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Extending Lévy's characterisation of Brownian motion

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Let $f(x,t)$ be a solution of the heat equation. Then if B_t is a real Brownian motion, a simple application of Ito's formula shows that $f(B_t,t)$ is a local martingale. Paul Lévy gave a converse of this using the parabolic function $f(x,t) = x^2 - t$, namely he showed that every continuous local martingale X_t for which $X_t^2 - t$ is also a local martingale has to be a Brownian motion. It is natural to ask if there exist other parabolic functions with this property. The short answer is yes; it is the rule, not the exception.

The purpose of this note is to prove the following extension of Lévy's characterisation. In the statement we use the notation m for Lebesgue measure on the line, and if D is measurable we write $m \perp D$ to mean that $m(D) = 0$.

Proposition Let X_t be a continuous local martingale which verifies the following condition.

(A) There exists a solution $f(x,t)$ of the heat equation $f_t + \frac{1}{2}f_{xx} = 0$ for which

1. The process $f(X_t,t)$ is a local martingale

   and

2. $m \perp \{t : f_t(X_t,t) = 0\}$.

Then X_t must be a Brownian motion.

The above question was originally posed, by K.R. Parthasarathy, in the context of the particular class of parabolic functions known as the Hermite polynomials

$H_n[x,t]$. These are defined by generating function

$$\exp\left(\gamma x - \frac{\gamma^2}{2}t\right) = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} H_n[x,t],$$

so when we take $f(x,t) = H_2[x,t] = x^2 - t$ then our result is precisely Lévy's characterisation (A.2 is now trivial).

The key is the following. It depends on the fact that every solution of the heat equation is analytic when $t > 0$. Remark how we do not require A_t to be continuous.

Observation Take f a solution of the heat equation such that $f_t \neq 0$. Then if B_t is a Brownian motion and A_t an adapted process of bounded variation we have $m \perp \{t : f_t(B_t, A_t) = 0\}$.

Proof: First recall 'if Y_t is a semimartingale then $\langle Y^c \rangle_t \perp \{t : Y_t = a\}$' (this is a direct consequence of the occupation density formula $\int g(a) L_t^a da = \int_0^t g(Y_s) d\langle Y^c \rangle_s$ for semimartingale local time L_t^a). Applying this to $Y_t = f_t(B_t, A_t)$, we know

$$\int_0^t f_{xt}^2(B_s, A_s) ds \perp \{t : f_t(B_t, A_t) = 0\}$$

which means $\{t : f_t(B_t, A_t) = 0\} \subseteq \{t : f_{xt}(B_t, A_t) = 0\}$ modulo m. Assuming $m \not\perp \{t : f_t(B_t, A_t) = 0\}$ and arguing inductively shows there exists at least one point where all the space derivatives of f_t vanish. By analyticity $f_t \equiv 0$, a contradiction.

This result is interesting in itself. For example it shows that A.2 is true for Brownian motion (A.1 is then automatic). The proof of the proposition now follows readily.

Proof: By Lévy's characterisation it suffices to prove $\langle X \rangle_t = t$. However Ito's formula and A.1 imply the process $\int_0^t f_s(X_s, s) d(\langle X \rangle_s - ds)$ is null. This means that $\int_0^t 1_{(f_s(X_s,s) \neq 0)}(d\langle X \rangle_s - ds) \equiv 0$; in particular by A.2 $\langle X \rangle_t$ is continuous strictly increasing, as is its inverse τ_t. Our claim $\langle X \rangle_t = t$ is now equivalent to $\langle X \rangle_t \perp \{t : f_t(X_t, t) = 0\}$, which by time-changing is just the statement $m \perp \{t : f_t(X_{\tau_t}, \tau_t) = 0\}$, this being true by our observation since X_{τ_t} is a Brownian motion.

Remarks (1) P.A. Meyer offered the suggestion that we look at general parabolic functions.

(2) Condition A.2 is rather interesting, and seems to arise naturally in this context. It may well be useful for describing other properties of the Brownian path.

(3) A.2 is essential. Note that if $f(x,t) = H_n[x,t]$, $n \geq 3$ and odd, then A.1 holds with $X_t = B_{t\wedge\tau}$ and τ any stopping time for which $B_\tau \equiv 0$.

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