D. P. Van der Vecht

Ultimateness and the Azéma-Yor stopping time

Séminaire de probabilités (Strasbourg), tome 20 (1986), p. 375-378

<http://www.numdam.org/item?id=SPS_1986__20__375_0>
Ultimateness and the Azéma-Yor stopping time

D.P. van der Vecht
Vrije Universiteit, Amsterdam

The purpose of this note is to give a correct proof of a result of Meilijson [3,p394], which was originally based on an identity proved wrong by Neil Falkner (theorem 2). Our proof uses a special property of the Azéma-Yor stopping time (theorem 1 and lemma 1).

Let denote standard Brownian Motion (started at zero) and for any stopping time define

\[ M_t := \sup_{0 \leq s \leq t} B_s. \]

A stopping time \( T \) is called standard, if whenever \( \sigma_1 \) and \( \sigma_2 \) are stopping times with \( \sigma_1 \leq \sigma_2 \leq T \), then

\[ E|B_{\sigma_1} - x| \leq E|B_{\sigma_2} - x| \quad \text{for all } x \in \mathbb{R}. \]

(As N. Falkner [2,p.386] showed, a stopping time \( T \) is standard if and only if the process \( (B_t) \) is uniformly integrable.)

Let \( X \) be a random variable with \( EX = 0 \) and define the function \( g_X \) on \( \mathbb{R} \) by

\[ g_X(x) := \begin{cases} E(X|X \geq x) & \text{if } P(X \geq x) > 0, \\ x & \text{otherwise.} \end{cases} \]

Azéma and Yor [1,p.95,p.625] showed that the stopping time \( T \) defined by

\[ T := \inf\{t: g_X(B_t)\} \]

embeds (the distribution of) \( X \), i.e. \( B_T \equiv X \), and is standard. We will refer to it as the A-Y stopping time (embedding \( X \) in \( (B_t) \)). It is also known that for any standard stopping time \( T \), that embeds \( X \) in \( (B_t) \),

\[ P(M_T \geq g_X(x)) \leq P(M_T \geq g_X(x)) = P(B_T \geq x) = P(X \geq x) \]

for \( x \in \mathbb{R} \).

For the inequality we refer to Azéma and Yor [1,p.632].

The first equality is easily seen from the definition of \( T \), while the second holds, because \( T \) embeds \( X \).

* I. Meilijson communicated this to me by letter.
Theorem 1.
Of all standard stopping times $\tau$ that embed $X$, the A-Y stopping time $T$ is essentially the only one with
\[ P(M_{\tau \geq g_X(x)}) = P(X \geq x), \quad x \in \mathbb{R}. \]

A standard stopping time $\tau$ is called ultimate, whenever $Y$ is a random variable with $E|Y-X| \leq E|B_{\tau}-X|$ for all $x \in \mathbb{R}$, then there exists a stopping time $\sigma \leq \tau$, that embeds $Y$.

Theorem 2. (I. Meilijson [3, p. 394])
Assume $\tau$ is a standard stopping time embedding $X$. If $\tau$ is ultimate, then there are $a \leq 0 \leq b$ with $P(X \in [a, b]) = 1$.

Proof of Theorem 1.
We write $g$ for $g_X$.
Let $\tau$ be a standard stopping time embedding $X$ such that (2) holds.
Define the stopping time $H^x$ by $H^x := \inf \{ t : B_{\tau} \geq g_X(x) \}$ and put $\tau^x := \tau \wedge H^x$. Then
\[ \{M_{\tau \geq g(x)} = \{H_x \leq \tau}. \]
For $z \leq x$
\[ E|B_{\tau} - z| \geq E|B_{\tau^x} - z| = \]
\[ (g(x) - z)P(H^x \leq \tau) + E|B_{\tau} - z| \mathbf{1}_{\{\tau < H^x\}} = \]
\[ E(X - z) \mathbf{1}_{\{X \leq x\}} + E|B_{\tau} - z| \mathbf{1}_{\{\tau < H^x\}} = \]
\[ E|B_{\tau} - z| + E|B_{\tau^x} - z| \{1_{\{B_{\tau} \geq x, \tau < H_x^x\}} - 1_{\{B_{\tau} < x, \tau \geq H_x^x\}}\}. \]
So
\[ E|B_{\tau} - z| \mathbf{1}_{\{B_{\tau} \geq x, \tau < H_x^x\}} \leq E|B_{\tau^x} - z| \mathbf{1}_{\{B_{\tau^x} < x, \tau \geq H_x^x\}}, \quad z \leq x. \]
Now using (2)
\[ P(B_{\tau} \geq x, \tau < H_x^x) = \]
\[ P(B_{\tau} \geq x) - P(B_{\tau} \geq x, \tau \geq H_x^x) = \]
\[ P(X \geq x) - P(\tau \geq H_x^x) + P(B_{\tau^x} < x, \tau \geq H_x^x) = \]
\[ P(B_{\tau^x} < x, \tau \geq H_x^x), \]
whence with $z \to -\infty$ in (3) it follows that
\[ P(B_{\tau} \geq x, \tau < H_x^x) = P(B_{\tau^x} < x, \tau \geq H_x^x) = 0. \]

