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Ultimateness and the Azéma-Yor stopping time

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The purpose of this note is to give a correct proof of a result of Meilijson [3,p394], which was originally based on an identity proved wrong by Neil Falkner (theorem 2). Our proof uses a special property of the Azéma-Yor stopping time (theorem 1 and lemma 1).

Let denote standard Brownian Motion (started at zero) and for any stopping time $T$ define

$$M_T := \sup_{0 \leq t \leq T} B_t.$$ 

A stopping time $T$ is called standard, if whenever $\sigma_1$ and $\sigma_2$ are stopping times with $\sigma_1 \leq \sigma_2 \leq T$, then

$$E|B_{\sigma_1}| < \infty, \quad i=1,2, \quad \text{and}$$

$$E|B_{\sigma_2} - x| \leq E|B_{\sigma_1} - x| \quad \text{for all } x \in \mathbb{R}.$$ 

(As N. Falkner [2,p.386] showed, a stopping time $T$ is standard if and only if the process $(B_{t \wedge T})$ is uniformly integrable.)

Let $X$ be a random variable with $EX = 0$ and define the function $g_X$ on $\mathbb{R}$ by

$$g_X(x) := \begin{cases} E(X|X \geq x) & \text{if } P(X \geq x) > 0, \\ x & \text{otherwise.} \end{cases}$$

Azéma and Yor [1,p.95,p.625] showed that the stopping time $T$ defined by

$$T := \inf\{t: g_X(B_t)\}$$

embeds (the distribution of) $X$, i.e. $B_T \overset{D}{=} X$, and is standard. We will refer to it as the A-Y stopping time (embedding $X$ in $(B_t)$). It is also known that for any standard stopping time $T$, that embeds $X$ in $(B_t)$,

$$(1) \quad P(M_T \geq g_X(x)) \leq P(M_T \geq g_X(x)) = P(B_T \geq x) = P(X \geq x)$$

for $x \in \mathbb{R}$.

For the inequality we refer to Azéma and Yor [1,p.632].

The first equality is easily seen from the definition of $T$, while the second holds, because $T$ embeds $X$.

\*\*I. Meilijson communicated this to me by letter.
Theorem 1.

Of all standard stopping times $\tau$ that embed $X$, the A-Y stopping time $T$ is essentially* the only one with

$$P(M_\tau \geq g(x)) = P(X \geq x), \quad x \in \mathbb{R}. \quad (2)$$

A standard stopping time $\tau$ is called ultimate, whenever $Y$ is a random variable with $E|Y - x| \leq E|B_\tau - x|$ for all $x \in \mathbb{R}$, then there exists a stopping time $\sigma \leq \tau$, that embeds $Y$.

Theorem 2. (I. Meilijson [3,p.394])

Assume $\tau$ is a standard stopping time embedding $X$. If $\tau$ is ultimate, then there are $a \leq 0 \leq b$ with $P(X \in [a,b]) = 1$. \hfill \Box

Proof of Theorem 1.

We write $g$ for $g_X$.

Let $\tau$ be a standard stopping time embedding $X$ such that (2) holds.

Define the stopping time $H_x$ by $H_x := \inf\{t : B_\tau \geq g(x)\}$ and put $\tau^*_x := \tau \wedge H_x$. Then 

$\{M_\tau \geq g(x)\} = \{H_x \leq \tau\}$.

For $z \leq x$

$$E|B_\tau - z| \geq E|B_\tau - z| =$$

$$(g(x) - z)P(H_x \leq \tau) + E|B_\tau - z| 1_{\{\tau < H_x\}} =$$

$$E(X - z) 1\{x \geq x\} + E|B_\tau - z| 1_{\{\tau < H_x\}} =$$

$$E|B_\tau - z| + E|B_\tau - z| 1\{B_\tau \geq x, \tau < H_x\} - 1\{B_\tau < x, \tau \geq H_x\} =$$

So

$$E|B_\tau - z| 1\{B_\tau \geq x, \tau < H_x\} \leq E|B_\tau - z| 1\{B_\tau < x, \tau \geq H_x\}, \quad z \leq x. \quad (3)$$

Now using (2)

