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The gauge and conditional gauge theorem

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Let \( \{X_t, \ t \geq 0\} \) be the Brownian motion in \( \mathbb{R}^d, \ d \geq 1 \). Let \( D \) be a bounded domain in \( \mathbb{R}^d \), \( \overline{D} \) its closure, \( \partial D \) its boundary; and let \( q \) be a Borel function defined in \( \mathbb{R}^d \) and satisfying the following condition:

\[
\limsup_{t \to 0} \sup_{x \in D} \mathbb{E}_{x}^{X} \left[ \int_{0}^{t} l_{D} |q(X_s)| ds \right] = 0
\]

where \( l_{D} \) is the indicator of \( D \). Such a function is said to belong to the Kato class \( K_d \). The equivalent condition (1) is given by Aizenman and Simon [1].

The gauge for \( (D,q) \) is defined to be the function \( u \) on \( \overline{D} \) below:

\[
u(x) = \mathbb{E}_{x}^{X} \left[ \exp \left( \int_{0}^{t} q(X_s) ds \right) \right].
\]

From here on we write for abbreviation:

\[
e_q(t) = \exp(\int_{0}^{t} q(X_s) ds).
\]

For a domain \( D \) with \( m(D) < \infty \) (where \( m \) denotes the Lebesgue measure), without any regularity hypothesis on \( \partial D \), and a bounded \( q \), we proved the following theorem in [3].

**The Gauge Theorem.** If \( u(\cdot) \neq \infty \) in \( D \), then \( u \) is bounded in \( \overline{D} \).

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Actually, if $\partial D$ is regular in the sense of the Dirichlet problem, then $u$ is continuous and strictly positive in $\overline{D}$. However, in this note we shall concentrate on the main thing, as stated above. Zhao [6] extended the theorem to $q \in K_d$ for a bounded domain in $\mathbb{R}^d$, $d \geq 3$; he also did the case $d = 2$ in yet unpublished notes. For $d = 1$ and $D$ a half-line, see [2]. Prior to Zhao's work, Falkner extended the theorem in another direction by considering the conditional gauge for $(D,q)$ as follows:

\begin{equation}
(4) \quad u(x,z) = E^x_z \{ e_q(\tau_D) \}, \quad (x,z) \in D \times \partial D;
\end{equation}

where $E^x_z$ is the expectation associated with the Brownian motion killed outside $D$, starting from $x$ and conditioned to converge to $z$ (at its life-time $\tau_D$). For a class of bounded domains including those with $C^2$ boundary, and bounded $q$, he proved the following theorem in [5].

**Conditional Gauge Theorem.** If $u(\cdot,\cdot) \neq \infty$ in $D \times \partial D$, then it is bounded there. This is the case if and only if $u(\cdot) \neq \infty$ in $D$, as in the gauge theorem.

I gave a simpler proof of Falkner's theorem in [4]. Subsequently Zhao [7] proved that if $u(\cdot) \neq \infty$, then $u(\cdot,\cdot) \neq \infty$, for bounded $C^2$ domains. He has since proved the conditional gauge theorem as stated above for bounded $C^{1,1}$ domains. In this note I shall show that the conditional gauge theorem actually follows in a general way and rather quickly from the gauge theorem.
The basic probabilistic argument turns out to be an old one in [2] (see the proof of Theorem 1 there), easily adapted to the multi-dimensional case. The sole difficulty encountered in extending the class of bounded $q$ to the class $K_d$ is contained in Lemma 1 below.

We begin by setting up the framework of the probabilistic argument involving a sequence of hitting times. Let $D_1$ and $D_2$ be subdomains of $D$ such that $\overline{D_1} \subset D_2$, $\overline{D_2} \subset D$, and $C \neq D - \overline{D_1}$ is connected and $m(C) < \varepsilon$ for an arbitrary $\varepsilon > 0$. This is possible if $\partial D$ is Lipschitzian for instance. For then each connected component of $\mathbb{R}^d - \overline{D}$ must contain a ball of fixed size, hence there are at most a finite number of "holes" inside the outer boundary of $D$. Since $D$ is connected, it is easy to see how to construct $D_1$ and $D_2$ as desired. A picture illustrates the result. I am indebted to Falkner for alerting me to the necessity of making $C$ connected.

