^4_k \not\geq N) \leq \mathcal{P}(C_b H_{\infty} \geq N)
\]

Hence
\[
\lim_{N \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \mathcal{P}(B^4_k)_{sk}^c \not\geq N = 0 \tag{2.7}
\]

5°
\[
|B^2_k| \leq \mathcal{P}(B^2_k \not\geq N) \leq \mathcal{P}(C_b H_{\infty} \geq N)
\]

Hence
\[
\lim_{n \to \infty} \lim_{k \to \infty} P_{B_k}^n(\lim_{k \to \infty} (B_k^n, 2^n)_{S_k}^{N(n)} = 0)
\]  
\[(2.9)\]

6° Take \(0 < \delta < 1 - b\)

\[
B_k^1 \leq 1_{\delta < \varphi < 1 - b} \left( \log \frac{1}{\delta} \right)^{\mu} + 1_{\varphi \leq \delta} \log \left( \log \frac{1}{\varphi} \right)^{\mu} = (1_{\delta < \varphi < 1 - b})^{\mu} + (1_{\varphi \leq \delta} \log \frac{1}{\varphi})^{\mu}
\]

\[
E_k(\left( B_k^1 \right)^{\mu} > N) \leq E_k((1_{\delta < \varphi < 1 - b})^{\mu} + \log \frac{N}{\log N} + E_k((1_{\varphi \leq \delta} \log \frac{1}{\varphi})^{\mu} < 0) \) (2.10)
\]

\[
1_{\delta < \varphi < 1 - b} \left( \log \frac{1}{\delta} \right)^{\mu} \leq 1_{\delta < \varphi < 1 - b} \left( \log \frac{1}{\delta} \right)^{\mu} + \log \left( \log \frac{1}{\varphi} \right)^{\mu} < \log \left( \log \frac{1}{\delta} \right)^{\mu} \leq \log \left( \log \frac{1}{\delta} \right)^{\mu} \leq C_b h
\]

By Lenglart's inequality

\[
E_k((1_{\delta < \varphi < 1 - b})^{\mu} \leq \log \left( \log \frac{1}{\delta} \right)^{\mu} + \log \left( \log \frac{1}{\varphi} \right)^{\mu} \leq C_b h
\]

\[(2.11)\]

On the other hand,

\[
1_{\delta < \varphi} \sim \frac{1}{k} \sum_{\lambda < \varphi} 1 + 1_{\varphi < \delta} \sim \varphi \sim \varphi \sim \varphi
\]

\[
\leq 1_{\varphi < \delta} \sim \varphi \sim \varphi \sim \varphi \sim \varphi \sim \varphi \sim \varphi
\]

\[
\leq 1_{\delta < \varphi < 1 - b} \left( \log \frac{1}{\delta} \right)^{\mu} + \log \left( \log \frac{1}{\varphi} \right)^{\mu} < \log \left( \log \frac{1}{\delta} \right)^{\mu} \leq \log \left( \log \frac{1}{\delta} \right)^{\mu} \leq C_b h
\]

\[(2.12)\]

Notice that \(\mu\) is integer-valued, again by Lenglart's inequality

\[
E_k((1_{\delta < \varphi < 1 - b})^{\mu} \leq \log \left( \log \frac{1}{\delta} \right)^{\mu} + \log \left( \log \frac{1}{\varphi} \right)^{\mu} \leq \log \left( \log \frac{1}{\delta} \right)^{\mu} \leq C_b h
\]

\[(2.13)\]

From (2.10) - (2.13) we get

\[
E_k((e^{B_k^1})^{\mu} > N) \leq L \log \frac{1}{\log N} + E_k(C_b h, \log N) + \log \left( \log \frac{1}{\delta} \right)^{\mu} + E_k((e^{B_k^1})^{\mu} > N)
\]

\[
= \eta + E_k((1_{\delta < \varphi < 1 - b})^{\mu} \leq \eta
\]

\[(2.14)\]

Set \(k \to \infty, n \to \infty, N \to \infty, L \to \infty, \delta \to 0, K \to \infty, \eta \to 0\) successively, we get

\[
\lim_{n \to \infty} \lim_{k \to \infty} E_k((e^{B_k^1})^{\mu} > N) = 0
\]

\[(2.14)\]

then (2.1) follows from (2.2) - (2.7), (2.9) and (2.14).

