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HOMOGENEOUS CHAOS REVISITED

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Let $(\theta, H, \#)$ be an abstract Wiener space. That is: θ is a separable real Banach space with norm $\|\cdot\|_\theta$; H is a separable real Hilbert space with norm $\|\cdot\|_H$; $H \subseteq \theta$, $\|h\|_\theta \leq C\|h\|_H$ for some $C < \infty$ and all $h \in H$, and H is $\|\cdot\|_\theta$ -dense in θ ; and $\#$ is the probability measure on $(\theta, \mathcal{B}_\theta)$ with the property that, for each $\ell \in \theta^*$, $\theta \in \theta \rightarrow \langle \ell, \theta \rangle$ under $\#$ is a Gaussian random variable with mean zero and variance $\|\ell\|_H^2 \equiv \sup\{\langle \ell, h \rangle^2 : h \in H \text{ with } \|h\|_H = 1\}$.

Let $\{\ell^k : k \in \mathbb{Z}^+\} \subseteq \theta^*$ be an orthonormal basis in H ; set

$\mathcal{A} = \{\alpha \in \mathcal{N}^{\mathbb{Z}^+} : |\alpha| = \sum_{k \in \mathbb{Z}^+} \alpha_k < \infty\}$; and for $\alpha \in \mathcal{A}$, define

$$\mathcal{H}_\alpha(\theta) = \prod_{k \in \mathbb{Z}^+} H_{\alpha_k}(\langle \ell^k, \theta \rangle), \quad \theta \in \theta,$$

where

$$H_m(\xi) = (-1)^m e^{\xi^2/2} \frac{d^m}{d\xi^m} (e^{-\xi^2/2}), \quad m \in \mathcal{N} \text{ and } \xi \in \mathbb{R}^1.$$

Then, $\{(\alpha!)^{-1/2} \mathcal{H}_\alpha : \alpha \in \mathcal{A}\}$ is an orthonormal basis in $L^2(\#)$.

Moreover, if, for $m \in \mathcal{N}$,

$$Z^{(m)} \equiv \overline{\text{span}\{\mathcal{H}_\alpha : |\alpha| = m\}} L^2(\#),$$

then: $Z^{(m)}$ is independent of the particular choice of the orthonormal basis $\{\ell^k : k \in \mathbb{Z}^+\}$; $Z^{(m)} \perp Z^{(n)}$ for $m \neq n$; and

$$L^2(\#) = \bigoplus_{m=0}^{\infty} Z^{(m)}. \quad \text{These facts were first proved by N. Wiener [6]}$$

and constitute the foundations on which his theory of homogeneous chaos is based.

The purpose of the present article is to explain how, for given $\phi \in L^2(\#)$, one can compute the orthogonal projection $\Pi_{Z^{(m)}} \phi$ of ϕ onto $Z^{(m)}$. In order to describe the procedure, it will be necessary to describe the elementary Sobolev theory associated with $(\theta, H, \#)$.

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To this end, let Y be a separable real Hilbert space and set $\mathcal{P}(Y) = \text{span}\{\#_\alpha y : \alpha \in \mathcal{A} \text{ and } y \in Y\}$. Then $\mathcal{P}(Y)$ is dense in $L^2(\mathcal{H}; Y)$. Next, for $m \in \mathcal{N}$ and $\Phi \in \mathcal{P}(Y)$, define $\theta \rightarrow D^m \Phi(\theta) \in H^{\otimes m} \otimes Y$ by

$$\begin{aligned} & (D^m \Phi(\theta), h^1 \otimes \dots \otimes h^m \otimes y)_{H^{\otimes m} \otimes Y} \\ &= \frac{\partial^m}{\partial t_1 \dots \partial t_m} (\Phi(\theta + \sum_{j=1}^m t_j h^j), y)_Y \Big|_{t_1 = \dots = t_m = 0} \end{aligned}$$

for $h^1, \dots, h^m \in H$ and $y \in Y$. Then D^m maps $\mathcal{P}(Y)$ into $\mathcal{P}(H^{\otimes m} \otimes Y)$ and $D^n = D^m \circ D^{n-m}$ for $0 \leq m \leq n$. Associated with the operator $D^m : \mathcal{P}(Y) \rightarrow \mathcal{P}(H^{\otimes m} \otimes Y)$ is its adjoint operator ∂^m . Using the Cameron-Martin formula [1], one can easily prove the following lemma.

