SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

SHENG-WU HE
JIA-GANG WANG

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Séminaire de probabilités (Strasbourg), tome 18 (1984), p. 256-267
<http://www.numdam.org/item?id=SPS_1984__18__256_0>

Two Results on Jump Processes

He Sheng Wu

East China Normal University, Shanghai, China and

University of Strasbourg, Strasbourg, France

Wang Jia Gang

Fu Dan University, Shanghai, China

1. Introduction. Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space, and $X = (X_t)_{t\geq 0}$ a jump process, i.e. all its trajectories are r.c.l.l. (right-continuous and with left limits) step functions and have only finitely many jumps in every finite interval. Denote by $(T_n)_{n\geq 1}$ the successive jump times of X, and by $(\Delta_n)_{n\geq 1}$ the successive jump sizes of X. By convention we have $T_0 = 0$ and $\Delta_0 = X_0$. Then X can be written as

$$X = X_0 + \sum_{n=1}^{\infty} \Delta_n I_{[\![T_n, \infty [\![} ,$$

and we have

1) $T_n \uparrow \infty$;
2) For all $n \geq 0$, $T_n < \infty \Rightarrow T_n < T_{n+1}$;
3) For all $n \geq 1$, $\Delta_n \neq 0 \Rightarrow T_n < \infty$.

Denote by $\underline{F} = (\underline{\underline{F}}_t)_{t\geq 0}$ the natural filtration of X:

$$\underline{\underline{F}}_t = \sigma\{ X_s, s \leq t, \underline{\underline{N}} \} ,$$

where $\underline{\underline{N}}$ is the family of P-null sets. It is well-known (see [3],[5] and [7]) that $\underline{F}$ is right-continuous, so $\underline{F}$ satisfies the usual conditions, and we have for any stopping time T

$$\underline{\underline{F}}_T = \sigma\{ X^T, \underline{\underline{N}} \} , \quad \underline{\underline{F}}_{T-} = \sigma\{ T, X^{T-}, \underline{\underline{N}}\} \tag{1}$$

in particular, for all $n \geq 1$

$$\underline{\underline{F}}_{T_n} = \sigma\{\Delta_0, T_1, \Delta_1, \ldots, T_n, \Delta_n, \underline{\underline{N}}\} , \ \underline{\underline{F}}_{T_n-} = \sigma\{\Delta_0, T_1, \Delta_1, \ldots, T_n, \underline{\underline{N}}\} \tag{2}$$

Denote by μ the jump measure induced by X:

$$\mu(dt, dx) = \sum_{n=1}^{\infty} \varepsilon_{(T_n, \Delta_n)}(dt, dx)\, I_{[T_n < \infty]}$$

where ε_a is the unite measure concentrating at point a, and by ν the predictable dual projection of μ. According to Jacod[7], we have

$$\nu(dt,dx) = \sum_{n=1}^{\infty} \frac{P(T_n \in dt,\ \Delta_n \in dx \mid \underline{\underline{F}}_{T_{n-1}})}{P(\ T_n \geq t \mid \underline{\underline{F}}_{T_{n-1}})} I_{]\!]T_{n-1},\ T_n]\!]} \tag{3}$$

The law of X is determined uniquely by that of $(T_n, \Delta_n)_{n\geq 0}$ and by ν as well. So it is natural to characterize the properties of X by the behaviour of $(T_n, \Delta_n)_{n\geq 0}$ or ν. In this note we show two simple but interesting results of this type.

