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Predictable local times and exit systems
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1. INTRODUCTION.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \mathcal{P}_X)$ be the canonical realization of a Hunt semi-group $(\mathcal{P}_t)$ on a state space $(E, \mathcal{E})$ and let $\bar{M}$ be the closure of the random set 
{$\{t > 0 : X_t \in B\}$, where $B$ is in $\mathcal{E}$.} We set $R = \inf\{t > 0 : t \in \bar{M}\} = \inf\{t > 0 : X_t \in B\}$. 
If $\bar{M}$ has no isolated point a.s., the predictable additive functional with 1-potential $P'(e^R)$ is a local time of $M$ (the set of its increase points is $\bar{M}$ a.s. by [5], p. 66). This restriction on $\bar{M}$ is essential, as proved by the following example of Azéma. Consider a process which stays at 0 for an exponential time and then jumps to 1 and moves to the right with speed 1. For $B = \{1\}$, $R$ is totally inaccessible and $M = \{R\}$ cannot have a predictable local time.

One can always define an optional local time for $M$, as recalled in section 2. One unpleasant feature of such a local time is that it may jump at times $t$ where $X_t \notin \bar{B}$, so that the associated time changed process is not necessarily $\bar{B}$ valued. Nevertheless, one can construct a local time which avoids this unpleasant feature by using the methods of [4] (see Remark 2). Here we shall give a direct construction by taking the $(\mathcal{F}_t)$ dual predictable projection of the process $\wedge_t$ of § 2, where as usual $D_t = \inf\{s > t : s \in \bar{M}\}$.

We shall also prove the existence of a related $(\mathcal{F}_t)$ predictable exit system in full generality, whereas the existence of an $(\mathcal{F}_t)$ predictable exit system requires some special assumptions as noted by Getoor and Sharpe [2] (see V of [8] for sufficient conditions). From this one can deduce conditioning formulae like in the optional case ([8]).

2. THE $(\mathcal{F}_t)$ PREDICTABLE LOCAL TIME.

Let $X$ be like previously and let $\bar{M}$ be an optional random closed set, homogeneous in $(0, \infty)$ and such that $\bar{M} = \bar{M} \setminus \{0\}$. The following notations are taken from [6]:
\[ R = \inf\{ s > 0 : s \in M \} \quad (\inf \phi = +\infty) , \]
\[ R_t = R + \delta_t , \quad D_t = t + R_t , \quad \hat{\delta}_t = \mathbb{D}_t , \]
\[ F = \{ x \in E : P_x^{R=0} = 1 \} , \]
\[ G = \{ t > 0 : R_{t-} = 0 , \quad R_t > 0 \} , \]
\[ G^{\Gamma} = \{ t \in G : X_t \notin F \} , \]
\[ G^1 = \{ t \in G : X_t \notin F \} . \]

For every homogeneous subset \( \Gamma \) of \( G \) we shall set
\[ \Lambda_t^\Gamma = \sum_{s \in \Gamma} (1 - e^{-R_s}) , \quad L_t^\Gamma = \sum_{s \in \Gamma} P_x^{s}(1 - e^{-R_s}) . \]

The process \( (\Lambda_t) \) defined by
\[ \Lambda_t = \int_0^t 1_M(s)ds + \Lambda_t^\Gamma , \quad t \geq 0 , \]
is an \( (\hat{\delta}_t) \) adapted additive functional with support (or set of increase) \( M \). Its \( (\hat{\delta}_t) \) dual optional projection \( (L_t^0) \) is a local time for \( M \) (i.e. an \( (\hat{\delta}_t) \) adapted additive functional with support \( M \) ). Its jump part is \( (L_t^{G^1}) \), as it follows easily from [6] for example. But this jump part is too big with respect to the discussion of section 1.

**Theorem 1.**

1) The set \( I \) of isolated points of \( M \) \( (I \subset G) \) is \( (\hat{\delta}_t) \) optional and \( (\hat{\delta}_t) \) predictable. Each \( (\hat{\delta}_t) \) stopping time \( T \) in \( I \cup \{ \infty \} \) is \( (\hat{\delta}_t) \) predictable and satisfies \( \hat{\delta}_{T^-} = \hat{\delta}_T \).