2.5. Remark. In Theorem 2.2 the condition (iii) can be substituted by

(iii') \(\forall A^n e \in \mathbb{R}^n
\]

\[
A^n e \to 0 \text{ in } (P^n) \Rightarrow A^n e \to 0 \text{ in } (P^n).
\]

Proof. (iii) \(\Rightarrow\) (iii')

\[
\lambda \delta \mu^n \leq \lambda \delta \mu^n + 1_{\lambda \delta \mu^n > \lambda \delta \mu^n} \sim \lambda \delta \mu^n \leq \lambda \delta \mu^n + 1_{\lambda \delta \mu^n > \lambda \delta \mu^n} \sim \lambda \delta \mu^n
\]

(iii') \(\Rightarrow\) (iii).
Hence for each sequence $N_n \to \infty$, take $A_n = \{N_n<\infty\}$, we have $
_{\infty}(N_n) \to 0$ in $(\tilde{P}_n)$, therefore (iii) holds.

2.6. Application to semimartingales. We make the assumptions, as in 1.6. Applying Theorem 2.2, we have the conclusion (see [5]):

$(\tilde{P}_n) \not\subseteq (P^n)$ iff

(i) $(\tilde{P}_n) \not\subseteq (P^n)$,

(ii) $\lim_{N \to \infty} \lim_{n \to \infty} \tilde{P}_n((P^n) \cdot \tilde{P}_n + (\sqrt{P} - \sqrt{\tilde{P}})^2 \cdot \tilde{P}_n + S_{\infty}(\sqrt{1-a_n} - \sqrt{1-\tilde{a}_n})^2 \geq N) = 0$,

(iii) $\forall \gamma > 0$,

$\lim_{N \to \infty} \lim_{n \to \infty} \tilde{P}_n((1_{n<\infty} \cdot \tilde{P}_n + S_{\infty}((1-\tilde{a}_n \cdot (1-a_n) < 1-a_n) \geq \gamma)) = 0$.

In fact, we have

$I(N) = \int_{\tilde{f}} \tilde{f}(1_{N < \infty} \cdot \tilde{P}_n + S((1-\tilde{a}_n \cdot (1-a_n) < (1-a_n)))$.

Similar to Remark 2.5, (iii) can be substituted by following

(iii') (a) $\forall A_n \in \tilde{P}_n$

$1_{A_n} \cdot \tilde{P}_n \to 0$ in $(\tilde{P}_n) \Rightarrow 1_{A_n} \cdot \tilde{P}_n \to 0$ in $(\tilde{P}_n)$,

(b) $\forall A_n \in P^n$

$(1_{A_n} \cdot S((1-a_n)1_{\tilde{a}_n > 0}))_{\infty} \to 0$ in $(\tilde{P}_n)$

$\Rightarrow (1_{A_n} \cdot S((1-a_n)1_{a_n > 0}))_{\infty} \to 0$ in $(\tilde{P}_n)$.

3. Convergence in Variation

3.1. Lemma. The following statements are equivalent:

(1) $\|P^n - \tilde{P}_n\| \to 0$.

(2) $(Z^n - 1)_{\infty} \to 0$ in $(P^n)$.

(3) $(\gamma^n - 1)_{\infty} \to 0$ in $(P^n)$, where $\gamma^n = \sqrt{Z^n} Z^n$.

Proof. Since $\|P^n - \tilde{P}_n\| = E^n[Z^n - Z^n_{\infty}] = 2E^n[Z^n_{\infty} - 1]$, $|Z^n_{\infty} - 1| \leq 1$, so $\|P^n - \tilde{P}_n\| \to 0 \iff |Z^n_{\infty} - 1| \to 0$ in $(Q^n)$.

(1) $\Rightarrow$ (2) By maximal inequality of martingales, for $\varepsilon > 0$

$Q^n((Z^n - 1)_{\infty} \geq \varepsilon) \leq \frac{1}{\varepsilon} E^n[Z^n_{\infty}^2 - 1]$.

Hence, $(Z^n - 1)_{\infty} \to 0$ in $(Q^n)$ and $(P^n)$. 


