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BROWNIAN MOTION ON A SURFACE OF NEGATIVE CURVATURE

Wilfrid S. Kendall

1. Introduction

Dynkin (1965; introduction, section 9) asked whether something could be said about the nonnegative harmonic functions on a simply-connected manifold of negative curvature bounded away from zero, and the relationship of this to the asymptotic behaviour of the angular component of Brownian motion on the manifold. Answers were found by Prat in the two-dimensional case (Prat (1971)) and in the general case (Prat (1975)). Kifer (1976) further elucidated the asymptotic behaviour of the angular component. However, in all these treatments extra conditions are required on the metric or on the curvature, beyond the simple condition that the sectional curvatures are bounded above by a negative constant. These extra conditions either bound the curvature below as well as above, or require the curvature to vary slowly at infinity, or place a bound on the angular variation of the metric.

In the two-dimensional case a geometric approach yields results on the behaviour of the angular component simply and without requiring additional conditions on the curvature or metric. The approach grows out of the work of Greene and Wu (1979), especially chapter 7 of that reference. The present paper provides an exposition, developing a note at the end of Kendall (1983). There it was noted that general Riemannian manifolds of negative curvature could be treated in a simple fashion if they contained totally geodesic sub-manifolds of codimension one. Generally this is a very restrictive condition. However in the special case of manifolds of dimension two it is always generously satisfied by the geodesics themselves. So Brownian motion on two-dimensional manifolds of negative curvature becomes particularly amenable to study.

Thus one has the theorem proved in this paper;

Theorem:

Let X be a Brownian motion on a simply-connected, complete Riemannian manifold, of dimension two, of negative curvature everywhere bounded above by a negative constant. The the limiting direction of X exists and has probability law of dense support on the whole absolute circle of directions.

The reader will wish to know whether these methods can be extended to higher dimensions. At present this seems a difficult problem. A closely related question in geometric function theory is reported as a conjecture by Greene and Wu (1979; page 3).

2. Preliminaries

A familiarity on the part of the reader with the basics of Riemannian manifolds will be assumed. A rapid introduction to that theory can be found in Cheeger and Ebin (1975).

For the purposes of this paper, Brownian motion X on a complete Riemannian manifold M is most conveniently defined by means of a martingale characterisation (see Stroock and Varadhan (1979 ch 6) and Williams (1981) for the martingale characterisation approach to diffusion theory). Thus a M-valued random process X, continuous up to a (possibly infinite) explosion time ζ, is said to be a Brownian motion on M if the following holds:

$$d\, f(X) = d\, C^f + (1/2)\, \Delta f\, (X)\, dt \qquad (2.1)$$

(Itô differentials)

implicitly defines a martingale C^f whenever f is a smooth function of compact support.

In symbols X is written as BM(M).

Here Δ is the Laplace-Beltrami operator of the Riemannian manifold M. A definition of Δ convenient for our purposes is given below.

In the sequel M is to be two-dimensional, of curvature bounded below by a negative constant $-H^2 < 0$. Thus by the Cartan-Hadamard theorem, Cheeger and Ebin (1975), it is possible to lay down normal polar co-ordinates (r,θ) in M about any specified point. If the curvature of M is not bounded below then it is possible for the explostion time ζ to be finite. This occurs for example if the metric is given by

$$ds^2 = dr^2 + r^2 \exp\,(2r^3/3)\; d\,\theta^2$$

working in polar co-ordinates (r, θ) about some specified point. For then the curvature K is given by

$$K = -4r\; -4r^4$$

and Azencott's condition (Azencott (1974 Prop. 7.9) applies. Following an analogy between Brownian motion and geodesics, if explosion cannot occur then M is said to be <u>stochastically complete</u>. The stochastic completeness of M is not assumed here; the case of explosion will require separate attention. Whether M is stochastically complete or not, the upper bound on the negative curvature forces the Brownian motion X to be transient (Prat (1971 Thm. 1)).

If f in (2.1) is merely C^2 then C^f is a local martingale defined up to the explosion time ζ. In this paper a particular $f = \tau$ is considered for which $C^f = C^\tau$ is actually a real-valued Brownian motion up to time ζ.