2) The set \( G^{-1} = \{ t \in G \setminus I : X_t \notin F \} \) is \( (\hat{\delta}_t) \) predictable. For each \( (\hat{\delta}_t) \) predictable stopping time \( T \) in \( G^{-1} \cup \{ \infty \} \) one has \( \hat{\delta}_{T^-} = \hat{\delta}_T \).

3) The set \( G^{-\Gamma} = \{ t \in G \setminus I : X_t \notin F \} \) is (a countable union of graphs of) \( (\hat{\delta}_t) \) totally inaccessible (stopping times).

**Theorem 2.** There exists an \( (\hat{\delta}_t) \) adapted local time \( (L_t^\Phi) \) for \( M \) which is, under each measure \( P^{\mu} \), the \( (\hat{\delta}_t) \) dual predictable projection of \( (\Lambda_t^\Phi) \). Its jump part is \( L_t^{G^{-\Gamma}} = L_t^{G^{-\Gamma}} \).

It will be convenient in the sequel to write simply \( o.p., s.t., d.p. \) for optional, predictable, stopping time(s), dual projection(s).

**Remark 1.** We know that \( T \notin G^\Gamma \) a.s. for each \( s.t. \) \( T \). Hence \( I \cup G^{-1} \subset G^1 \).
a.s. by Theorem I, and \( L^d \) is less than the jump part of \( L^0 \). When \( M \) is related to a Borel set \( B \) like in § I, we have \( X_t \in B \) for \( t \in I \cup G^{-1} \) a.s., since \( X_T = X_{T^-} \in B \) a.s. on \( \{ T < \infty \} \) for each p.s.t. \( T \) in \( G^{-1} \cup \{ \infty \} \). Therefore our local time \( L \) is really local.

**Proof.** (a) The set \( I \) is \((\mathcal{I})\) optional (see (3.3) of [7]) and can be written as a countable union of graphs of \((\xi_t)\) s.t. Let \( T \) be one of these s.t. and let 
\[
g_T = \sup \{ s < T : s \in \mathbb{M} \} \quad (\sup \varphi = 0) .
\]
By (2.4) of [7], \( g_T \) is an \((\xi_t)\) s.t. Consider 
\[
T_n = \inf \{ t \geq g_T : R_t = \frac{1}{n} \}
\]
for \( n \in \mathbb{N} \). Since \( T_n < T \) on \( \{ T < \infty \} \) and \( T_n \uparrow T \), \( T \) is \((\xi_t)\) predictable (it is announced by the sequence \((T_n \wedge \infty)\)). In addition 
\[
\hat{\xi}_{T^-} = V_{n} \hat{\xi}_{T_n \wedge n} = V_{n} \hat{\xi}_{T_n} = V_{n} \hat{\xi}_{D_{T_n}} \quad \text{and} \quad D_{T_n} = T \quad \text{on} \quad \{ T < \infty \},
\]
so that \( \hat{\xi}_{T^-} \cap \{ T < \infty \} = \hat{\xi}_{T} \cap \{ T < \infty \} \) and \( \hat{\xi}_{T^-} = \hat{\xi}_{T} \). The first part of Theorem I is established.

(b) Let \( T \) be an \((\xi_t)\) p.s.t. which is a left accumulation point of \( M \) on \( \{ T < \infty \} \). If \( T \) is announced by a sequence \((T_n)\), it is also announced by the sequence 
\( (D_{T_n}) \) of \((\xi_t)\) s.t., so that \( T \) is \((\xi_t)\) predictable and satisfies 
\[
\hat{\xi}_{T^-} = V_{n} \hat{\xi}_{T_n} = V_{n} \hat{\xi}_{D_{T_n}} = \hat{\xi}_{T} \text{ on } \{ T < \infty \},
\]
the last equality following from the quasi-left continuity of \((\xi_t)\).