We introduce another useful notations. Put

$$\Lambda_t = \nu([0,t] \times \mathbb{R}\ )\ ,\quad a_t = \Delta\Lambda_t = \nu(\{t\} \times \mathbb{R}\ )\ .$$

It is easy to see that (Λ_t) is the predictable dual projection of the simple point process $N = \sum_{n=1}^{\infty} I_{[\![T_n,\infty[\![}$, (a_t) is the predictable projection of I_D, where $D = [\ \Delta X \neq 0\ ] = \bigcup_{n=1}^{\infty}[\![\ T_n\ ]\!]$ is the set of the jumps of X, and $J = [\ a \neq 0\ ]$ is the predictable support of D. Suppose that on $\{\ T_n < \infty\ \}$

$$P(\ \Delta_n \in dx \mid \underline{\underline{F}}_{T_n-}) = G_n(dx;\Delta_0,T_1,\Delta_1,\ldots,T_{n-1},\Delta_{n-1},T_n) \quad \text{a.s.}$$

Then we have

$$\nu(dt,dx) = G(t,dx)d\Lambda_t\ ,$$
$$G(t,dx) = \sum_{n=1}^{\infty} G_n(dx;\Delta_0,T_1,\Delta_1,\ldots,T_{n-1},\Delta_{n-1},t)I_{]\!]T_{n-1},T_n]\!]}(t) \tag{4}$$

Our first result is concerned with the predictable representation property. We recall that X (or $\underline{F}$) has the predictable representation property if there exists a $\underline{F}$-local martingale M such that every $\underline{F}$-local martingale L, with $L_0 = 0$, can be represented as a predictable stochastic integral H.M. In [4], under the assumption that $\underline{F}$ is quasi-left-continuous we showed that X has the predictable representation property if and only if for every $n \geq 1$, Δ_n is a.s. a measurable function of $(\Delta_0,T_1,\Delta_1,\ \ldots,T_n)$, or equivalently, $\underline{F}$ is exactly the natural filtration of the simple point process $\Delta_0 + N$. But we know (see Chow and Meyer[1]) that the process $\Delta_0 + N$ has always the predictable representation property. It is not reasonable to assume that the natural filtration $\underline{F}$ is quasi-left-continuous. Now we get the general result as follows.

Theorem 1. The following statements are equivalent:

1° X has the predictable representation property;

2° For every $n \geq 1$, there exist two Borel functions $f_n^{(i)}(x_o,t_1,x_1,\ldots,t_{n-1},x_{n-1},t_n)$, $i = 1,2$, such that on the set $\{T_n < \infty\}$ we have

1) $\Delta_n = f_n^{(1)}(\Delta_o,T_1,\Delta_1,\ldots,T_{n-1},\Delta_{n-1},T_n)$ a.s. on $\{a_{T_n} < 1\}$,

2) $\Delta_n \in \{f_n^{(i)}(\Delta_o,T_1,\Delta_1,\ldots,T_{n-1},\Delta_{n-1},T_n),\ i = 1,2\}$ a.s. on $\{a_{T_n} = 1\}$.

In other words, the conditional distribution of Δ_n with respect to $\underline{\underline{F}}_{T_n-}$ on the set $\{T_n < \infty\}$ is a two-valued discrete distribution, furthermore, it reduces to an unite one on the set $\{a_{T_n} < 1\}$;

3° There exist four predictable processes $(C_t^{(i)})$, $(\alpha_t^{(i)})$, $i = 1,2$, with $C^{(1)} \geq 0$, $C^{(2)} \geq 0$, $C^{(1)} + C^{(2)} = 1$, such that

$$\nu(dt,dx) = G(t,dx)dA_t,\quad G(t,dx) = C_t^{(1)}\varepsilon_{(\alpha_t^{(1)})}(dx) + C_t^{(2)}\varepsilon_{(\alpha_t^{(2)})}(dx)I_{[a_t=1]} \quad (5)$$

Our next result is concerned with the Markov property. Note that if a jump process is Markovian, it is strong Markovian automatically because of its sample function property.

Theorem 4. The following statements are equivalent:

1° X is Markovian;