* apart from disagreement on a null set.
Therefore
\[ \{ B_t \geq x \} = \{ M_t \geq g(x) \} \text{ for all } x \in \mathbb{Q} (= \text{the rational numbers}) \text{ a.s.}. \]

As for all \( x \in \mathbb{R} \) we can find a sequence \( (x_n) \) in \( \mathbb{Q} \) increasing to \( x \) and \( g \) is left-continuous, we get
\[ \{ B_t \geq x \} = \{ M_t \geq g(x) \} \text{ for all } x \in \mathbb{R} \text{ a.s.,} \]
whence
\[ M_t \geq g(B_t) \text{ a.s.} \]

(Simply observe that
\[ B_t \in [x, x + \frac{1}{n}) \iff M_t \in [g(x), g(x + \frac{1}{n})) \]
for all \( x \in \mathbb{R} \) and all \( n \in \mathbb{N} \) a.s.)

Now \( t < T \) implies \( M_t < g(B_t) \) and therefore \( T \geq t \text{ a.s.} \). As \( T \) is standard, it follows that for any stopping time \( \sigma \) with \( T \leq \sigma \leq T \text{ a.s.} \),
\[ 
E[|B_{\sigma} - x|] = E[X - x] \quad \text{for all } x \in \mathbb{R},
\]
which can only happen if \( T = \tau \text{ a.s.} \]

Let \( T^- \) be the \( \mathcal{A} \)-\( \mathcal{Y} \) stopping time embedding \(-X\) in \((-B_t)\), then
with \( m_t = \inf_{0 \leq s \leq t} B_s \)
\[ T^- = \inf\{ t: m_t \leq -g_{-X}(-B_t) \} \]
and
\[ B_{T^-} \overset{D}{=} X. \]

**Lemma 1.**
If \( T = T^- \text{ a.s.} \), then there are \( a \leq 0 \leq b \) with \( P(X \in \{a,b\}) = 1. \]

**Proof.**
First observe that
\[ -g_{-X}(-x) \leq x \leq g_{-X}(x) \quad (x \in \mathbb{R}) \]

Now for a path \((B_t)\) with \( T = T^- \) and \( B_T = B_{T^-} = x \) we have
\[ M_T \geq g_{-X}(x) \quad (\geq x), \quad \text{and} \quad m_T \leq -g_{-X}(-x) \quad (\leq x). \]

That implies however that
\[ -g_{-X}(-x) = x \text{ or } g_{-X}(x) = x. \]

[If such a path first reaches level \( M_T \) and then level \( m_T \), it is forced to cross level \( x \) in between (continuity of paths) and 'T stops to soon', unless \( -g_{-X}(-x) = x \); conversely if level \( m_T \) is reached before level \( M_T \), 'T- stops to soon', unless \( g_{-X}(x) = x. \]
Now (4) implies \( x \leq \inf X =: a(\leq 0) \), or \( x \geq \sup X =: b(\geq 0) \).

As \( T = T^- \text{ a.s.} \), we can conclude
\[
B_T \leq a \text{ or } B_T \geq b \text{ a.s.}
\]

As \( X \overset{D}{=} B_T \), it follows that \( P(X \notin (a,b)) = 1 \).

By definition of \( a \) and \( b \) \( P(X \in [a,b]) = 1 \).

It follows that \( a \) and \( b \) are finite and \( P(X \in \{a,b\}) = 1 \).

\[ \square \]

**Proof of theorem 2.**

By lemma 1 it is enough to prove \( \tau = T \text{ a.s.} \) and \( \tau = T^- \text{ a.s.} \).

As \( T^- \) is the A-Y stopping time embedding \(-X\) in \((-B_T^-)\), it is sufficient to prove, that an ultimate stopping time is equal to the A-Y stopping time a.s., i.e. \( \tau = T \text{ a.s.} \).

With \( H \) as in the proof of theorem 1 we have for all \( x \in \mathbb{R} \) by (1)
\[
P(\tau \geq H_x) \leq P(T \geq H_x) = P(X \geq x).
\]

As \( \tau \) is ultimate and \( T \) is standard, there is a stopping time \( \sigma_x \leq \tau \) with
\[
B_{\sigma_x} \overset{D}{=} B_{TH_x} \; \text{. But then}
\]
\[
P(M_{\tau} \geq g_X(x)) \geq P(B_{\sigma_x} \geq g_X(x)) = P(B_{TH_x} \geq g_X(x)) = P(T \geq H_x),
\]
and so
\[
P(M_{\tau} \geq g_X(x)) = P(X \geq x).
\]

By theorem 1 it follows that \( \tau = T \text{ a.s.} \).

\[ \square \]

**References.**

[1] J. AZEMA et M. YOR,

a. Une solution simple au problème de Skorokhod.

b. Le problème de Skorokhod: compléments à l'exposé précédent.


[2] N. FALKNER,

On Skorokhod embedding in n-dimensional Brownian Motion by means of natural stopping times.


[3] I. MEILIJSON,

There exists no ultimate solution to Skorokhod's problem.