(2) \( \Rightarrow \) (1). Obviously, \( \varphi^n - 1 \to 0 \) in \( \mathbb{P}^n \). For given \( \varepsilon > 0 \), and \( 0 < \delta < \varepsilon < 1 \),
\[
\mathbb{Q}^n \left( |\varphi^n - 1| < \varepsilon \right) \geq \int_{|\varphi^n - 1| < \varepsilon} \frac{1}{\varphi^n} \, d\mathbb{P}^n \geq \frac{1}{(1+\delta)} \mathbb{P}^n \left( |\varphi^n - 1| < \varepsilon \right)
\]
Set \( n \to \infty \), \( \delta \to 0 \) successively, we get \( Z^n_{\infty} - 1 \to 0 \) in \( \mathbb{Q}^n \).

Note that \( 1 - (\gamma^n)^2 = (1 - Z^n)^2 \) and \( 0 \leq \gamma^n \leq 1 \), we have
\[
(1 - \gamma^n)^* \leq (1 - Z^n)^* \leq 2(1 - \gamma^n)^*.
\]
(2) \( \Leftrightarrow \) (3) follows.

3.2 Theorem([4]). The following statements are equivalent:

(1) \( \varphi^n - n \to 0 \).

(2) (a) \( \varphi^n - 0 \to 0 \),

(b) \( H_n^\infty - 1 \to 0 \) in \( \mathbb{Q}^n \).

(3) (a) \( \varphi^n - 0 \to 0 \),

(b) \( H_n^\infty - 1 \to 0 \) in \( \mathbb{P}^n \).

Proof. (1) \( \Rightarrow \) (2). (a) is trivial. Suppose that the Doob-Meyer decomposition

of \( \gamma^n = M^n + A^n \)

is
\[
\gamma^n = \gamma^n + M^n - A^n
\]

