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ON THE UNBOUNDEDNESS OF MARTINGALE TRANSFORMS

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The starting point for our investigation is an observation of Stein and Weiss (1959) or more precisely Davis' (1973) proof of this fact. To state their results and explain our motivation, we will need a number of definitions:

Let B_t be a two dimensional Brownian motion.
Let $D = \{z: |z| < 1\}$
Let $\tau = \inf\{t: B_t \notin D\}$
Let $E \subset \partial D$ and let $u(x) = P_x(B_\tau \in E)$.

Finally let $v(x)$ be the "harmonic conjugate" of u: i.e. the unique function with $v(0) = 0$ which makes $u + iv$ an analytic function.

The function u is an object which has been much studied by probabilists (see e.g. Port and Stone (1978), F. Knight (1981), or Chung (1982)) and it is well known that u is harmonic in D and

(1) $$\lim_{t\uparrow\tau} u(B_t) = 1_E(B_\tau) \quad \text{a.s.}$$

Stein and Weiss' result shows that u's harmonic conjugate is also special.

(2) $\lim_{t\uparrow\tau} v(B_t)$ exists a.s. and furthermore the distribution of the limit depends only on $P_0(B_\tau \in E)$.

Stein and Weiss proved (2) by supposing E was a finite union of intervals and then patiently finding the places where $v(e^{i\theta}) > y$. See pp. 273-274. In (1973) Davis gave the following proof of their result which makes the conclusion obvious.

Proof of (2). Itô's formula implies that if $t < \tau$

$$u(B_t) = \int_0^t \nabla u(B_s) \cdot dB_s$$

$$v(B_t) \doteq \int_0^t \nabla v(B_s) \cdot dB_s$$

and the Cauchy Riemann equations:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

imply $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ so it follows from Lévy's theorem (see Meyer

(1976), or Durrett (1984), Section 2.11) that $(u(B_t),v(B_t))t < \tau$ is a time change of a Brownian motion $\bar{B}_u$ run for a random amount of time $u < \sigma$.

To prove (2) we will show that $\sigma = T \equiv \inf\{u: \bar{B}_u^1 \notin (0,1)\}$. If we discard the trivial cases $P_0(B_\tau \in E) = 0$ or 1 then $0 < u(x) < 1$ for $x \in D$ and hence $u(B_t) \in (0,1)$ for $t < \tau$ so $\sigma \geq T$. (1) shows we cannot have $\sigma > T$ so we must have $\sigma = T$.

To motivate our generalization we begin by redescribing the relationship between u and v. It is well known (see Meyer (1976) or Durrett (1984), Section 2.14) that

(3) If $X \in \sigma(B_t\ t \geq 0)$ has $EX = 0$ and $EX^2 < \infty$ then

$$X = \int_0^\infty H_s \cdot dB_s$$

where

$$EX^2 = E \int_0^\infty |H_s|^2\, ds.$$

Let A be a $d \times d$ matrix. Since

$$E \int_0^\infty |AH_s|^2\, ds \leq E \int_0^\infty C|H_s|^2\, ds = CEX^2 < \infty$$

the Burkholder Gundy inequalities (see Meyer (1976) or Durrett (1984), Section 6.3) imply that $\int_0^t AH_s \cdot dB_s$ is an L^2 bounded martingale so we can define a new random variable by setting

$$A * X = \int_0^\infty AH_s \cdot dB_s$$

(see Durrett (1984), Section 6.6 for more details).

$A * X$ is called a martingale transform. If $d = 2$ and we let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $X = 1_{(B_\tau \in E)}$ then using the notation introduced in the proof of (2) the Cauchy Riemann equations can be written as $\nabla v = A\nabla u$, and it follows that $A * X = v(B_\tau)$.

With conjugation identified as a martingale transform, it becomes natural to ask when (2) holds for martingale transforms. Tracing back through the proof of (2) gives the following result:

(4) Suppose A satisfies (a) $y \cdot Ay = 0$ and (b) $|y| = |Ay|$ for all $y \in R^d$ then the distribution of $A * 1_B$ depends only on $P(B)$.

Unfortunately matrices which satisfy both (a) and (b) are rare. There are none if d is odd because such matrices must have a real eigenvector and yet

(a) $\Rightarrow$ there is no nonzero real eigenvalue

(b) $\Rightarrow$ A is invertible $\Rightarrow$ 0 is not an eigenvalue.

In even dimensions the situation is somewhat better but not much. It is easy to see that there are examples

(5) $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) \quad \ldots$$

and it is also easy to see that these are the only ones.

(6) Any matrix satisfying (4) can after a change of basis be written in the form given in (5).