The Laplace-Beltrami operator can be defined by

$$f(m) = \frac{d^2}{dt^2}\, f \circ \gamma^1\,(t)\Big|_{t=0} + \frac{d^2}{dt^2}\, f \circ \gamma^2(t)\Big|_{t=0} \qquad (2.2)$$

where γ^1 and γ^2 are an orthonormal pair of geodesics emanating from m. See Greene and Wu (1979) for a discussion of (2.2) and related formulae. This definition can be shown to be independent of the choice of the orthonormal pair γ^1, γ^2.

3. Results

Let M be a simply-connected two-dimensional manifold of negative curvature bounded away from zero. Consider a geodesic λ in M. The Cartan-Hadamard theorem, Cheeger and Ebin (1975), asserts that the exponential map at λ (0) is a diffeomorphism

$$\exp : T_{\lambda(o)} M \rightarrow M.$$

Thus M-Im λ forms two components M_1 and M_2, since its preimage under exp is a plane with a line deleted.

LEMMA 1

If M is stochastically complete then with probability one the Brownian motion X will eventually select one of the components of M - Im λ and stay in it for ever. That is,

$$P(\exists T : X(t) \text{ misses Im } \lambda \text{ for all } t > T) = 1 \qquad (3.1)$$

Proof

The proof depends heavily on the geometrical arguments of Greene and Wu (1979: ch 7). As a consequence of the comparison arguments underlying the proof of the Cartan-Hadamard theorem (see for example the discussion of focal points following the Rauch and Berger theorems in Cheeger and Ebin (1975; Thms 1.28, 1.29)) it can be shown that

$$\tau : M \rightarrow R$$

is smooth, where τ is given by $\tau(x)$ = dist (x, Im λ) if x is in M_1 or Im λ, and $\tau(x)$ = -dist(x, Im λ) if x is in M_2. This all follows from the work in the proof of Greene and Wu (1979; ch 7 Prop 7.1). The gradient of τ has length 1, since at x in M one can choose γ^1 parallel and γ^2 perpendicular to the minimising geodesic from x to Im λ, and check that the first derivatives of $\tau\circ\gamma^1$ and $\tau\circ\gamma^2$ at x are 1 and 0 respectively.

A further application of comparison arguments following those of Greene and Wu (1979; chapter 7) shows that

$$\frac{d^2}{dt^2} \tau \circ \gamma^2 (t)\Big|_{t=0} \geq H^{-1} \tanh (H^{-1} \tau(x)/2) \quad \text{on } M_1 \qquad (3.2)$$

$$\leq -H^{-1} \tanh (H^{-1} \tau(x)/2) \quad \text{on } M_2$$

Moreover since the derivative of $\tau \circ \gamma^1$ is always 1 it follows that the corresponding second derivative of $\tau \circ \gamma^1$ is zero. Thus bounds can be obtained for the Laplacian of τ, via formula (2.2).

It is reasonable that the second derivative of $\tau \circ \gamma^2$ should be bounded as in (3.2). By the properties of negative curvature, since γ^2 is parallel to λ at x it should bend away from λ as it leaves x. This forces the second derivative of $\tau \circ \gamma^2$ to have the sign of $\tau \circ \gamma^2$ itself. The bounds of (3.2) quantify this intuitive argument.

These calculations on derivatives allow conclusions as follows. Because the gradient of τ is of unit length, the local martingale C^τ has brackets-process

$$\int_0^{t \wedge \zeta} |\text{grad } \tau|^2 = t \wedge \zeta$$

Therefore C^τ is real-valued Brownian motion stopped at ζ. (This identification of the brackets-process follows from a comparison of the Itô formula with the martingale characterisation of (2.1), both applied to $\{\tau(X)\}^2$). On the other hand the bounds on the Laplacian show that $L = \tau(X) - C^\tau$ has derivative bounded away from zero whenever τ is greater than 1 in absolute value. In fact for some constant c

$dL/dt > c$ on $\{\tau > 1\}$

$dL/dt < -c$ on $\{\tau < -1\}$

Hence a comparison with Brownian motion of constant non-zero drift allows the conclusion that $\tau(X_t)$ must diverge to plus infinity or minus infinity as t increases to infinity. See for example Ikeda and Watanabe (1971; ch VI Thm 1.1).