**Lemma 1.** If $\partial D$ is sufficiently smooth, then for any given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that if the $C$ described above has $m(C) \leq \delta(\varepsilon)$, then
In [7], Zhao proved that $C^2$ boundary is sufficient for the lemma to hold; more recently he has improved this result to require only $C^{1,1}$ boundary. In this connection it should be mentioned that the gauge theorem for an arbitrary bounded domain $D$, and $q \in K_d$, follows quickly from an easier analogue of (5) for a small ball $B$, as follows:

$$\sup_{x \in B} E^X_z \left\{ \int_0^{\tau_B} |q|(X_t) dt \right\} \leq \varepsilon. \quad (7)$$

This was proved in Zhao [6]. The deduction of (6) from (5) is standard Markovian calculation.

Lemma 2 is a strengthened form of an argument I have indicated elsewhere (see [5], Remark 2.13). The constants $a_1, a_2, \ldots$ below are strictly positive, depending only on $D_1, D_2$ and $D$. We assume $\partial D$ to be Lipschitzian below.

**Lemma 2.** For all $y \in \partial D_2$ and $z \in \partial D$, we have

$$a_1 \leq E^Y_z \{ \tau_C = \tau_D, e_q(\tau_D) \} \leq a_2. \quad (8)$$
Proof: Recall that

\[(9) \quad p^y_z[\tau_C < \tau_D] = \frac{f(y,z)}{K(y,z)}\]

where \(K\) is the Poisson kernel for \(D\), and

\[f(y,z) = E^y[\tau_C < \tau_D; K(X(\tau_C), z)] .\]

For each \(y \in \partial D_2\), \(f(y, \cdot)\) is continuous on \(\partial D\), because on \(\{\tau_C < \tau_D\}\) we have \(X(\tau_C) \in \partial D_1\) almost surely, and \(K\) is bounded continuous in \(\partial D_1 \times \partial D\). For each \(z \in \partial D\), \(f(\cdot, z)\) is harmonic in \(C\). Hence \(f\) is continuous on \(\partial D_2 \times \partial D\) since \(\partial D_2\) and \(\partial D\) are disjoint closed sets. It follows that the function of \((y, z)\) in (9) is continuous and positive on \(\partial D_2 \times \partial D\). The function \(K(\cdot, z) - f(\cdot, z)\) is harmonic in \(C\) and unbounded in the neighborhood of \(z\), because \(K\) is unbounded while \(f\) is bounded. Hence it is strictly positive in \(C\) by harmonicity, because \(C\) is connected and \(z \in \partial C\). Therefore we have by continuity

\[(10) \quad b = \inf_{y \in \partial D_2, z \in \partial D} p^y_z[\tau_C = \tau_D] > 0 .\]

Now it follows by Jensen's inequality and (15) that for \((y, z) \in \partial D_2 \times \partial D:\)

\[(11) \quad E^y_z[e^{-\varepsilon \tau_D} | \tau_C = \tau_D] \geq E^y_z[e^{-\varepsilon \tau_D} | \tau_C = \tau_D] \geq \exp\left\{ -E^y_z[\int_0^{\tau_D} |q|(X_t) dt | \tau_C = \tau_D] \right\} \geq \exp\left\{ -\frac{1}{b} \int_0^{\tau_C} |q|(X_t) dt \right\} \geq e^{-\varepsilon/b} .\]
Combining (10), (11) and (16), we have proved (8) with 
\[ a_1 = b \exp^{-\varepsilon/b}, \]
\[ a_2 = \frac{1}{1-\varepsilon}. \]

We are ready to prove the conditional gauge theorem for a bounded Lipschitzian domain for which the conclusions of Lemma 1 hold true, thus at least when \( \partial D \) belongs to \( C^{1,1} \). Put \( T_0 = 0 \), and for \( n \geq 1 \):
\[ T_{2n-1} = T_{2n-2} + \tau_{D_2} \circ \theta_{T_{2n-2}} \]
\[ T_{2n} = T_{2n-1} + \tau_{C} \circ \theta_{T_{2n-1}}. \]

For any \( (x,z) \in D \times \partial D \), we have \( P^x_z[\tau_D < \infty] = 1 \). This nontrivial result has recently been proved by M. Cranston for a bounded Lipschitzian domain; for a bounded \( C^1 \)-domain \( D \) it follows from the fact that \( K(\cdot,z) \) is integrable over \( D \), by a remark communicated to me by Kenig. It follows that for some \( n \geq 1 \), \( X(T_{2n}) \in \partial D \).

