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PREDICTABLE LOCAL TIMES AND EXIT SYSTEMS

Haya Kaspi
Department of Industrial Engineering
Technion, Haifa 32000
ISRAEL

Bernard Maisonneuve
I. M. S. S.
47-X 38040 Grenoble Cedex
FRANCE

1. INTRODUCTION.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the canonical realization of a Hunt semi-group (P_t) on a state space $(E, \mathcal{E})$ and let M be the closure of the random set $\{t > 0 : X_t \in B\}$, where B is in $\mathcal{E}$. We set $R = \inf\{t>0 : t \in M\} = \inf\{t>0 : X_t \in B\}$. If M has no isolated point a.s., the predictable additive functional with 1-potential $P^\cdot(e^{-R})$ is a local time of M (the set of its increase points is M a.s. by [5], p.66). This restriction on M is essential, as proved by the following example of Azéma. Consider a process which stays at 0 for an exponential time and then jumps to 1 and moves to the right with speed 1. For $B = \{1\}$, R is totally inaccessible and $M = \{R\}$ cannot have a predictable local time.

One can always define an optional local time for M, as recalled in section 2. One unpleasant feature of such a local time is that it may jump at times t where $X_t \notin \bar{B}$, so that the associated time changed process is not necessarily $\bar{B}$ valued. Nevertheless, one can construct a local time which avoids this unpleasant feature by using the methods of [4] (see Remark 2). Here we shall give a direct construction by taking the $(\mathcal{F}_{D_t})$ dual predictable projection of the process Λ_t of § 2, where as usual

$$D_t = \inf\{s>t : s \in M\} .$$

We shall also prove the existence of a related $(\mathcal{F}_{D_t})$ predictable exit system in full generality, whereas the existence of an $(\mathcal{F}_t)$ predictable exit system requires some special assumptions as noted by Getoor and Sharpe [2] (see V of [8] for sufficient conditions). From this one can deduce conditioning formulae like in the optional case ([8]).

2. THE $(\mathcal{F}_{D_t})$ PREDICTABLE LOCAL TIME.

Let X be like previously and let M be an optional random closed set, homogeneous in $(0,\infty)$ and such that $M = \overline{M\backslash\{0\}}$. The following notations are taken from [6]:

$R = \inf\{s>0 : s \in M\}$ $(\inf \phi = +\infty)$,

$R_t = R \circ \theta_t$, $D_t = t + R_t$, $\hat{\mathcal{F}}_t = \mathcal{F}_{D_t}$,

$F = \{x \in E : P^x\{R=0\} = 1\}$,

$G = \{t>0 : R_{t-} = 0,\ R_t>0\}$,

$G^r = \{t \in G : X_t \in F\}$,

$G^i = \{t \in G : X_t \notin F\}$.

For every homogeneous subset Γ of G we shall set

$$\Lambda_t^\Gamma = \sum_{\substack{s\in\Gamma \\ s\le t}} (1-e^{-R_s}) , \qquad L_t^\Gamma = \sum_{\substack{s\in\Gamma \\ s\le t}} P^{X_s}(1-e^{-R}) .$$

The process (Λ_t) defined by

$$\Lambda_t = \int_0^t 1_M(s)ds + \Lambda_t^G , \qquad t \ge 0 ,$$

is an $(\hat{\mathcal{F}}_t)$ adapted additive functional with support (or set of increase) M . Its $(\mathcal{F}_t)$ dual <u>optional</u> projection (L_t^0) is a <u>local time</u> for M (i.e. an $(\mathcal{F}_t)$ adapted additive functional with support M). Its jump part is $(L_t^{G^i})$, as it follows easily from [6] for example. But this jump part is too big with respect to the discussion of section 1.

THEOREM 1. 1) <u>The set</u> I <u>of isolated points of</u> M $(I \subset G)$ <u>is</u> $(\mathcal{F}_t)$ <u>optional and</u> $(\hat{\mathcal{F}}_t)$ <u>predictable. Each</u> $(\mathcal{F}_t)$ <u>stopping time</u> T <u>in</u> $I \cup \{\infty\}$ <u>is</u> $(\hat{\mathcal{F}}_t)$ <u>predictable and satisfies</u> $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$.

2) <u>The set</u> $G^{-i} = \{t \in G\backslash I : X_{t-} \notin F\}$ is $(\mathcal{F}_t)$ <u>predictable. For each</u> $(\mathcal{F}_t)$ <u>predictable stopping time</u> T <u>in</u> $G^{-i} \cup \{\infty\}$ <u>one has</u> $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$.

3) <u>The set</u> $G^{-r} = \{t \in G\backslash I : X_{t-} \in F\}$ <u>is</u> (a countable union of graphs of) $(\hat{\mathcal{F}}_t)$ <u>totally inaccessible</u> (stopping times).

THEOREM 2. <u>There exists an</u> $(\mathcal{F}_t)$ <u>adapted local time</u> (L_t) <u>for</u> M <u>which is, under each measure</u> P^μ , <u>the</u> $(\hat{\mathcal{F}}_t)$ <u>dual predictable projection of</u> (Λ_t) . <u>Its jump part is</u> $L^d = L^{I \cup G^{-i}}$.