where \( M^n \) is a martingale with \( M_0^n = 0 \), \( A^n = \gamma^n - H^n \). By Lemma 3.1, \( (\gamma^n - \gamma^n)^* \to 0 \) in 
\( \mathbb{Q}^n \). \( A^n \) is dominated by \( (\gamma^n - \gamma^n)^* \). \( \Delta(\gamma^n - \gamma^n)^* \leq |\Delta \gamma^n| \leq 1 \). By Lenglart's inequality, we have \( A_\infty^n \to 0 \) in \( \mathbb{Q}^n \). On \( \{ \inf \gamma^n_t < \frac{1}{2} \} \),
\[
H_n^\infty = (1/\gamma^n - 1) \times A_0^n + A_\infty^n \leq 2(\gamma^n - 1) \times A_\infty^n + A_\infty^n.
\]
On \( \{ \inf \gamma^n_t < \frac{1}{2} \} \), \( (\gamma^n - 1)^* \geq \frac{1}{2} \). Therefore, \( \forall \varepsilon > 0 \)
\[
\mathbb{Q}^n \left( H_\infty^n \geq \varepsilon \right) \leq \mathbb{Q}^n \left( (\gamma^n - 1)^* \geq \frac{1}{2} \right) + \mathbb{Q}^n \left( 2(\gamma^n - 1)^* + 1 \right) \times \varepsilon
\]
Hence, \( H_\infty^n \to 0 \) in \( \mathbb{Q}^n \). (2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1). At first, observe that
\[
2H_\infty^n \geq (\lambda - \overline{\lambda})^2 \times \nu_\infty \geq \lambda (\sqrt{\lambda - \lambda})^2 \times \nu_\infty
\]
\[
(1/\lambda - 1)^2 \times 1_{N_\lambda \times \lambda} \times \nu_\infty
\]
Hence, \( 1_{N_\lambda \times \lambda} \times \nu_\infty \to 0 \) in \( \mathbb{P}^n \). Applying Theorem 2.2, we have
\[
\mathbb{P}^n \in (\mathbb{P}^n). \text{ Since } \mathbb{P}^n(T^n_k < \infty) \leq 1/k, \mathbb{P}^n(S^n_k < \infty) \leq 1/k,
\]
\[
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}^n(S^n_k < \infty) = 0
\]
Now define \( L = 1/\gamma^n(X) = 1/\gamma^n \cdot M - H \). Using Itô's formula. on \( \gamma^n \cap \Gamma \) we get
\[
(1/Y_\cdot M = \frac{1}{2}((1/Z_\cdot Z + (1/Z_\cdot Z)) - \frac{1}{2}(\bar{1}/\bar{X} - \bar{1}/\bar{X})^2(\nu - \mu))
\]
\[
= \frac{1}{2}(1/Z_\cdot - 1/Z_\cdot)Z^{c, P} + (\sqrt{\bar{1}/\bar{X} - 1})*1 - \mu, P, P + \\
+ \frac{1}{2}(1/Z_\cdot - 1/Z_\cdot)(1/Z_\cdot)Z^{c, P} + (\sqrt{\bar{1}/\bar{X} - 1})*\nu \quad (3.3)
\]
where \(Z^{c, P}\) is the continuous local martingale part of \(Z\) under \(P\). It is easy to see, on \([0, T]\),
\[
\frac{1}{2}(1/Z_\cdot - 1/Z_\cdot)1/Z_\cdot Z^{c, P} + (\sqrt{\bar{1}/\bar{X} - 1})*1 - \mu, P, P \leq \frac{1}{2}(1/Z_\cdot + 1/Z_\cdot)^2. <Z^c> \leq 8H \quad (3.4)
\]
Under \(P\) we have
\[
\frac{1}{2}(1/Z_\cdot - 1/Z_\cdot)Z^{c, P} + (\sqrt{\bar{1}/\bar{X} - 1})*1 - \mu, P, P \leq \frac{1}{2}(1/Z_\cdot + 1/Z_\cdot)^2. <Z^c> + (2k+ 1)(\sqrt{\bar{1}/\bar{X} - 1})*\nu \leq 2(1+2k)H \quad (3.5)
\]
From (3.3) - (3.6) and Lenglart's inequality, we obtain
\[
(S_n)_{S_k} \rightarrow 0 \quad \text{in } (P^n) \quad (3.7)
\]
By exponential formula, on \([0, T]\),
\[
Y = Y_0 \exp \left\{ L + S(\log(1 + AL) - AL) \right\}
\]
Note that \(0 < x - \log(1+x) < x^2\) for \(|x| < \frac{1}{2}\), and \(AL = \sqrt{(1+2Z/Z_\cdot)(1+2Z_\cdot/Z)} - 1\) we have
\[
0 \leq \sum (AL - \log(1 + AL)) \leq \sum (AL)^2 \quad |AL| < \frac{1}{2} \\
\leq (\sqrt{\bar{1}/\bar{X} - 1})*\mu_\infty < (\sqrt{\bar{1}/\bar{X}})^2*\mu_\infty
\]
Since \((\sqrt{\bar{1}/\bar{X}})^2*\mu_\infty < 2(1+2k)H, using Lenglart's inequality again, we obtain
\[
(S(AL^n - \log(1 + AL^n)) 1_{\left|AL\right| < \frac{1}{2}})_{S_k} \rightarrow 0 \quad \text{in } (P^n) \quad (3.8)
\]
\(\forall \varepsilon > 0, \quad (S(1 + AL - AL))_{S_k} \rightarrow \varepsilon \}
\]
\[
\subseteq (S(1 + AL - AL))_{S_k} \rightarrow \varepsilon \}
\]
According to (3.7), \((AL^n)_{S_k} \rightarrow 0 \quad \text{in } (P^n), and from (3.8), (3.9) we get
\[
(L^n)_{S_k} \rightarrow 0 \quad \text{in } (P^n) \quad (3.10)
\]
By (1), \(Y_0^n - 1 \rightarrow 0 \quad \text{in } (P^n), now
\[
Y^n - 1 = Y_0^n - 1 + \exp(L^n + S(\log(1 + AL^n) - AL^n)) - 1 \}
\]
and from (3.10) we have
(\(Y^n - 1\)\(^*\))_{n}^{\infty} \to 0 \text{ in } (P^n) \quad (3.11)

For given \(\varepsilon > 0\),
\[
P^n((Y^n - 1)_{\infty}^{\infty} \geq \varepsilon) \leq P^n(S^n_{\infty} < \infty) + P^n((Y^n - 1)_{\infty}^{\infty} \geq \varepsilon)
\]

Set \(n \to \infty\) and \(k \to \infty\) successively, from (3.11) and (3.2) we know
\[n \to \infty \text{ and } k \to \infty \text{ successively}, \text{ from (3.11) and (3.2) we know} \]

At last, \(\|P^n - P^m\| \to 0\) follows from Lemma 3.1.

References


He Sheng Wu  
Department of Mathematical Statistics  
East China Normal University  
Shanghai, China  

Wang Jia Gang  
Institute of Mathematics  
Fudan University  
Shanghai, China