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A maximal inequality for martingale local times

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1. Introduction

Let $M$ and $N$ be continuous local martingales, let $\hat{M}$, $\hat{N}$ denote $M-M_0$ and $N-N_0$ respectively, and let $L^a_t(M)$, $L^a_t(N)$ denote the local times of $M$ and $N$ respectively.

It was shown in [3] that

$$K_p \sup_{a, \tau} \sup_{t} \left| L^a_t(M) - L^a_t(N) \right| \overset{p}{\leq} \left\| \hat{M} - \hat{N} \right\|_\infty,$$

or equivalently,

$$C_p \sup_{a, \tau} \sup_{t} \left| L^a_t(M) - L^a_t(N) \right| \overset{p}{\geq} \left\| \hat{M} - \hat{N} \right\|_\infty,$$

(1.1)

for all $p \in (0, \infty)$, whilst Barlow and Yor established in [2] that

$$\left| \sup_{a, \tau} \sup_{t} \left| L^a_t(M) - L^a_t(N) \right| \right| \overset{p}{\leq}
C_p \left\| (M-N) \right\|_\infty \left| M^* + N^* \right|_p^{1/2} \left( 1 + 1 \ln \left\{ \left\| (M-N) \right\|_\infty \right\} \right)^{1/2}.$$

In this note we prove the following:

**Theorem 1** For all $p \in (1, \infty)$ there is a universal constant $c_p$ such that for all continuous martingales $M, N \in \mathcal{M}$
2. Some preliminaries. We recall some properties of local times.

For a continuous semi-martingale \((X_t; t \geq 0)\) we may define (c.f. [1]) its family of local times by means of Tanaka's formula:

\[
|X_{t \downarrow} - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s) dX_s + L^a_t(X)
\]

where

\[
\text{sgn}(x) = \begin{cases} 
1 & : x > 0 \\
-1 & : x \leq 0
\end{cases}
\]

Note that \(L^a_t(X)\) is increasing in \(t\) and increases only on \(\{t: X_t = a\}\) (c.f. [4]).

Furthermore it has been shown in [5] that if \(X\) is a continuous local martingale then \(L^a_t(X)\) has a bi-continuous version and we shall assume, without loss of generality, that we are working with such a version.

To simplify notation we fix \(M\) and \(N\), two continuous martingales, and their filtration \((F_t; t \geq 0)\) and define

\[
U(a, t) = (L^a_t(M) - L^a_t(N))
\]

\[
A_t = \sup_a (L^a_t(M) - L^a_t(N)) = \sup_a U(a, t)
\]
\[ B_t = \sup_{a} (L^a_t(N) - L^a_t(M)) = -\inf_{a} U(a,t) \]
\[ D_t = \sup_a |L^a_t(M) - L^a_t(N)| \]

and for any \((X_t; t\geq 0)\)

\[ X_t^* = \sup_{s\leq t} |X_s|, \quad X_t = X_t - X_0. \]

3. **Proof of Theorem 1.** The crucial result is contained in the following lemma:

**Lemma 2** Define

\[ \sigma_x = \inf\{t \geq 0 : A_t \geq 2x\} \]
\[ \tau_x = \inf\{t \geq \sigma_x : U(M_t, t) \leq x\} \]

where, as is usual \(\inf\phi\) is taken as \(\infty:\) then, if \(M\) and \(N\) are in \(H^1\)

\[ \mathbb{E}\left[(2(M-N)_{\tau_x} + A_{\tau_x})I(\sigma_x < \infty, \tau_x = \infty)\right] \geq x IP(\sigma_x < \infty) \quad (3.1) \]

**Proof** It was shown in [3] that \(A_t\) is continuous, so on \(\sigma_x\), \(A_{\tau_x} = 2x\).

Now \(M\) and \(N\) are in \(H^1\) so \(M_{\sigma_x}, N_{\sigma_x} < \infty\) a.s., so a.s. \(U(a, \sigma_x)\) is zero off a compact set (since \(L^a_t(X)\) only increases when \(X\) is at a) and continuous and we may conclude that \(\sup_a U(a, \sigma_x)\) is attained.

