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Ultimateness and the Azéma-Yor stopping time

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The purpose of this note is to give a correct proof of a result of Meilijson [3,p394], which was originally based on an identity proved wrong by Neil Falkner (theorem 2). Our proof uses a special property of the Azéma-Yor stopping time (theorem 1 and lemma 1).

Let \( (B_t)_{t \geq 0} \) denote standard Brownian Motion (started at zero) and for any stopping time \( \tau \) define
\[
M_\tau := \sup_{0 \leq t \leq \tau} B_t.
\]
A stopping time \( \tau \) is called standard, if whenever \( \sigma_1 \) and \( \sigma_2 \) are stopping times with \( \sigma_1 \leq \sigma_2 \leq \tau \), then
\[
E|B_{\sigma_1}| < \infty, \quad i=1,2, \text{ and}
E|B_{\sigma_2} - x| \leq E|B_{\sigma_1} - x| \quad \text{for all } x \in \mathbb{R}.
\]
(As N. Falkner [2,p.386] showed, a stopping time \( \tau \) is standard if and only if the process \( (B_t)_{t \leq \tau} \) is uniformly integrable.)

Let \( X \) be a random variable with \( EX = 0 \) and define the function \( g_X \) on \( \mathbb{R} \) by
\[
g_X(x) := \begin{cases} 
E(X|X \geq x) & \text{if } P(X \geq x) > 0, \\
x & \text{otherwise}. 
\end{cases}
\]
Azéma and Yor [1,p.95,p.625] showed that the stopping time \( T \) defined by
\[
T := \inf\{t: g_X(B_t)\}
\]
embeds (the distribution of) \( X \), i.e. \( B_T \overset{D}{=} X \), and is standard. We will refer to it as the A-Y stopping time (embedding \( X \) in \( (B_t) \)). It is also known that for any standard stopping time \( \tau \), that embeds \( X \) in \( (B_t) \),
\[
P(M_\tau \geq g_X(x)) \leq P(M_T \geq g_X(x)) = P(B_T \geq x) = P(X \geq x)
\]
for \( x \in \mathbb{R} \).
For the inequality we refer to Azéma and Yor [1,p.632].
The first equality is easily seen from the definition of \( T \), while the second holds, because \( T \) embeds \( X \).

* I. Meilijson communicated this to me by letter.
Theorem 1.

Of all standard stopping times $\tau$ that embed $X$, the $A\text{-}Y$ stopping time $T$ is essentially* the only one with

\[(2) \quad P(M_{\tau} \geq g(x)) = P(X \geq x), \quad x \in \mathbb{R}.\]

A standard stopping time $\tau$ is called ultimate, whenever $Y$ is a random variable with $E|Y-X| \leq E|B_{\tau}-x|$ for all $x \in \mathbb{R}$, then there exists a stopping time $\sigma \leq \tau$, that embeds $Y$.

Theorem 2. (I. Meilijson [3, p.394])

Assume $\tau$ is a standard stopping time embedding $X$. If $\tau$ is ultimate, then there are $a \leq 0 \leq b$ with $P(X \in [a,b]) = 1$. \[\Box\]

Proof of Theorem 1.

We write $g$ for $g_X$.

Let $\tau$ be a standard stopping time embedding $X$ such that (2) holds.

Define the stopping time $H_x$ by $H_x := \inf\{t: B_{\tau_t} \geq g(x)\}$ and put $\tau_x := \tau \wedge H_x$. Then \[
\{M_{\tau} \geq g(x)\} = \{H_x \leq \tau\}.
\]

For $z \leq x$

\[
E|B_{\tau_t} - z| \geq E|B_{\tau_x} - z| =
(g(x) - z)P(H_x \leq \tau) + E|B_{\tau} - z| \mathbb{1}_{\{\tau < H_x\}} =
E(X - z) \mathbb{1}_{\{X \geq x\}} + E|B_{\tau} - z| \mathbb{1}_{\{\tau < H_x\}} =
E|B_{\tau} - z| + E|B_{\tau} - z| (\mathbb{1}_{\{B_{\tau} \geq x, \tau < H_x\}} - \mathbb{1}_{\{B_{\tau} < x, \tau \geq H_x\}}).
\]

So

\[(3) \quad E|B_{\tau} - z| \mathbb{1}_{\{B_{\tau} \geq x, \tau < H_x\}} \leq E|B_{\tau} - z| \mathbb{1}_{\{B_{\tau} < x, \tau \geq H_x\}}, \quad z \leq x.
\]

Now using (2)

\[
P(B_{\tau} \geq x, \tau < H_x) =
P(B_{\tau} \geq x) - P(B_{\tau} \geq x, \tau \geq H_x) =
P(X \geq x) - P(\tau \geq H_x) + P(B_{\tau} < x, \tau \geq H_x) =
P(B_{\tau} < x, \tau \geq H_x),
\]

whence with $z \to -\infty$ in (3) it follows that

\[
P(B_{\tau} \geq x, \tau < H_x) = P(B_{\tau} < x, \tau \geq H_x) = 0.
\]