Therefore we have by the strong Markov property of the conditioned process:
\[ (12) \quad E^X_z[e_q(\tau_D)] = \sum_{n=1}^{\infty} E^X_z[T_{2n} = \tau_D; e_q(\tau_D)] \]
\[ = \sum_{n=1}^{\infty} E^X_z[T_{2n-2} < \tau_D; e_q(T_{2n-1}) E^X_z[T_{2n-1}] \{\tau_C = \tau_D; e_q(\tau_D)\}] \]

Observe that \( \partial C = \partial D_1 \cup \partial D \). On the set \( \{T_{2n-2} < \tau_D\}, \ X(T_{2n-1}) \in \partial D_2 \).

Hence by Lemma 2
\[ (13) \quad a_1 \sum_{n=1}^{\infty} E^X_z[T_{2n-2} < \tau_D; e_q(T_{2n-1})] \leq E^X_z[e_q(\tau_D)] \]
\[ \leq a_2 \sum_{n=1}^{\infty} E^X_z[T_{2n-2} < \tau_D; e_q(T_{2n-1})]. \]
The general term in the series above is explicitly:

\[(14) \quad \frac{1}{K(x,z)} E^\chi \{ T_{2n-2} < \tau_D; e_q(T_{2n-1}) K(X(T_{2n-1}), z) \} \]

Since \( K \) is continuous and strictly positive on \( \overline{D}_2 \times \partial D \), we have for \((x,z)\) and \((x',z')\) in \( \overline{D}_2 \times \partial D \), almost surely

\[(15) \quad \frac{K(X(T_{2n-1}), z')}{K(x', z')} \leq \frac{K(X(T_{2n-1}), z)}{K(x, z)} \leq \frac{K(X(T_{2n-1}), z')}{K(x', z')} \]

where \( a_3 \) and \( a_4 \) depend only on \( \overline{D}_2 \) and \( D \). It follows from (13), (14) and (15) that

\[(16) \quad \sup_{x \in \overline{D}_2} \sup_{z \in \partial D} u(x, z) \leq \frac{a_2 a_4}{a_1 a_3} \inf_{x \in \overline{D}_2} \inf_{z \in \partial D} u(x, z) \cdot \]

Since \( u(x) \) is a probability average of \( u(x, z) \) over \( z \in \partial D \), we have

\[(17) \quad \inf_{z \in \partial D} u(x, z) \leq u(x) \leq \sup_{z \in \partial D} u(x, z) \cdot \]

Now by hypothesis of the theorem, there exists \((x_0, z_0) \in D \times \partial D\) such that \( u(x_0, z_0) < \infty \). Without loss of generality we may suppose \( x_0 \in D_2 \). Hence by (16)

\[(18) \quad \sup_{z \in \partial D} u(x_0, z) \leq \frac{a_2 a_4}{a_1 a_3} u(x_0, z_0) < \infty \cdot \]

Next by (17), \( u(x_0) < \infty \). Hence by the gauge theorem, \( \sup_{x \in \overline{D}} u(x) < \infty \). It follows then by (16) and (17) that

\[(19) \quad \sup_{x \in \overline{D}_2} \sup_{z \in \partial D} u(x, z) < \infty \cdot \]
For we use the old argument in [3] adapted to the conditioned process, as follows:

\[ u(x,z) = E^x_z[\tau_C = \tau_D, e^{q(\tau_C)}] + E^x_z[\tau_C < \tau_D, u(X(\tau_C),z)] \]

\[ \leq \frac{1}{1-\varepsilon} + \sup_{x \in \mathbb{D}_1} \sup_{z \in \mathbb{D}} u(x,z) < \infty. \]

This establishes the first assertion of the conditional gauge theorem. The second assertion has also been proved between the lines above.

Remark: Conditional gauge theorem is also true for a bounded \( C^1 \) domain, and bounded \( q \), using a hard inequality of Kenig's to prove lemma 1.

References