(1) Lemma: The operator ∂^m does not depend on the choice of orthonormal basis $\{\ell^k : k \in Z^+\}$, $\mathcal{P}(H^{\otimes m} \otimes Y) \subseteq \text{Dom}(\partial^m)$, and $\partial^m : \mathcal{P}(H^{\otimes m} \otimes Y) \rightarrow \mathcal{P}(Y)$. Moreover, if $m \in Z^+$, $K = (k_1, \dots, k_m) \in (Z^+)^m$, and $\ell^K = \ell^{k_1} \otimes \dots \otimes \ell^{k_m}$, then

$$(2) \quad \partial^m \ell^K = \#_{\alpha(K)}$$

where $\alpha(K)$ is the element of $\mathcal{A}$ defined by

$$(\alpha(K))_k = \text{card}\{1 \leq j \leq m : k_j = k\}, \quad k \in Z^+.$$

In particular, $H^{\otimes m} \subseteq \text{Dom}(\partial^m)$.

Since ∂^m is densely defined, it has a well-defined adjoint $(\partial^m)^*$. Set $\mathcal{W}_m^2(Y) = \text{Dom}((\partial^m)^*)$ and use $\|\cdot\|_{\mathcal{W}_m^2(Y)}$ to denote the associated graph norm on $\mathcal{W}_m^2(Y)$. The following lemma is an easy application of inequalities proved by M and P. Kree [3].

(3) Lemma: $\mathcal{W}_m^2(H^{\otimes m} \otimes Y) \subseteq \text{Dom}(\partial^m)$, $\|\partial^m \psi\|_{L^2(\mathcal{H}; Y)} \leq C_m \|\psi\|_{\mathcal{W}_m^2(H^{\otimes m} \otimes Y)}$, and $\partial^m = ((\partial^m)^*)^*$. Moreover, $\mathcal{P}(Y)$ is $\|\cdot\|_{\mathcal{W}_m^2(Y)}$ -dense in $\mathcal{W}_m^2(Y)$.

Finally, $W_{m+1}^2(Y) \subseteq W_m^2(Y)$ and $\|\cdot\|_{W_m^2(Y)} \leq C_m \|\cdot\|_{W_{m+1}^2(Y)}$ for all $m \geq 0$.

Warning: In view of the preceding, the use of D^m to denote its own closure $(\partial^m)^*$ is only a mild abuse of notation. Because it simplifies the notation, this abuse of notation will be used throughout what follows.

Now set $W_{-m}^2(Y) = W_m^2(Y)^*$, $m \geq 0$, and $W_{\infty}^2(Y) = \bigcap_{m=0}^{\infty} W_m^2(Y)$. Then, when $W_{\infty}^2(Y)$ is given the Fréchet topology determined by $\{\|\cdot\|_{W_m^2(Y)} : m \geq 0\}$, $(W_{\infty}^2(Y))^*$ is $W_{-\infty}^2(Y) \equiv \bigcup_{m=0}^{\infty} W_{-m}^2(Y)$. Moreover, $L^2(\mathcal{W}; Y)$ becomes a subspace of $W_{-\infty}^2(Y)$ when $\phi \in L^2(\mathcal{W}; Y)$ is identified with the linear functional $\psi \in W_{\infty}^2(Y) \rightarrow E^{\#}[(\phi, \psi)_Y]$; and in this way $W_{\infty}^2(Y)$ becomes a

dense subspace of $W_{-\infty}^2(Y)$. Finally, D^m has a unique continuous extension as a map from $W_{-\infty}^2(Y)$ into $W_{-\infty}^2(H^{\otimes m} \otimes Y)$. In particular, for $T \in W_{-\infty}^2(R^1)$, there is a unique $D^m T(1) \in H^{\otimes m}$ defined by:

$$(4) \quad (D^m T(1), h)_{H^{\otimes m}} = T(\partial^m h), \quad h \in H^{\otimes m}.$$

Note that when $\phi \in W_{\infty}^2(R^1)$,

$$(5) \quad D^m \phi(1) = E^{\#}[D^m \phi].$$

(6) Theorem: Let $\phi \in L^2(\mathcal{W})$ be given. Then, for each $m \geq 0$:

$$(7) \quad \pi_{Z(m)} \phi = \frac{1}{m!} \partial^m (D^m \phi(1)).$$

Hence,

$$(8) \quad \phi = \sum_{m=0}^{\infty} \frac{1}{m!} \partial^m (D^m \phi(1)).$$

In particular, when $\phi \in W_{\infty}^2(R^1)$:

$$(7') \quad \pi_{Z(m)} \phi = \frac{1}{m!} \partial^m E^{\#}[D^m \phi]$$

and

$$(8') \quad \phi = \sum_{m=0}^{\infty} \frac{1}{m!} \partial^m E^{\#}[D^m \phi].$$