We may deduce that, on \((\sigma_x < \infty)\), \(\sup_a U(a, \sigma_x)\) is attained at \(b = M_{\sigma_x}\), for, suppose not, then \(\exists b \neq M_{\sigma_x} \) s.t. \(2x = U(b, \sigma_x) > U(b, t)\) for all \(t < \sigma_x\) but,
since \( b \neq M \), \( \exists t < \sigma \) s.t. \( L_t^b(M) = L_t^b(M) \) whilst (since \( L_t^b(N) \) is increasing in \( s \)) \( L_t^b(N) \leq L_t^b(N) \) so that \( U(b, t) \geq U(b, \sigma) \) which contradicts the definition of \( \sigma \). We conclude that, on \( (\sigma \wedge \omega) \), \( U(M, \sigma) = 2x \) whilst \( M \) is in \( H^1 \) so has a limit variable \( M_\infty \) and so

\[
\mathbb{E}[U(M, \sigma) - U(M, \tau)] = \mathbb{E}[(2x - U(M, \tau))I(\sigma < \omega)]
\]  
(3.2)

(since \( \tau > \sigma \) so, on \( (\sigma \wedge \omega) \), \( \sigma = \tau = \omega \)).

Similarly, we may see that, on \( (\tau \wedge \omega) \), \( U(M, \sigma) = 2x \) so that (3.2) is

\[
\mathbb{E}[2xI(\sigma < \omega) - xI(\tau \wedge \omega) - U(M, \tau)]I(\sigma < \omega, \tau = \omega)
\]
(3.3)

Conversely, (3.2) is

\[
\mathbb{E}(L_{x}^N(M) - L_{x}^N(M)) = (L_{x}^N(M) - L_{x}^N(M))
\]
(3.4)

Applying Tanaka's formula to the two \( (F_{\sigma + t} : t \geq 0) \) martingales, \( m_t = M_{x + t} \) and \( n_t = N_{x + t} \), we obtain the formulae

\[
L_{x}^N(M) - L_{x}^N(M) = L_{x}^N(m)
\]

\[
= |M_{\sigma} - M_{\sigma}| + \int_{\sigma}^{x} \operatorname{sgn}(M_{s} - M_{\sigma})dM_{s}
\]
(3.5.i)

\[
L_{x}^N(N) - L_{x}^N(N) = L_{x}^N(n)
\]

\[
= |N_{\sigma} - N_{\sigma}| - |N_{\sigma} - N_{\sigma}|
\]

\[
+ \int_{\sigma}^{x} \operatorname{sgn}(N_{s} - N_{\sigma})dN_{s}
\]
(3.5.ii)
Now $M$ and $N$ are in $H^1$ and $|\text{sgn}(x)| = 1$ so the two stochastic integrals in (3.5) are uniformly integrable and so we may apply the optional sampling theorem to obtain:

\[
\begin{align*}
\mathbb{E}[L^X_x(M) - L^X_x(M)] &= \mathbb{E} \left| \frac{M_{\sigma_x}}{M_{\sigma_x}} \right| \quad (3.6.i) \\
\mathbb{E}[L^X_x(N) - L^X_x(N)] &= \mathbb{E} \left( |N_{\sigma_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| \right) \quad (3.6.ii)
\end{align*}
\]

Substituting equations (3.6) in (3.4), and equating (3.2), (3.3) and (3.4) we see that

\[
\mathbb{E}[2xI_{(\sigma_x < \omega)} - xI_{(\tau_x < \omega)} - U(M_{\sigma_x}, \tau_x)I_{(\sigma_x \leq \tau_x \leq \omega)}] = \mathbb{E} \left[ |N_{\sigma_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| \right] \quad (3.7)
\]

Now, by a similar argument to that given above, we may see that, on $(\tau_x < \omega), N_{\tau_x} = M_{\sigma_x}$, so on $(\tau_x < \omega)$ the term inside the expectation on the RHS of (3.7) is non-positive whilst on $(\sigma_x = \omega)$ it disappears so that the RHS is dominated by

\[
\mathbb{E}[U(M_{\sigma_x}, \tau_x) + 2(\bar{M} - N)^{\omega}_{\sigma_x} I_{(\sigma_x < \omega, \tau_x = \omega)}]
\]