Proof. Let x have norm 1 and let $y = Ax$. (a) and (b) imply $x \cdot y = 0$ and $|y| = 1$. Using (a) twice more gives

$$0 = (x + y) \cdot A(x + y) = y \cdot Ax + x \cdot Ay$$

so

$$x \cdot Ay = -y \cdot y = -1$$

and since $|Ay| = |y| = 1$ it follows that $Ay = -x$.

The last result shows that the behavior observed by Stein and Weiss is very rare among martingale transforms and in fact distinguishes "conjugation" and its generalizations to R^{2n} (the Hilbert transforms of Varopoulos (1980)) from the other martingale transforms. Faced with this situation, if we want to prove something for more general matrices we have to settle for something less than the conclusion of (4). The next result shows that we can weaken the condition on the matrices quite a bit without sacrificing too much in the conclusion.

(7) If A has no real eigenvalue then there are C and γ which depend on A and $P(B)$ so that $P(\sup_t |(A * 1_B)_t| > y) \geq Ce^{-\gamma y}$

Before proving this we would like to make two remarks which explain the condition and the conclusion.

1. The result is false if A has a real eigenvalue for if v R^d is an associated real eigenvector and we

let $Y_t = \frac{1}{2} + \int_0^t v \cdot dB_s$

let $\sigma = \inf \{t: Y_t \notin (0,1)\}$

and let $X_t = Y_{t \wedge \sigma}$

then $X_\infty = 1_{(Y_\sigma = 1)}$ but

$$(A * X) = \int_0^\sigma Av \cdot dB_s$$

$$= \lambda \int_0^\sigma v \cdot dB_s = \lambda(Y_\sigma - Y_0)$$

so $A * X$ is bounded.

2. Well known formulas for Brownian motion show that when $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the left hand side of (7) is $\sim Ce^{-\gamma y}$ (here C, γ $(0,\infty)$ are constants whose values may change from line to line) and the John Nirenberg inequality (see Meyer (1976) or Durrett (1984), Section 7.6) implies that for any matrix A

$$P\left(\sup_t |(A * 1_B)_t| > y\right) \leq Ce^{-\gamma y}$$

where C, γ depend only on A and $P(B)$ so we cannot hope for a better lower bound.

Proof of (7). Let $X = 1_B$, $X_t = E(X|\mathcal{F}_t)$. We will prove (7) by showing that although $(X_t, (A*X)_t)$ may not be a time change of Brownian motion, there is a part of $A * X$ which is independent of X and which is a time change of a Brownian mition run for an amount of time $\geq \varepsilon T$ where T is the time defined in the proof of (2).

To isolate the part of $A * X$ we want, we introduce the following orthogonal decomposition of Ax

$$Ax = C(x)x + F(x)$$

where $C(x)$ is a number and $F(x) \in R^d$ has

$$F(x) \cdot x + 0.$$

It is easy to see that the last two equations specify $C(x)$ and $F(x)$ and we have $|F(x)| \leq |Ax|$. To prove (7) we need a bound in the other direction. To do this we observe that if A has no real eigenvalues then $F(x) \neq 0$ for all $x \neq 0$ and scaling implies that for $y \neq 0$

$$|F(y)| = |y| \left|F\left(\frac{y}{|y|}\right)\right|$$

so we have

$$\text{(8)} \qquad \inf_{y \neq 0} \frac{|F(y)|}{|y|} = \inf_{z,|z|=1} |F(z)| > 0.$$

With (8) established our next step is to decompose $(A * X)_t$. If

$$X_t = \int_0^t H_s \cdot dB_s$$

then

$$(A * X) = \int (AH_s) \cdot dB_s$$

so we let

$$Y_t = \int_0^t C(H_s)H_s \cdot dB_s$$

$$Z_t = \int_0^t F(H_s) \cdot dB_s .$$

The formula for the covariance of two stochastic integrals (see Meyer (1976) or Durrett (1984) Chapter 2) implies

$$\langle X,Z\rangle_t = \int_0^t F(H_s) \cdot H_s \, ds = 0$$

and (8) tells us that

$$\langle Z\rangle_t \equiv \langle Z,Z\rangle_t = \int_0^t |F(H_s)|^2 \, ds$$

$$\geq \varepsilon^2 \langle X\rangle_t$$

At this point we have found the part of $A * X$ we referred to at the beginning of the proof. The next step is to show Z has the desired properties. To do this we let

$$\gamma(u) = \inf \{t: \langle Z\rangle_t > u\} \quad \text{for} \quad u < \langle Z\rangle_\infty$$

and define

$$W_u = \begin{cases} Z_{\gamma(u)} & u < \langle Z\rangle_\infty \\ Z_\infty + \hat{B}_{u-Z_\infty} & u \geq \langle Z\rangle_\infty \end{cases}$$

where $\hat{B}$ is a one dimensional Brownian motion which is independent of the d-dimensional Brownian motion B. We have added $\hat{B}$ after the end of Z so that the following holds.