* apart from disagreement on a null set.
Therefore
\[ \{B_T \geq x\} = \{M_T \geq g(x)\} \text{ for all } x \in \mathbb{Q} (= \text{the rational numbers}) \text{ a.s.}. \]
As for all \( x \in \mathbb{R} \) we can find a sequence \( (x_n) \) in \( \mathbb{Q} \) increasing to \( x \) and \( g \) is left-continuous, we get
\[ \{B_T \geq x\} = \{M_T \geq g(x)\} \text{ for all } x \in \mathbb{R} \text{ a.s.}, \]
whence
\[ M_T \geq g(B_T) \text{ a.s.}. \]
(Simply observe that
\[ B_T \in [x, x + \frac{1}{n}] \iff M_T \in [g(x), g(x + \frac{1}{n})] \]
for all \( x \in \mathbb{R} \) and all \( n \in \mathbb{N} \) a.s.)
Now \( t < T \) implies \( M_T < g(B_T) \) and therefore \( T \geq t \) a.s.. As \( T \) is standard, it follows that for any stopping time \( \sigma \) with \( T \leq \sigma \leq t \) a.s..
\[ E|B_\sigma - x| = E|X - x| \text{ for all } x \in \mathbb{R}, \]
which can only happen if \( T = \sigma \) a.s.

Let \( T^- \) be the A-Y stopping time embedding \( -X \) in \((B_T)\), then
with \( m_T = \inf_{0 \leq s \leq T} B_s \),
\[ T^- = \inf \{ t : m_t \leq g_{-X}(-B_t) \} \]
and
\[ B_{T^-} \overset{D}{=} X. \]

Lemma 1.
If \( T = T^- \) a.s., then there are \( a \leq 0 \leq b \) with \( P(X \in \{a, b\}) = 1. \)

Proof.
First observe that
\[ -g_{-X}(-x) \leq x \leq g_X(x) \quad (x \in \mathbb{R}) \]
Now for a path (of \((B_t)\)) with \( T = T^- \) and \( B_T = B_{T^-} = x \) we have
\[ M_T \geq g_X(x) \quad (\geq x), \text{ and} \]
\[ m_T \leq -g_{-X}(-x) \quad (\leq x). \]
That implies however that
\[ -g_{-X}(-x) = x \text{ or } g_X(x) = x. \]
[If such a path first reaches level \( M_T \) and then level \( m_T \), it is forced to cross level \( x \) in between (continuity of paths) and '\( T \) stops to soon', unless \( -g_{-X}(-x) = x \); conversely if level \( m_T \) is reached before level \( M_T \), '\( T^- \) stops to soon', unless \( g_X(x) = x \).]
Now (4) implies $x \leq \inf X =: a(\leq 0)$, or $x \geq \sup X =: b(\geq 0)$.

As $T = T^{-a.s.}$, we can conclude

$$B_T^{-} \leq a \quad \text{or} \quad B_T^{+} \geq b \quad \text{a.s.}$$

As $X \overset{D}{=} B_T$, it follows that $P(X \notin (a,b)) = 1$.

By definition of $a$ and $b$, $P(X \in [a,b]) = 1$.

It follows that $a$ and $b$ are finite and $P(X \in \{a,b\}) = 1$.  

Proof of theorem 2.

By lemma 1 it is enough to prove $\tau = T$ a.s. and $\tau = T^{-}$ a.s.

As $T^{-}$ is the $A$-$Y$ stopping time embedding $-X$ in $(-B^{-}_{T})$, it is sufficient to prove, that an ultimate stopping time is equal to the A-$Y$ stopping time a.s., i.e. $\tau = T$ a.s.

With $H$ as in the proof of theorem 1 we have for all $x \in \mathbb{R}$ by (1)

$$P(\tau \geq H^{-}_{X}) \leq P(T \geq H^{-}_{X}) = P(X \geq x).$$

As $\tau$ is ultimate and $T$ is standard, there is a stopping time $\sigma^{-}_{X} \leq \tau$ with

$$B_{\sigma^{-}_{X}}^{+} \overset{D}{=} B_{T^{-}}^{+}.$$  But then

$$P(M_{\tau} \geq g_{X}(x)) \geq P(B_{\sigma^{-}_{X}} \geq g_{X}(x)) = P(B_{T^{-}}^{+} \geq g_{X}(x)) = P(T \geq H^{-}_{X}),$$

and so

$$P(M_{\tau} \geq g_{X}(x)) = P(X \geq x).$$

By theorem 1 it follows that $\tau = T$ a.s..  

References.

[1] J. AZEMA et M. YOR,

a. Une solution simple au problème de Skorokhod.

b. Le problème de Skorokhod: compléments à l'exposé précédent.


[2] N. FALKNER,

On Skorokhod embedding in n-dimensional Brownian Motion by means of natural stopping times.


[3] I. MEILIJSON,

There exists no ultimate solution to Skorokhod's problem.