Observing that $|X_{\omega}| - |X_{\sigma_x}| \leq 2x^\omega$ and rearranging terms in (3.7) we achieve the inequality:

\[
\mathbb{E}[U(M_{\sigma_x}, \tau_x) + 2(\bar{M} - N)^{\omega}_{\sigma_x} I_{(\sigma_x < \omega, \tau_x = \omega)}] \\
\geq 2x \mathbb{P}(\sigma_x < \omega) - x \mathbb{P}(\tau_x < \omega) \quad (3.8)
\]
All that remains, to complete the proof, is to see that, since
\[ x \geq \sigma_x, \quad \mathbb{P}(\tau < \infty) \leq \mathbb{P}(\sigma < \infty), \] whilst on \( (\tau = \infty) \)
\[ U(M, \tau_x) = U(M, \infty) \leq A_\infty. \]

**Lemma 3**  If \( M \) and \( N \) are martingales in \( H^1 \)

\[ \mathbb{E}(2(M-N)_\infty^* + A_\infty) I_{(A_\infty \geq x)} \geq x \mathbb{P}(A_\infty \geq 2x) \] \hspace{1cm} (3.9)

**Proof**  On \( (\sigma < \infty, \tau = \infty) \), \( A_\infty \geq x \) whilst \( (\sigma < \infty) \) \( (A_\infty \geq 2x) \) so (3.9) follows immediately from (3.1).

We may now establish the theorem:

**Proof of the theorem:** multiplying both sides of (3.9) by \( px^{p-2} \) and integrating with respect to \( x \) we obtain, by Fubini's theorem:

\[ \frac{p}{p-1} \mathbb{E}(2(M-N)_\infty^* + A_\infty) A_\infty^{p-1} \geq \mathbb{E}(A_\infty^*)^{p/2p} \] \hspace{1cm} (3.10)_A

whilst reversing the roles of \( M \) and \( N \) in (3.9) we obtain:

\[ \frac{p}{p-1} \mathbb{E}(2(M-N)_\infty^* + B_\infty) B_\infty^{p-1} \geq \mathbb{E}(B_\infty^*)^{p/2p} \] \hspace{1cm} (3.10)_B

Clearly \( D = A \vee B \), so that, since \( A \) and \( B \) are non-negative,

\[ 2D^p \geq A^p + B^p \geq D^p. \]

Thus, adding (3.10)_A and (3.10)_B,
Applying Holder's inequality to the first term on the left, we obtain,

$$\frac{2^{p+1}}{(p-1)^p} \mathbb{E}[(2(M-N)^* + D^*)D^{p-1}] \geq \mathbb{E}(D^*)^p/2^p$$  \hspace{1cm} (3.11)$$

Now, by (1.1), \(\|\hat{(M-N)^*}\|_p \leq c_p \|D^*\|_p\), so substituting this inequality in (3.11):

$$\frac{2^{p+1}}{(p-1)^p} \left(\|D^*\|_p^p + 2c_p \|D^*\|_p \|D^\|_p^{p-1}\right) \geq \|D^*\|_p^p, \hspace{1cm} (3.12)$$

and dividing both sides of (3.12) by \(\|D^\|_p^p\) we obtain the result that

$$\|D^\|_p \leq K_p \|D^\|_p$$

where \(K_p\) is the largest zero of

$$f_p(x) = x^p - \frac{2^{p+1}}{(p-1)^p} (2c_p x+1)$$

\hspace{1cm} \Box

**Corollary 4** If \(M\) is in \(\mathcal{H}^1\) then for all \(p \in (1, \infty)\), \(\text{ac} \ \mathbb{R}\)

$$\|\hat{(M-M_0)^*}\|_p \leq K_p \inf_{a} \sup_{x \in \mathbb{R}} \|L^a(M) - L_{\infty}^a(M)\|$$

This follows immediately from theorem 1 and (1.1) by setting \(N = x-M\).
Remarks

(1) Theorem 8 of [1] enables us to extend the range of $p$ in Theorem 1 to $(1, \infty]$.

(2) Corollary 4 is a specific case of the more general result that

$$||\widehat{(M-N)_{\infty}}||_p \leq K \inf_p \left\{ \sup_{x \in \mathbb{R}} |L^a_{\infty}(M) - L^a_{\infty}(N)| \right\}.$$ 

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References