Proof: Simply observe that, by Lemma (1):

$$\begin{aligned} \partial^m(D^m \Phi(1)) &= \sum_{K \in (Z^+)^m} E^{\#}[\Phi \partial^m e^K] \partial^m e^K \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha} E^{\#}[\Phi \#_{\alpha}] \#_{\alpha} = m! \pi_{Z(m)} \Phi \end{aligned}$$

The classic abstract Wiener space is the Wiener space associated with a Brownian motion on R^1 . Namely, define $H_1(R^1)$ and $\theta(R^1)$ to be, respectively, the completion of $C_0^{\infty}((0, \infty); R^1)$ with respect to

$$\|\psi\|_{H_1(R^1)} \equiv \left(\int_0^{\infty} |\psi'(t)|^2 dt \right)^{1/2}$$

and

$$\|\psi\|_{\theta(R^1)} \equiv \sup_{t \geq 0} \frac{1}{1+t} |\theta(t)|.$$

Then Wiener's famous existence theorem shows that there is a probability measure on $\theta(R^1)$ such that $(\theta(R^1), H_1(R^1), \#)$ is an abstract Wiener space. For $(\theta(R^1), H_1(R^1), \#)$, K. Itô [2] showed how to cast Wiener's theory of homogeneous chaos in a particularly appealing form. To be precise, set $\square_m = [0, \infty)^m$; and, for $f \in L^2(\square_m)$, define

$$\begin{aligned} \int_{\square_m} f d^m \theta &= \sum_{\sigma \in \Pi_m} \int_0^{\infty} d\theta(t_m) \int_0^{t_m-1} d\theta(t_{m-2}) \dots \\ &\quad \int_0^{t_2} f(t_{\sigma(1)}, \dots, t_{\sigma(m)}) d\theta(t_1) \end{aligned}$$

where Π_m denotes the permutation group on $\{1, \dots, m\}$ and the $d\theta(t)$ -integrals are taken in the sense of Itô. What Itô discovered is that, for given $\phi \in L^2(\#)$, there exists a unique symmetric $f_{\phi}^{(m)} \in L^2(\square_m)$ such that

$$(9) \quad \pi_{Z(m)} \phi = \frac{1}{m!} \int_{\square_m} f_{\phi}^{(m)} d^m \theta$$

In order to interpret Itô's result in terms of Theorem (5), let $\{\psi^k : k \in Z^+\} \subseteq C_0^{\infty}((0, \infty); R^1)$ be an orthonormal basis in $L^2(\square_1)$ and

define $\ell^k \in \square (R^1)^*$ by $\ell^k(dt) = (\int_0^t \psi^k(s)ds)dt$. Then $\langle \ell^k, \theta \rangle = \int_{\square_1} \psi^k d^1\theta$. Moreover, by using, on the one hand, the generating function for the Hermite polynomials and, on the other hand, the uniqueness of solutions to linear stochastic integral equations (cf. H. P. McKean [5]), one finds that for $K = (k_1, \dots, k_m) \in (Z^+)^m$:

$$\int_{\square_m} \psi^K d^m\theta = \mathcal{H}_{\alpha(K)}$$

where $\psi^K = \psi^{k_1} \otimes \dots \otimes \psi^{k_m}$ and $\alpha(K) \in \mathcal{A}$ is defined as in Lemma (1).

Hence, by Lemma (1):

$$(10) \quad \partial^m \ell^K = \int_{\square_m} \psi^K d^m\theta, \quad K \in (Z^+)^m.$$

Finally, for $(t_1, \dots, t_m) \in \square_m$, define $h_{(t_1, \dots, t_m)}(s_1, \dots, s_m) =$

$(s_1 \wedge t_1) \dots (s_m \wedge t_m)$. Then, for each $h \in H_1(R^1)^{\otimes m}$, there is a unique $h' \in L^2(\square_m)$ such that $(h, h_{(t_1, \dots, t_m)})_{H_1(R^1)^{\otimes m}}$

$\int_0^t \dots \int_0^r h'(s_1, \dots, s_m) ds_1, \dots, ds_m$ for all $(t_1, \dots, t_m) \in \square_m$

(11) Theorem: Given $\Phi \in L^2(\mathcal{H})$ and $m \geq 1$, the $f_{\Phi}^{(m)}$ in (9) is $(D^m \Phi(1))'$.