$$P(B_\tau \geq x, \tau < H_x) =$$

$$P(B_\tau \geq x) - P(B_\tau \geq x, \tau \geq H_x) =$$

$$P(X \geq x) - P(\tau \geq H_x) + P(B_\tau < x, \tau \geq H_x) =$$

$$P(B_\tau < x, \tau \geq H_x)$$

whence with $z \rightarrow -\infty$ in (3) it follows that

$$P(B_\tau \geq x, \tau < H_x) = P(B_\tau < x, \tau \geq H_x) = 0.$$

* apart from disagreement on a null set.
Therefore
\begin{align*}
\{B_T \geq x\} &= \{M_T \geq g(x)\} \text{ for all } x \in \mathbb{Q} (= \text{the rational numbers}) \text{ a.s.}, \\
\end{align*}
As for all \(x \in \mathbb{R}\) we can find a sequence \((x_n)\) in \(\mathbb{Q}\) increasing to \(x\) and \(g\) is left-continuous, we get
\begin{align*}
\{B_T \geq x\} &= \{M_T \geq g(x)\} \text{ for all } x \in \mathbb{R} \text{ a.s.},
\end{align*}
whence
\begin{align*}
M_T \geq g(B_T) \text{ a.s.}
\end{align*}
(Simply observe that
\begin{align*}
B_T \in [x, x + \frac{1}{n}] \iff M_T \in [g(x), g(x + \frac{1}{n})]
\end{align*}
for all \(x \in \mathbb{R}\) and all \(n \in \mathbb{N}\) a.s..)

Now \(t < T\) implies \(M_T < g(B_T)\) and therefore \(T \geq T\) a.s.. As \(T\) is standard, it follows that for any stopping time \(\sigma\) with \(T \leq \sigma \leq T\) a.s.,
\begin{align*}
E|B_{\sigma} - x| &= E|X - x| \text{ for all } x \in \mathbb{R},
\end{align*}
which can only happen if \(T = \tau\) a.s..<br />

Let \(T^-\) be the A-Y stopping time embedding \(-X\) in \((-B_T)\), then
\begin{align*}
\inf_{0 \leq s \leq t} B_s &= T^- = \inf\{t: m_t \leq -g_X(-B_t)\}
\end{align*}
and
\begin{align*}
B_{T^-} &= X.
\end{align*}
Lemma 1.
If \(T = T^-\) a.s., then there are \(a \leq 0 \leq b\) with \(P(X \in \{a,b\}) = 1\).
Proof.
First observe that
\begin{align*}
-g_X(-x) \leq X \leq g_X(x) \quad (x \in \mathbb{R})
\end{align*}
Now for a path (of \((B_t)\)) with \(T = T^-\) and \(B_T = B_{T^-} = x\) we have
\begin{align*}
M_T \geq g_X(x) \quad (\geq x), \text{ and } \\
M_T \leq -g_X(-x) \quad (\leq x).
\end{align*}
That implies however that
\begin{align*}
-g_X(-x) &= x \text{ or } g_X(x) = x.
\end{align*}
[If such a path first reaches level \(M_T\) and then level \(m_T\) it is forced to cross level \(x\) in between (continuity of paths) and 'T stops to soon', unless \(-g_X(-x) = x\); conversely if level \(m_T\) is reached before level \(M_T\), 'T- stops to soon', unless \(g_X(x) = x\).]
Now (4) implies \( x \leq \inf X =: a (\leq 0) \), or \( x \geq \sup X =: b (\geq 0) \).

As \( T = T^- \text{ a.s.} \), we can conclude
\[
B_T^T \leq a \text{ or } B_T^T \geq b \text{ a.s.}
\]

As \( X \overset{D}{=} B_T^T \), it follows that \( P(X \notin (a,b)) = 1 \).

By definition of \( a \) and \( b \) \( P(X \in [a,b]) = 1 \).

It follows that \( a \) and \( b \) are finite and \( P(X \in \{a,b\}) = 1 \).

Proof of theorem 2.

By lemma 1 it is enough to prove \( \tau = T \text{ a.s.} \) and \( \tau = T^- \text{ a.s.} \).

As \( T^- \) is the A-Y stopping time embedding \(-X \) in \( (-B_T^-) \), it is sufficient to prove, that an ultimate stopping time is equal to the A-Y stopping time a.s., i.e. \( \tau = T \text{ a.s.} \).

With \( H \) as in the proof of theorem 1 we have for all \( x \in \mathbb{R} \) by (1)
\[
P(\tau \geq H_x^x) \leq P(T \geq H_x^x) = P(X \geq x).
\]

As \( \tau \) is ultimate and \( T \) is standard, there is a stopping time \( \sigma \leq \tau \) with
\[
B_{\sigma_X}^X \overset{D}{=} B_{T \wedge H_X^x}^X.
\]

But then
\[
P(\sigma \geq g_x^x(x)) \leq P(B_{\sigma_X}^X \geq g_x^x(x)) = P(B_{T \wedge H_X^x}^X \geq g_x^x(x)) = P(T \geq H_x^x),
\]

and so
\[
P(\sigma \geq g_x^x(x)) = P(X \geq x).
\]

By theorem 1 it follows that \( \tau = T \text{ a.s.} \).

References.

[1] J. AZEMA et M. YOR,
   a. Une solution simple au problème de Skorokhod.
   b. Le problème de Skorokhod: compléments à l'exposé précédent.

[2] N. FALKNER,
   On Skorokhod embedding in n-dimensional Brownian Motion by means of natural stopping times.

[3] I. MEILIJSON,
   There exists no ultimate solution to Skorokhod's problem.