2° $(T_n,X_{T_n})_{n\geq o}$ is a homogeneous Markovian chain with state space $\overline{\mathbb{R}}_+ \times \mathbb{R}$, and its transition probability $Q(s,x;dt,dy)$ satisfies the following conditions:

$$1)\ Q(s,x;dt,dy) = Q(s,x;]u,\infty] \times \mathbb{R})Q(u,x;dt,dy) \qquad 0 \leq s \leq u \leq t \quad (6)$$

$$2)\ Q(s,x;]0,s] \times \mathbb{R}) = Q(s,x;\mathbb{R}_+\times\{x\}) = 0\,, \qquad s < \infty \quad (7)$$
$$Q(s,x;\{\infty\},dy) = Q(s,x;\{\infty\} \times \mathbb{R})\,\varepsilon_{(x)}(dy)$$

$$3)\ Q(\infty,x;dt,dy) = \varepsilon_{(\infty)}(dt)\,\varepsilon_{(x)}(dy) \quad (8)$$

$$3°\ \nu(dt,dx) = Q(t,X_{t-};X_{t-}+dx)\Lambda(X_{t-},dt) \quad (9)$$

where 1) $Q(t,x;dy)$ is a transition probability from $\mathbb{R}_+\times\mathbb{R}$ to $\mathbb{R}$ and $Q(t,x;\{x\}) = 0$; 2) (i) $\Lambda(x,dt)$ is a σ-finite transition measure from $\mathbb{R}$ to $\mathbb{R}_+$ and $\Lambda(x,\{t\}) \leq 1$, (ii) There exist two sequences of Borel functions $f_n(x)$ and $g_n(x)$ such that for every x, $\mathbb{R}_+$ is the union of disjoint intervals $\bigcup_{n=1}^{\infty}[f_n(x),g_n(x)[$, and for $t \in]f_n(x),g_n(x)[$

$$\Lambda(x,]f_n(x),t[) < \infty\,. \quad (10)$$

This problem was firstly discussed by Jacobsen[6] in a slightly different form and under the hypothesis that the state space is denumerable. Gihman and Skorohod [2] essentially showed that the statements 1° and 2° are equivalent, though their proof utilized rather complicated calculation. In fact, one can use the following formulas of jump processes to simplify the calculation. If $(W_t)_{t\geq 0}$ is an intgrable process, then its optional and predictable projections respectly are:

$$^{o}W_t = \sum_{n=1}^{\infty} \frac{E(W_t I_{[T_n>t]} \mid \underline{\underline{F}}_{T_{n-1}})}{E(I_{[T_n>t]} \mid \underline{\underline{F}}_{T_{n-1}})} I_{[T_{n-1}\leq t < T_n]}$$

and

$$^{p}W_t = \begin{cases} \sum_{n=1}^{\infty} \frac{E(W_t I_{[T_n\geq t]} \mid \underline{\underline{F}}_{T_{n-1}})}{E(I_{[T_n\geq t]} \mid \underline{\underline{F}}_{T_{n-1}})} I_{[T_{n-1}< t \leq T_n]}, & t > 0, \\ W_0, & t = 0. \end{cases}$$

We observe some particular cases. 1) In order that X is homogeneous Markovian it is necessary and sufficient that $Q(t,x;dy)$ are independent of t, and $A(x,dt)= q(x)dt$, with $q(x) \geq 0$. Hence we have

$$\nu(dt,dx) = Q(X_{t-}; X_{t-} + dx)q(X_{t-})dt \ .$$

This is well-known for the homogeneous Markovian processes with discrete state space (see Jacod[8]). 2) In order that X is a process with independent increments it is necessary and sufficient that $Q(t,x;dy)$ and $A(x,dt)$ are independent of x. Hence we have

$$\nu(dt,dx) = Q(t;dx)dA_t$$

In addition, if X is stationary, then

$$\nu(dt,dx) = \lambda Q(dx)dt \ , \quad \lambda > 0 \ .$$

These are the results of [9].

2. Predictable representation property. Note that in our case all local martingales are purely discontinuous, and we can deduce the following lemma from the relevant results in Jacod[8].

Lemma 1. Let M be a local martingale. Then every local martingale L, with $L_0 = 0$, can be represented as a predictable stochastic integral $H.M$ if and only if the following conditions are satisfied:

1) For every totally inaccessible stopping time T, $[\![T]\!] \subset [\, \Delta M \neq 0]$;

2) For every stopping time T, $\underline{\underline{F}}_T = \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \}$;

3) There exist two predictable processes $(\alpha_t^{(i)})$, $i = 1,2$, such that ΔM equals to $\alpha^{(1)}$ or $\alpha^{(2)}$.