(c) Consider the \((\xi_t)\) p. part \( G^i,p \) and the \((\xi_t)\) totally inaccessible part 
\( G^i,1 \) of the \((\xi_t)\) o. set \( G^i \) :
\[
G^i,p = \{ t \in G^i \setminus I : X_{t^-} = X_t \},
\]
\[
G^i,1 = \{ t \in G^i \setminus I : X_{t^-} \neq X_t \}.
\]
It follows from b) that \( \hat{\xi}_{T^-} = \hat{\xi}_T \) for each \((\xi_t)\) p.s.t. in \( G^i,p \cup \{ \infty \} \) and that \( G^i,1 \) is \((\xi_t)\) totally inaccessible.

(d) If follows from (a), (c) that \( L^1 \) and \( L^{G^i,p} \) are the \((\xi_t)\) d.p.p. of \( L^1 \) and \( G^{i,p} \) under each measure \( P^{i,1} \). Now consider under \( P^{i,1} \), the \((\xi_t)\) d.p.p. of 
\( G^{i,p} \cup G^{i,1} \) : it is continuous since \( G^i \) and \( G^i,1 \) are \((\xi_t)\) totally inaccessible (for \( G^i \) see (3.2) of [7]) and carried by \( M \) (recall that \( M \setminus \{0\} = \{ t > 0 : R_t = 0 \} \) is \((\xi_t)\) p. ), hence it is \((\xi_t)\) adapted ([5], p. 56 or [9], p. 229) and thus it is \( P^{i,1} \)-indistinguishable from the continuous additive functional \( (K_t) \) which is the \((\xi_t)\) d.p.p. of \( G^{i,p} \cup G^{i,1} \).

Therefore the \((\xi_t)\) adapted additive functional 
\[
L_t = \int_0^t 1_M(s)ds + K_t + \int_0^t G^{i,p}
\]
is the \((\xi_t)\) d.p.p. of \((\Lambda_t)\) under \( P^{i,1} \). Since the support of \( \Lambda \) is the \((\xi_t)\) p. set \( M \), the support of \( L \) is \( M \) a.s. The proof of both theorems will be complete if we
show that $G^r \cup G^{1,1} = G^r$ a.s. and $G^{1,p} \subset G^{-1}$ a.s. But the continuous part $L^c$ of $L$ is carried by $F$ since $\{t \in M : X_t \notin F\}$ is a.s. countable. Therefore $X_{t^-} \notin F$ for $t \in G^r \cup G^{1,1}$ a.s.; on the other hand $X_{t^-} = X_t \notin F$ for $t \in G^{1,p}$ a.s. •

Remark 2. We indicate here how to construct a local time by using the methods of [4]. Consider the local time of equilibrium of order 1 $(\bar{L}_t)$ (see [5]) for the perfect kernel of $\mathfrak{M}$, and define $\bar{G}^1 = \{t \in G, \Delta \bar{L}_t > 0 \text{ or } t \in \bar{G}^\mathfrak{F}\}$, where $\bar{G}^\mathfrak{F}$ is the left closure of $\mathfrak{I}$. Then $L' = L^c + L\bar{G}^1$ is a local time such that $\{t : t \notin I, \Delta L'_t > 0\}$ is $(\mathcal{F}_t)$ predictable and thus is good with respect to the discussion of §1. One can even show that $L^c$ is absolutely continuous with respect to $L^c$, and that $I \cup G^{-1}$ and $\bar{G}^1$ are indistinguishable.

### 3. THE $(\mathcal{S}_{D_t})$ PREDICTABLE EXIT SYSTEM.

In this section we shall assume that $R$ is $\mathfrak{S}^*$ measurable, where $\mathfrak{S}^*$ is the universal completion of $\mathfrak{S}^0 = \sigma(X_t, t \in \mathbb{R}_+)$. The universal completion of $\mathfrak{E}$ will be denoted by $\mathfrak{E}^*$. 