Proof: By (9):

$$\begin{aligned} \partial^m (D^m \Phi(1)) &= \partial^m \left[\sum_{K \in (Z^+)^m} (D^m \Phi(1), \ell^K)_{H_1(R^1)^{\otimes m}} \ell^K \right] \\ &= \sum_{K \in (Z^+)^m} ((D^m \Phi(1))', \psi^K)_{L^2(\square_m)} \int_{\square_m} \psi^K d^m\theta \\ &= \int_{\square_m} (D^m \Phi(1))' d^m\theta. \end{aligned}$$

Thus, by (7):

$$\pi_{Z(m)} = \frac{1}{m!} \int_{\square_m} (D^m \Phi(1))' d^m\theta.$$

(12) Remark: It is intuitively clear that the $f_{\Phi}^{(m)}$ in (9) must be given by $f_{\Phi}^{(m)}(t_1, \dots, t_m) = E^{\#}[\Phi \dot{\theta}(t_1) \dots \dot{\theta}(t_m)]$, where $\dot{\theta}(t)$ is white noise. What Theorem (11) does is provide a rigorous meaning for this equation.

(13) Remark: Given $d \geq 2$, define $H_1(R^d)$ and $\theta(R^d)$ by analogy with $H_1(R^1)$ and $\theta(R^1)$. Then $(\theta(R^d), H_1(R^d), \#)$ becomes an abstract Wiener space when $\#$ is the Wiener measure associated with the Brownian motion in R^d . To provide an Itô interpretation in this case, let $\{\psi^k : k \in Z^+\} \subset C_0^\infty((0, \infty); R^1)$ be chosen as before and set $\ell^{(k, i)} = \psi^k e_i$, $k \in Z^+$ and $i \in \mathcal{D} \equiv \{1, \dots, d\}$, where $\{e_1, \dots, e_d\}$ is a standard basis for R^d . Next, for $f = \sum_{I \in \mathcal{D}^m} f_I e_I \in L^2(\square_1; (R^d)^{\otimes m})$,

define

$$\int_{\square_m} f d^m \theta = \sum_{I \in \mathcal{D}^m} \int_{\square_m} f_I d^m \theta_I$$

where

$$\int_{\square_m} f_I d^m \theta_I = \sum_{\sigma \in \Pi_m} \int_0^\infty d\theta_{i_m}(t_m) \int_0^{t_m} d\theta_{i_{m-1}}(t_{m-1}) \dots \int_0^{t_2} f_I(t_{\sigma(1)}, \dots, t_{\sigma(m)}) d\theta_{i_1}(t_1)$$

for $I = (i_1, \dots, i_m) \in \mathcal{D}^m$. One can then check that

$$\partial^m \ell^{(K, I)} = \int_{\square_m} \psi^K d^m \theta_I.$$

Finally, after associating with each $h \in H_1(R^d)^{\otimes m}$ the unique $h' \in L^2(\square_1^m; (R^d)^{\otimes m})$ satisfying

$$h(t_1, \dots, t_m) = \int_0^{t_m} \int_0^{t_1} h'(s_1, \dots, s_m) ds_1 \dots ds_m,$$

we again arrive at the equation

$$\pi_{Z(m)} \Phi = \int_{\square_m} (D^m \Phi(1))' d^m \theta.$$

(14) Remark: Theorem (11) is little more than an exercise in formalism unless $\Phi \in W_\infty^2(R^1)$. Fortunately, many interesting functions are in $W_\infty^2(R^1)$. For example, let $\sigma : R^1 \rightarrow R^1$ and $b : R^1 \rightarrow R^1$ be smooth functions having bounded first derivatives and slowly increasing derivatives of all orders. Define $X(\cdot, x)$, $x \in R^1$, to be the solution to

$$X(T, x) = x + \int_0^T \sigma(X(t, x)) d\theta(t) + \int_0^T b(X(t, x)) dt, \quad T \geq 0.$$

Then, for each $(T, x) \in (0, \infty) \times \mathbb{R}^1$, $X(T, x) \in W_{\infty}^2(\mathbb{R}^1)$. In fact, $DX(\cdot, x)$ satisfies:

$$DX(T, x) = \int_0^T \sigma'(X(t, x)) DX(t, x) d\theta(t) + \int_0^T b'(X(t, x)) DX(t, x) dt \\ + \int_0^{\cdot \wedge T} \sigma(X(t, x)) dt;$$

an equation which can be easily solved by the method of variation of parameters. Moreover, $D^m X(T, x)$, $m \geq 2$, can be found by iteration of the preceding.

(15) Remark: In many ways, the present paper should be viewed as an outgrowth of P. Malliavin's note [4]. Indeed, it was only after reading Malliavin's note that the ideas developed here occurred to the present author.

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