It will be convenient in the sequel to write simply o.,p.,s.t.,d.p. for optional, predictable, stopping time(s), dual projection(s).

<u>Remark 1.</u> We know that $T \notin G^r$ a.s. for each s.t. T. Hence $I \cup G^{-i} \subset G^i$

a.s. by Theorem 1, and L^d is less than the jump part of L^0. When M is related to a Borel set B like in § 1, we have $X_t \in \bar{B}$ for $t \in I \cup G^{-i}$ a.s., since $X_T = X_{T-} \in \bar{B}$ a.s. on $\{T<\infty\}$ for each p.s.t. T in $G^{-i} \cup \{\infty\}$. Therefore our local time L is really local.

Proof. (a) The set I is $(\mathcal{F}_t)$ optional (see (3.3) of [7]) and can be written as a countable union of graphs of $(\mathcal{F}_t)$ s.t. . Let T be one of these s.t. and let $g_T = \sup\{s<T : s \in M\}$ ($\sup \emptyset = 0$). By (2.4) of [7], g_T is an $(\hat{\mathcal{F}}_t)$ s.t. . Consider $T_n = \inf\{t \geq g_T : R_t \leq \frac{1}{n}\}$ for $n \in \mathbb{N}$. Since $T_n < T$ on $\{T<\infty\}$ and $T_n \uparrow T$, T is $(\hat{\mathcal{F}}_t)$ predictable (it is announced by the sequence $(T_n \wedge n)$). In addition $\hat{\mathcal{F}}_{T-} = \bigvee_n \hat{\mathcal{F}}_{T_n \wedge n} = \bigvee_n \hat{\mathcal{F}}_{T_n} = \bigvee_n \mathcal{F}_{D_{T_n}}$ and $D_{T_n} = T$ on $\{T<\infty\}$, so that $\hat{\mathcal{F}}_{T-} \cap \{T<\infty\} = \mathcal{F}_T \cap \{T<\infty\}$ and $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$. The first part of Theorem 1 is established.

(b) Let T be an $(\hat{\mathcal{F}}_t)$ p.s.t. which is a left accumulation point of M on $\{T<\infty\}$. If T is announced by a sequence (T_n), it is also announced by the sequence (D_{T_n}) of (F_t) s.t., so that T is $(\mathcal{F}_t)$ predictable and satisfies $\hat{\mathcal{F}}_{T-} = \bigvee_n \hat{\mathcal{F}}_{T_n} = \bigvee_n \mathcal{F}_{D_{T_n}} = \mathcal{F}_{T-} = \mathcal{F}_T$ the last equality following from the quasi-left continuity of $(\mathcal{F}_t)$.

(c) Consider the $(\mathcal{F}_t)$ p. part $G^{i,p}$ and the $(\mathcal{F}_t)$ totally inaccessible part $G^{i,i}$ of the $(\mathcal{F}_t)$ o. set $G^i \backslash I$:

$$G^{i,p} = \{t \in G^i\backslash I : X_{t-} = X_t\},$$

$$G^{i,i} = \{t \in G^i\backslash I : X_{t-} \neq X_t\}.$$

It follows from b) that $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$ for each $(\mathcal{F}_t)$ p.s.t. in $G^{i,p} \cup \{\infty\}$ and that $G^{i,i}$ is $(\hat{\mathcal{F}}_t)$ totally inaccessible.

(d) It follows from (a), (c) that L^I and $L^{G^{i,p}}$ are the $(\hat{\mathcal{F}}_t)$ d.p.p. of Λ^I and $\Lambda^{G^{i,p}}$ under each measure P^μ. Now consider under P^μ, the $(\hat{\mathcal{F}}_t)$ d.p.p. of $\Lambda^{G^r \cup G^{i,i}}$: it is continuous since G^r and $G^{i,i}$ are $(\hat{\mathcal{F}}_t)$ totally inaccessible (for G^r see (3.2) of [7]) and carried by M (recall that $M\backslash\{0\} = \{t>0 : R_{t-}=0\}$ is $(\hat{\mathcal{F}}_t)$ p.), hence it is $(\mathcal{F}_t)$ adapted ([5], p. 56 or [9], p. 229) and thus it is P^μ-indistinguishable from the continuous additive functional (K_t) which is the $(\mathcal{F}_t)$ d.p.p. of $\Lambda^{G^r \cup G^{i,i}}$. Therefore the $(\mathcal{F}_t)$ adapted additive functional

$$L_t = \int_0^t 1_M(s)ds + K_t + L_t^{I \cup G^{i,p}}$$

is the $(\hat{\mathcal{F}}_t)$ d.p.p. of (Λ_t) under P^μ. Since the support of Λ is the $(\hat{\mathcal{F}}_t)$ p. set M, the support of L is M a.s. The proof of both theorems will be complete if we show that $G^r \cup G^{i,i} = G^{-r}$ a.s. and $G^{i,p} = G^{-i}$ a.s. But the continuous part L^c of L is carried by F since $\{t \in M : X_t \notin F\}$ is a.s. countable. Therefore $X_{t-} \in F$ for $t \in G^r \cup G^{i,i}$ a.s. ; on the other hand $X_{t-} = X_t \notin F$ for $t \in G^{i,p}$ a.s. ∎