(9) W is a Brownian motion which is independent of $\sigma(X_t, t \geq 0)$.

Proof. This is a consequence of a theorem of F. Knight (1971) but the proof is short so we will prove it. It is easy to check that W_u $u \geq 0$ is a local martingale and $W_u \equiv u$ (for more details see Meyer (1976), or Durrett (1984)

Section 2.11) so it follows from Lévy's characterization that W_u is a Brownian motion. To check the independence

let $$U = \int f_s \, dX_s$$

and let $$V = \int g_s \, dW_s$$

be stochastic integrals with

$$\int |f_s|^2 \, d\langle X\rangle_s, \quad \int |g_s|^2 \, ds < \infty$$

and $g_s = 0$ for $s \geq \langle Z\rangle_\infty$. Unscrambling the definitions we see that

$$\int f_s \, dX_s = \int (f_s H_s) \cdot dB_s$$

and

$$\int g_s \, dW_s = \int g_s \, dZ_{\gamma(s)}$$

$$= \int g(\langle Z\rangle_t) \, dZ_t$$

$$= \int g(\langle Z\rangle_s) F(H_s) \cdot dB_s ,$$

so it follows from the formula for the covariance of two stochastic integrals that

$$EUV = E \int f_s g(\langle Z\rangle_s) H_s \cdot F(H_s) \, ds = 0.$$

It is trivial that we have $EUV = 0$ if $g_s = 0$ for $s \leq \langle Z\rangle_\infty$ so the last equality holds for any f,g which satisfy (*) and hence for any $V \in L^2(\sigma(X_t: t \geq 0))$ and $V \in L^2(\sigma(W_u: u \geq 0))$ which proves (9).

With (9) established the rest is simple but requires a little trickery. Z_t is a time change of W_u $u < \langle Z\rangle_\infty$ and by (8) $\langle Z\rangle_\infty \geq \varepsilon^2 \langle X\rangle_\infty$, so we have

$$\sup_t |Z_t| \geq \sup_{u \leq \varepsilon^2 \langle X\rangle_\infty} |W_u| .$$

where $\langle X\rangle_\infty \in \sigma(X_t: t \geq 0)$ is independent of W. To get a lower bound on $\sup_t |(A * X)_t|$ find the first point $u_0 \leq \varepsilon^2 \langle X\rangle_\infty$ where the sup on the right occurs. If we let $t_0 = \gamma(u_0)$ (which is finite since $u_0 \leq \varepsilon^2 \langle X\rangle_\infty \leq \langle Z\rangle_\infty$ then

$$(A * X)_{t_0} = Y_{t_0} + Z_{\gamma(u_0)} .$$

At this point we could get unlucky and Y_{t_0} could cancel $Z_{\gamma(u_0)}$, but the sign of $Z_{\gamma(u_0)}$ is independent of the sign of Y_{t_0} so at least ½ of the time Y_{t_0} will make $|(A * X)_{t_0}| \geq |Z_{\gamma(u_0)}|$ and it follows that

$$P(|(A * X)_{t_0}| > y) \geq \tfrac{1}{2} P\left(\sup_{u \leq \varepsilon^2 \langle X \rangle_\infty} |W_u| > y \right)$$

To compute the quantity on the right hand side and complete the proof of (8) we observe that since X_∞ is independent of W_u

$$\sup_{u \leq \varepsilon^2 \langle X \rangle_\infty} |W_u| \overset{d}{=} \varepsilon \sup_{u \leq \langle X \rangle_\infty} |W_u|$$

and the distribution of the right hand side is given in Remark 2.

Having proved (7) for one matrix, it is natural, especially if you have heard of Janson's (1977) theorem (see e.g. Durrett (1984), Section 6.7.), to ask what happens if we have a family of matrices without a common real eigenvector. The answer is just what you should expect:

(10) If $A^1, \ldots, A^m$ are matrices without a common real eigenvector then there are constants C and γ which only depend on $P(B)$ (and the matrices) so that

$$P(\sup_i \sup_t |(A^i * 1_B)_t| \quad y) \geq Ce^{-\gamma y}$$

Since this is a rather straightforward generalization of (7) we will explain why we want to prove this before we describe how to do it. In our discussion of (4) above we observed that if d is odd then A must have a real eigenvector, so the hypothesis of (7) cannot be satisfied in this case. With the matrices of (5) in mind you might realize that in the first nontrivial case $(d = 3)$ it is easy to write down two matrices which have no common real eigenvector

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Congratulations, you have just (re)discovered the Riesz transforms. Gundy and Varopolous (1979) (and later by a different method Gundy and Silverstein (1982)) have shown that if we define a process W_t $-\infty < t \leq 0$ in $H = R^n \times (0,\infty)$ which is a Brownian motion which "starts at time $-\infty$ from Lebesgue measure on $R^n \times \{\infty\}$ and exits H at time 0," then the Reisz transforms are related to martingale transforms of W.