Lemma 2. $K = [\, a = 1 \,]$ is the largest predictable set contained in $D = [\Delta X \neq 0]$.

Proof. Let B be a predictable set contained in D, and S a predictable stopping time, with $[\![S]\!] \subset B$. Then

$$a_S I_{[S<\infty]} = E[I_D(S) I_{[S<\infty]} \mid \underline{\underline{F}}_{S-}] = I_{[S<\infty]} \,.$$

Hence, $[\![S]\!] \subset K$, and $B \subset K$. $K \subset D$ is evident.

Proof of theorem 1. No loss generality we can suppose that X is locally integrable, i.e. its predictable dual projection X^p exists. Otherwise, we can replace X by another jump process $\tilde{X}$ without change of its jump times and natural filtration as follows.

$$\tilde{X} = X_o + \sum_{n=1}^{\infty} \tilde{\Delta}_n I_{[\![T_n,\infty [\![} \,, \quad \tilde{\Delta}_n = \operatorname{arctg} \Delta_n \,.$$

Then $\tilde{X}$ is locally integrable, since $(\tilde{\Delta}_n)_{n\geq 1}$ is bounded.

1° ⇒ 2°. Suppose that every local martingale can be represented as a predictable stochastic integral with respect to a local martingale M. Then $X - X^p = H.M$, where H is a predictable process. By lemma 1 there exist two predictable processes $(\tilde{\alpha}_t^{(i)})$, $i = 1,2$, such that ΔM equals to $\tilde{\alpha}^{(1)}$ or $\tilde{\alpha}^{(2)}$. Put

$$\bar{\alpha}^{(i)} = \Delta X^p + H\tilde{\alpha}^{(i)}, \quad i = 1,2,$$

and

$$\alpha^{(1)} = \bar{\alpha}^{(1)} I_{[|\bar{\alpha}^{(1)}| \geq |\bar{\alpha}^{(2)}|]} + \bar{\alpha}^{(2)} I_{[|\bar{\alpha}^{(1)}| < |\bar{\alpha}^{(2)}|]} \,,$$
$$\alpha^{(2)} = \bar{\alpha}^{(2)} I_{[|\bar{\alpha}^{(1)}| \geq |\bar{\alpha}^{(2)}|]} + \bar{\alpha}^{(1)} I_{[|\bar{\alpha}^{(1)}| < |\bar{\alpha}^{(2)}|]} \,.$$

Then ΔX equals to $\alpha^{(1)}$ or $\alpha^{(2)}$, and $|\alpha^{(2)}| \leq |\alpha^{(1)}|$. Hence we obtain

$$[\, |\alpha^{(2)}| > 0 \,] \subset [\, \Delta X \neq 0 \,].$$

Since $[\, |\alpha^{(2)}| > 0 \,]$ is predictable, by lemma 2 we have

$$[\, |\alpha^{(2)}| > 0 \,] \subset [\, a = 1 \,] \,.$$

Now it is easy to see that for $n \geq 1$ on the set $\{ T_n < \infty \}$

$$\Delta_n = \Delta X_{T_n} \in \{ \alpha_{T_n}^{(1)}, \alpha_{T_n}^{(2)} \} \,.$$

But on $\{ a_{T_n} < 1 \}$, $\alpha^{(2)}_{T_n} = 0$, it must be $\Delta_m = \alpha^{(1)}_{T_n}$. On the other hand, since $\alpha^{(i)}$, i= 1,2, are predictable, we have $\alpha^{(i)}_{T_n} \in \underline{\underline{F}}_{T_n-}$. So by (2) $\alpha^{(i)}_{T_n}$ can be represented as

$$\alpha^{(i)}_{T_n} = f^{(i)}_n(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{m-1},T_n) \quad \text{a.s.} \quad i = 1,2,$$

where $f^{(i)}_n$, $i = 1,2$, are Borel measurable.

2° ⇒ 1°. It suffices to verify that the local martingale $M = X - X^p$ satisfies the conditions in lemma 1.