Remark 2. We indicate here how to construct a local time by using the methods of [4]. Consider the local time of equilibrium of order 1 $(\bar{L}_t)$ (see [5]) for the perfect kernel of M, and define $\bar{G}^i = \{t \in G\ ,\ \Delta \bar{L}_t > 0$ or $t \in \bar{I}^g\}$, where $\bar{I}^g$ is the left closure of I. Then $L' = \bar{L}^c + L^{\bar{G}^i}$ is a local time such that $\{t : t \notin I\ ,\ \Delta L'_t > 0\}$ is $(\mathcal{F}_t)$ predictable and thus is good with respect to the discussion of § 1. One can even show that L^c is absolutely continuous with respect to $\bar{L}^c$, and that $I \cup G^{-i}$ and $\bar{G}^i$ are indistinguishable.

3. THE $(\mathcal{F}_{D_t})$ PREDICTABLE EXIT SYSTEM.

In this section we shall assume that R is $\mathcal{F}^*$ measurable, where $\mathcal{F}^*$ is the universal completion of $\mathcal{F}^0 = \sigma(X_t\ ,\ t \in \mathbb{R}_+)$. The universal completion of $\mathcal{E}$ will be denoted by $\mathcal{E}^*$.

THEOREM 3. <u>There exists an $\mathcal{E}^*$ measurable positive function ℓ on E, carried by F, and a kernel ${}_*P$ from $(E,\mathcal{E}^*)$ to $(\Omega,\mathcal{F}^*)$ such that</u> (L is defined as in Theorem 2)

(i) $\int_0^t 1_M(s)ds = \int_0^t \ell \circ X_s dL_s$,

(ii) $P^{\cdot} \sum_{s \in G} Z_s f \circ \theta_s = P^{\cdot} \int_0^\infty Z_s\, {}_*P^{X_s}(f) dL_s$

<u>for all positive $(\hat{\mathcal{F}}_t)$ predictable Z and $\mathcal{F}^*$ measurable f</u> ,

(iii) $\ell + {}_*P^{\cdot}(1-e^{-R}) \equiv 1$ <u>on</u> E <u>and</u>
${}_*P^{\cdot} \equiv P^{\cdot}/P^{\cdot}(1-e^{-R})$ <u>on</u> $E \backslash F$.

The system $(L, {}_*P)$ will be called the $(\mathcal{F}_{D_t})$ <u>predictable "exit system"</u> (according to the terminology of [6]). Note that in (ii) X_s can be replaced by Y_{s-} , where $Y_s = X_{D_s}$.

<u>Proof.</u> - Let ${}_*P^{\cdot}$ be defined on $E \backslash F$ as in (iii). The equality (ii) is immediate with $I \cup G^{-i}$ and L^d instead of G and L , due to Theorem 1. By the arguments of [6] we then establish the existence of a kernel N from $(E,\mathcal{E}^*)$ into $(\Omega,\mathcal{F}^*)$ such

that $N^{\cdot}\{R=0\} = 0$ and

$$P^{\cdot} \sum_{s\in G^{-r}} Z_s((1-e^{-R})f)\circ\theta_s = P^{\cdot}\int_0^\infty Z_s N^{X_s}(f)dL_s^c$$

for all positive $(\mathcal{F}_t)$ p. Z. This formula extends to positive $(\hat{\mathcal{F}}_t)$ p. Z by the argument of (d) of Section 2. If ℓ is a Motoo density of $(\int_0^t 1_M(s)ds)$ relative to (L_t^c) , the kernel N can be modified in such a way that $\ell + N^{\cdot}(1) = 1$. We can also assume that ℓ is carried by F . Setting $_*P^{\cdot}(f) = N^{\cdot}(\frac{f}{1-e^{-R}})$ on F , we get (ii) with G^{-r} and L^c instead of G and L and the proof is complete.

From this result one can extend some results of [8] and [3] (based on the $(\mathcal{F}_t)$ p. exit system). For analogous results without duality see Boutabia's thesis [1].

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<u>Note</u>. There is an error in Theorem V.3 of p. 64 of [5]. The functionnal (A_t) should be assumed $(\hat{\mathcal{F}}_t)$ p. and the condition $H^{\lambda}_U \Phi(y) < \Phi(y)$ should be required for each $(\hat{\mathcal{F}}_t)$ s.t. U such that $P^y\{U>0\} > 0$. For the proof of the converse part (1.3 of p. 65) one considers the predictable s.t. $T = S_{\{A_S>0\}}$ and a sequence (T_n) that announces T . One has $A_{T_n \wedge S} \leq A_{S-} = 0$. Hence $H^{\lambda}_{T_n \wedge S}\Phi(y) = \Phi(y)$ by (13) and $T_n \wedge S = 0$ P^y-a.s. by assumption. Since $T_n \wedge S \uparrow T \wedge S = S$, we have $S = 0$ P^y - a.s. and the proof is complete. Note also that Definition V.7, should be modified accordingly.