To explain the relationship we need some notation. Let f be a function on ∂H which is in L^2 and let

$$u(z) = E_z f(B_\tau)$$

where $\tau = \inf\{t: B_t \notin H\}$. If we let A^i be the matrix which has

$$A^i_{jk} = \begin{cases} 1 & j = 1 \quad k = i \\ -1 & j = i \quad k = 1 \\ 0 & \text{otherwise} \end{cases}$$

then the ith Riesz transform may be written as

$$R_i f(w_0) = E\left(\int_{-\infty}^{0} A^i \nabla u(w_s) \cdot dw_s \,\middle|\, w_0 \right).$$

Since the Riesz transforms are (for the theory of Hardy spaces at least) the appropriate generalization to H of conjugation in D, it is natural to ask if

$$|\{x: \sup_i R_i 1_B(x,0) > \lambda\}| \geq Ce^{-\gamma\lambda}$$

where C and γ are constants which depend only on $|B|$. (10) shows that the analogous result is true for martingale transforms and that the stochastic integral in (11) is unbounded. Unfortunately the conditional expectation might convert the integral into a bounded function so we have not been able to use this to solve the (still open) question posed above.

Proof. For simplicity, we will give the proof only for $m = 2$. The reader can obtain a proof of the general result by changing 2 to m and inserting ... at appropriate points. As in the proof of (7) we begin by introducing orthogonal decompositions

$$A^1 x = c^1(x)x + F^1(x)$$

$$A^2 x = c_0^2(x)x + c_1^2(x)F^1(x) + F^2(x)$$

where $F^i(x) \cdot x = 0 \quad i = 1, 2$ and $F^1(x) \cdot F^2(x) = 0$.

Now if A^1 and A^2 have no common real eigenvector then

$$\{F^1(x) = 0\} \cap \{c_1^2(x)F^1(x) + F^2(x) = 0\} = \emptyset$$

i.e. $\{F^1(x) = 0\} \cap \{F^2(x) = 0\} = \emptyset$ and repeating the proof of (8) shows

$$\text{(11)} \qquad \inf_{x \neq 0} \frac{|F^1(x) + F^2(x)|}{|x|} \equiv \varepsilon > 0.$$

The next step is the decompose the $(A^i * X)_t$ and time change some of the pieces to produce independent Brownian motions.

Let $Z_t^i = \int_0^t F^i(B_s) \cdot dB_s$

let $Y_t^i = (A^i * X)_t - Z_t^i$

let $\gamma_i(u) = \inf\{t: \langle Z \rangle_t > u\}$

and let

$$W_u^i = \begin{cases} Z^i_{\gamma_i(u)} & u < \langle Z^i \rangle_\infty \\ Z^i_\infty + \hat{B}^i_{u-\langle Z^i \rangle_\infty} & u \geq \langle Z^i \rangle_\infty \end{cases}$$

where $\hat{B}^1$ and $\hat{B}^2$ are independent Brownian motions which are independent of B.

A simple generalization of (9) (or invoking Knight's theorem) implies that W_u^1 and W_u^2 are independent Brownian motions which are independent of $\sigma(x_t: t \geq 0)$ and (11) implies that $\langle Z^1 \rangle_\infty + \langle Z^2 \rangle_\infty \geq \varepsilon^2 \langle X \rangle_\infty$ so now we can complete the proof almost as before.

Let $j = \inf\{i: \langle Z^i \rangle_\infty \geq \varepsilon^2/2 \langle X \rangle_\infty\}$

Let u_0 be the first point at which

$$\sup_{u \leq (\varepsilon^2/2)\langle X \rangle_\infty} |W_u^j| \text{ is attained.}$$

Let $t_0 = \gamma^j(u_0)$

$$(A^j * X)_{t_0} = Y^j_{t_0} + Z^j_{\gamma^j(u_0)}$$

and again the signs of the two terms on the right are independent so

$$P(|(A^j * X)_{t_0}| > y) \geq \tfrac{1}{2} P(\sup_{u \leq (\varepsilon^2/2)\langle X \rangle_\infty} |W_u^j| > y)$$

proving the desired result.

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