If M is not stochastically complete then more sophisticated arguments are required.

LEMMA 2

If M is not necessarily stochastically complete then P{either $\tau(X)$ diverges to plus infinity, or to minus infinity, or it converges to zero} = 1.

Proof

In the decomposition

$\tau(X) = C^\tau + L$

the process C^τ is now only a real Brownian motion up to the Markov time ζ of explosion. If explosion time is finite then the limit $C^\tau(\zeta -)$ exists. All depends on the behaviour of the drift process L.

If X is eventually bounded away from zero then so is dL/dt and therefore L and so X must diverge to plus or minus infinity.

Suppose X were not bounded away from zero. Then a proof by contradiction shows it must converge to zero. For otherwise for some $b > 0$ it would have to make an infinite number of upcrossings of $(-b,0)$

or downcrossings of (0,b). The quantity τ (x) dL/dt being always of one sign, this would force the existence of an infinite bounded sequence.

$$t_0 < t_1 < \ldots < t_n < \ldots t, \text{ with } t_n \to t,$$

such that

$$|C^\tau (t_{2n+1}) - C^\tau (t_{2n})| > b$$

with positive probability. Hence

$$E \{C^\tau(t)\}^2 = \Sigma E\{C^\tau (t_{2n+1}) - C^\tau (t_{2n})\}^2 = \infty$$

But C^τ is a real-valued Brownian motion (possibly stopped at a finite time ζ), so $E \{C^\tau (t)\}^2 < \infty$. This contradiction shows that if $\tau(X)$ does not diverge to infinity then it must converge to zero.

In fact a more sophisticated argument using 'horocycles' can exclude the possibility of convergence to zero for $\tau(X)$. But this is not necessary for the purposes of this paper. All that is needed is the following corollary.

COROLLARY

$$P \{\exists T : X(t) \text{ is in } M_1 \text{ for all } t > T\} > 0.$$

and similarly for M_2.

Proof

That X has a positive chance of hitting M_1 follows from observing that the stochastic differential equation for τ (X) before explosion has smooth coefficients with a martingale term of diffusion coefficient one (Stroock and Varadhan (1972)).

If X(t) is in M_1 then there is a positive chance that

$$C^\tau(t + s) - C^\tau(t) > - \tau (X(t))/2$$

for all s with $s < \zeta - t$. For the increment in C^τ is a stopped Brownian motion. Since the drift term is positive on M_1 this establishes that X has a positive chance of ending up permanently in M_1. The case of M_2 is similar.

The lemmas and corollary above enable the proof of the main theorem.

THEOREM

Let X be a Brownian motion on a simply-connected, complete Riemannian manifold, of dimension two, of negative curvature everywhere bounded above by a negative constant. Then the limiting direction of X exists and has probability law of dense support on the whole absolute circle of directions.

Proof

If M is not stochastically complete then the possibility that X converges towards some geodesic has not been excluded. But comparison arguments and the negative curvature of M show that geodesics intersecting at the origin diverge apart at a super-exponential rate. This, together with the transience of X (Prat (1971; Thm 1)), means that if dist $(X, \lambda) \to 0$ for a geodesic λ then X must converge to one of the limiting directions of λ.

Thus it suffices to consider the case when X does not converge to any geodesic of a countable family under consideration. Given normal (polar) coordinates for M, (r, θ), and a real number α, let λ^{α} be the geodesic defined by $\theta = \alpha \mod \pi$. An application of the lemmas above to each of the dissections

$M\text{-Im}\ (\lambda^{k\pi/n})$ for $k = 1,\ldots,n$

shows that eventually X must have direction lying permanently in one of the sectors

$((k-1)\ \pi/n,\ k\ \pi/n)\ :\ k = 1,\ldots,2n$

The convergence of the direction of X follows by letting n tend to infinity.

The law of the limiting direction must charge every open are of the absolute circle of directions. For suppose that (θ, ψ) is an open sub-arc of a specified arc, with closure contained in the super-arc. Then there is a geodesic λ with the two directions θ, ψ as its limiting directions. By the corollary there is a positive chance that X eventually remains permanently in the component of M-Im λ bounded by the arc (θ, ψ), and this forces the limiting direction of X to lie in (θ, ψ), a closed sub-arc of the original arc specified.

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