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A COMPARISON THEOREM FOR SEMIMARTINGALES
AND ITS APPLICATIONS

by YAN Jia-an

We work on a filtered probability space $(\Omega, \underline{F}, P; (\underline{F}_t))$ satisfying the usual conditions. Let X be a semimartingale such that $\sum_{0 < s \leq t} |\Delta X_s| < \infty$ for $t < \infty$ (as usual, we allow an evanescent exceptional set in our inequalities without mentioning it) : this is the class of semimartingales for which Yor (Astérisque 52-53 , Temps Locaux, p.23-35) has shown the existence of local times $L_t^a(X)$ continuous in t , and cadlag in a . On the other hand, X has a unique decomposition

$$X = X_0 + M + A$$

where M is a continuous local martingale, and A is of finite variation. We denote by A^c the continuous part of A .

LEMMA 1. Assume the following conditions

$$(i) L^0(X) = 0 \quad (ii) \int_0^t I_{\{X_{s-} > 0\}} dA_s^c \leq 0 \quad (iii) \Delta X \leq 0.$$

Then we have $X \leq 0$ on the set $\{X_0 \leq 0\}$.

Proof. We have from Tanaka-Meyer's formula

$$X_t^+ = X_0^+ + \sum_{0 < s \leq t} I_{\{X_{s-} \leq 0\}} X_s^+ + \frac{1}{2} L_t^0 + \sum_{0 < s \leq t} I_{\{X_{s-} > 0\}} X_s^- + \int_0^t I_{\{X_{s-} > 0\}} dX_s$$

On $\{X_0 \leq 0\}$ the first term vanishes. The second one vanishes because of (iii) and the third one because of (i). Therefore on $\{X_0 \leq 0\}$

$$X_t^+ = \sum_{0 < s \leq t} I_{\{X_{s-} > 0\}} (X_s^- + \Delta X_s) + \int_0^t I_{\{X_{s-} > 0\}} (dM_s + dA_s^c)$$

We have $X^- + \Delta X = X^+ - X_- \leq 0$ on $\{X_- > 0\}$ by (iii) and $\int_0^t I_{\{X_{s-} > 0\}} dA_s^c \leq 0$ by (ii). Therefore

$$\int_0^t I_{\{X_{s-} > 0\}} dM_s \geq 0 \text{ on } \{X_0 \leq 0\}$$

Since this is a continuous local martingale starting from 0, it must be equal to 0, and from this we deduce $X_t^+ \leq 0$, and finally $X \leq 0$.

We apply this lemma to a generalization of the comparison lemma given by Ikeda-Watanabe ([1], p.352). One might extract from the proof a slightly more general version of lemma 1, but we shall not give it explicitly.

THEOREM 1. Let X^1, X^2 be solutions of two stochastic differentials equations

$$X_t^i = X_0^i + \int_0^t \sigma(s, X_{s-}^i) dM_s + \int_0^t b^i(s, X_{s-}^i) dB_s + \int_0^t c^i(s, X_{s-}^i) dC_s \quad (i=1,2)$$

where M is a continuous local martingale, B is a continuous increasing process and C an increasing process (B and C adapted). We assume

- $\sigma(s, x)$ is Borel measurable, $|\sigma(s, x) - \sigma(s, y)| \leq \rho(|x - y|)$, where ρ is an increasing function on \mathbb{R}_+ such that $\int_0^+ \rho^{-2}(u) du = +\infty$.
- $b^i(s, x), c^i(s, x)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}$ given the product of the right topology on \mathbb{R}_+ and the ordinary topology on \mathbb{R} .
- $b^1(s, x) < b^2(s, x)$ and $c^1(s, x) < c^2(s, x)$
- $x < y \Rightarrow c^1(s, x) \leq c^2(s, y)$.

Then we have $X^1 \leq X^2$ on the set $\{X_0^1 \leq X_0^2\}$.