**THEOREM 3.** There exists an $\mathfrak{E}^*$ measurable positive function $\bar{\varepsilon}$ on $E$, carried by $F$, and a kernel $\bar{P}$ from $(E, \mathfrak{E}^*)$ to $(\mathfrak{N}, \mathfrak{S}^*)$ such that $(L$ is defined as in Theorem 2)

\[
\begin{align*}
(i) & \quad \int_0^t \mathfrak{M}(s) ds = \int_0^t \bar{\varepsilon} X_s dL_s, \\
(ii) & \quad \bar{P} \sum_{g \in G} Z_s \mathfrak{G} = P' \int_0^\infty Z_s \mathfrak{G} X_{s'}(f) dL_s
\end{align*}
\]

for all positive $(\bar{\mathcal{F}}_t)$ predictable $Z$ and $\mathfrak{S}^*$ measurable $f$.

\[
\begin{align*}
(iii) & \quad \bar{\varepsilon} + \mathfrak{P}'(1-\bar{\varepsilon}^{-1}R) = 1 \text{ on } E \text{ and } \\
& \quad \mathfrak{P}' = \mathfrak{P}' / \mathfrak{P}'(1-\bar{\varepsilon}^{-1}R) \text{ on } E \setminus F.
\end{align*}
\]

The system $(L, \bar{P})$ will be called the $(\mathcal{S}_{D_t})$ predictable "exit system" (according to the terminology of [6]). Note that in (ii) $X_s$ can be replaced by $Y_s$, where $Y_s = X_{D_s}$.

**Proof.** Let $\mathfrak{P}'$ be defined on $E \setminus F$ as in (iii). The equality (ii) is immediate with $I \cup G^{-1}$ and $L^d$ instead of $G$ and $L$, due to Theorem 1. By the arguments of [6] we then establish the existence of a kernel $N$ from $(E, \mathfrak{E}^*)$ into $(\mathfrak{N}, \mathfrak{S}^*)$ such
that $N'[R=0] = 0$ and

$$P' \sum_{s \in G^{-R}} Z_s ((1-e^{-R})f), \theta_s = P' \int_0^\infty Z_s N \frac{X_s(f)}{s} dL^C_s$$

for all positive $(\xi_t)$ p. Z. This formula extends to positive $(\xi_t)$ p. Z by the argument of (d) of Section 2. If $\xi$ is a Motoo density of $(\int_0^x 1_{s \lessgtr b}(s)ds)$ relative to $(L^C_t)$, the kernel $N$ can be modified in such a way that $\xi + N'(1) = 1$. We can also assume that $\xi$ is carried by $F$. Setting $s^+ P'(f) = N' \left( \frac{f}{1-e^{-R}} \right)$ on $F$, we get (ii) with $G^{-R}$ and $L^C$ instead of $G$ and $L$ and the proof is complete.

From this result one can extend some results of [8] and [3] (based on the $(\xi_t)$ p. exit system). For analogous results without duality see Boutabia's thesis [1].

REFERENCES.


Note. There is an error in Theorem V.3 of p. 64 of [5]. The functional \((\lambda_t)\) should be assumed \((\hat{\lambda}_t)\) and the condition \(H^\lambda_U \hat{\xi}(y) < \xi(y)\) should be required for each \((\hat{\lambda}_t)\) s.t. \(U\) such that \(P^y[U>0] > 0\). For the proof of the converse part (1.3 of p. 65) one considers the predictable s.t. \(T = S_{\{A_s > 0\}}\) and a sequence \((T_n)\) that announces \(T\). One has \(A_{T_n \wedge S} \leq A_{S^-} = 0\). Hence \(H^\lambda_{T_n \wedge S} \xi(y) = \xi(y)\) by (13) and \(T_n \wedge S = 0\text{ }P^y\text{-a.s.}\) by assumption. Since \(T_n \wedge S = T \wedge S = S\), we have \(S = 0\text{ }P^y\text{-a.s.}\) and the proof is complete. Note also that Definition V.7, should be modified accordingly.