Proof. We may assume $X_0^1 \leq X_0^2$ everywhere. Consider the stopping time

$$T = \inf\{t > 0 : X_t^1 - X_t^2 > 0\}$$

We assume $P\{T < \infty\} > 0$ and derive a contradiction. First of all, we have $X_{T-}^1 \geq X_{T-}^2$ (on $\{T < \infty\}$), and $X_{T-}^1 \leq X_{T-}^2$ on $\{0 < T < \infty\}$. We cannot have $X_{T-}^1 < X_{T-}^2$ on $\{0 < T < \infty\}$, because $\Delta X_{T-}^1 \leq \Delta X_{T-}^2$ (last hypothesis) would then imply $X_{T-}^1 < X_{T-}^2$. Therefore $X_{T-}^1 = X_{T-}^2$ on $\{0 < T < \infty\}$. On $T=0$, we have by convention $X_{T-}^1 = X_{T-}^2$, and it is clear that $X_{T-}^1 = X_{T-}^2$ on this set.

Let X be the semimartingale $(X^1 - X^2)_{T+t}$ on $\{T < \infty\}$, relative to the family (\mathbb{F}_{T+t}) . From the above, we have $X_0 = 0$. X belongs to the class of semimartingales considered at the beginning, and we set $X = M + A$ as before. There is an interval $[0, U(\omega)[$ on which $\Delta X \leq 0$, $\int_0^\cdot \mathbb{1}_{\{X_{s-} > 0\}} dA_s^C \leq 0$, due to the third hypothesis, and the right continuity of $b^i(T+s, X_{T+s-}^i)$, $c^i(T+s, X_{T+s-}^i)$. Finally, the first hypothesis will imply, exactly as in LeGall's paper [2], that $L^0(X) = 0$ (this is the key point of the proof).

Then we apply lemma 1, not to X , but to X stopped at $U-$, where

$$U = \inf\{t > 0 : \Delta X_t > 0 \text{ or } \int_0^t \mathbb{1}_{\{X_{s-} > 0\}} dA_s^C > 0\}$$

which is a.s. $> T$ due to the above: we deduce that $X \leq 0$ on $[0, U[$, which contradicts the definition of T .

REMARKS. 1) The first hypothesis can be weakened as

- $\sigma(s, x)$ is Borel measurable, and for any x there is a $\delta(x) > 0$ such that

$$|\sigma(s, x) - \sigma(s, y)| \leq \rho(|x - y|) \text{ for } y \in [x - \delta(x), x + \delta(x)].$$

In fact, if we set $V = \inf\{t > 0 : |\sigma(t, X_{t-}^1) - \sigma(t, X_{t-}^2)| > \rho(|X_{t-}^1 - X_{t-}^2|)\}$
 $L^0(X^{V-}) = 0$ and we may apply lemma 1 to $X^{(U \wedge V)-}$.

2) As we mentioned, the key point of the proof is to check $L^0(X)=0$, and we deduced this from our first hypothesis as in [2]. Similar conditions ensuring that $L^0(X)=0$ (see [2], Corollaire 1.2) will lead to the same conclusion $X^1 \leq X^2$.

Similarly, we can prove the following theorem.

THEOREM 2. Let X^i be solutions of the following stochastic differential equations

$$X_t^i = X_0^i + \int_0^t \sigma(s, X_{s-}^i) dW_s + \int_0^t b^i(s, X_{s-}^i) ds + \int_0^t \int_{U_0} f(s, X_{s-}^i, u) \hat{N}_p(ds, du) \\ + \int_0^t \int_{U \setminus U_0} g^i(s, X_{s-}^i, u) N_p(ds, du) .$$

Here (W_t) is a Wiener process, N_p is the counting measure of a quasi-left continuous point process p on a standard measurable space U , $U_0 \subset U$ is a measurable subset such that $E[N_p(t, U \setminus U_0)] < \infty$ for t finite, and $\hat{N}_p = N_p - \tilde{N}_p$ ($\tilde{}$ denoting compensation as usual).

We may assert that $X^1 \leq X^2$ on $\{X_0^1 \leq X_0^2\}$ if the following hypotheses are satisfied :

- σ and b^i are as in the preceding theorem.
- f^i, g^i are measurable functions on $\mathbb{R}_+ \times \mathbb{R} \times U$, and for any fixed $u \in U$, $f^i(s, x, u)$ and $g^i(s, x, u)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}$ in the same topology as in theorem 1.
- $(x \leq y) \Rightarrow (f^1(s, x, u) \leq f^2(s, y, u) \text{ and } g^1(s, x, u) \leq g^2(s, y, u))$.

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