1) For every totally inaccessible stopping time T, we have $[\![T]\!] \subset D$. Therefore, on the set $\{ T < \infty \}$, $\Delta X_T \neq 0$, $\Delta X^p_T = 0$, because X^p is predictable. This yields $\Delta M_T \neq 0$, i.e. $[\![T]\!] \subset [\Delta M \neq 0]$.

2) For every stopping time T, we have $\Delta X^p_T I_{[T<\infty]} \in \underline{\underline{F}}_{T-}$. So by (1)

$$\Delta X_T I_{[T<\infty]} \in \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \},$$

$$\underline{\underline{F}}_T = \underline{\underline{F}}_{T-} \vee \sigma \{ \Delta X_T I_{[T<\infty]} \} = \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \} .$$

3) Put

$$\begin{aligned} \tilde{\alpha}^{(1)} &= \sum_{n=1}^{\infty} f^{(1)}_n(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{n-1}, t)\, I_{]\!] T_{n-1},T_n]\!]} \\ \tilde{\alpha}^{(2)} &= I_{[a=1]}\sum_{n=1}^{\infty} f^{(2)}_n(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{n-1},t)\, I_{]\!] T_{m-1},T_n]\!]} \end{aligned} \tag{11}$$

Then $\tilde{\alpha}^{(i)}$, $i = 1,2$, are predictable and ΔX equals to $\tilde{\alpha}^{(1)}$ or $\tilde{\alpha}^{(2)}$. In reality, if $\Delta X_t = 0$, it must be $a_t \le 1$, and $\tilde{\alpha}^{(2)}_t = 0$; if $\Delta X_t \neq 0$, there exists an $n \ge 1$ such that $t = T_n$, then $\Delta X_t = \Delta_n \in \{ f^{(i)}_n(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{n-1}, T_n), i = 1,2 \} = \{ \tilde{\alpha}^{(i)}_{T_n}, i = 1,2 \} = \{ \tilde{\alpha}^{(i)}_t, i = 1,2 \}$. Now set

$$\alpha^{(i)} = - \Delta X^p + \tilde{\alpha}^{(i)}, \quad i = 1,2,$$

$\alpha^{(i)}$, $i = 1,2$, are predictable, and ΔM equals to $\alpha^{(1)}$ or $\alpha^{(2)}$.

2° ⇒ 3°. For $n \ge 1$, put

$$P(\Delta_n = f^{(i)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n) \mid \underline{\underline{F}}_{T_n-}) = c^{(i)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n)$$

$$C^{(i)} = \sum_{n=1}^{\infty} c^{(i)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},t) I_{]\!] T_{n-1},T_n]\!]}, \qquad i = 1,2.$$

Then $C^{(i)}$, $i = 1,2$, are predictable, and $C^{(1)} \ge 0$, $C^{(2)} \ge 0$, $C^{(1)} + C^{(2)} = 1$. On the set $\{ T_n < \infty \}$ we have

$$P(\Delta_n \in dx \mid \underline{\underline{F}}_{T_n-}) = c^{(1)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n)\mathcal{E}_{(f^{(1)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n))}(dx)$$

$$+c^{(2)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n)\mathcal{E}_{(f^{(2)}_n(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{m-1},T_n))}(dx) I_{[a_{T_n} = 1]} .$$

By (4) we obtain

$$G(t,dx) = C_t^{(1)}\mathcal{E}_{(\alpha_t^{(1)})}(dx) + C_t^{(2)}\mathcal{E}_{(\alpha_t^{(2)})}(dx)I_{[a_t=1]} \quad ,$$

where predictable processes $\alpha^{(i)}$, $i = 1,2$, are defined as above.

3° ⇒ 2°. It suffices to see that for every $n \geq 1$ on the set $\{ T_n < \infty \}$

$$P(\Delta_n \in dx | \underline{F}_{T_n-}) = G(T_n, dx) = C_{T_n}^{(1)}\mathcal{E}_{(\alpha_{T_n}^{(1)})}(dx) + C_{T_n}^{(2)}\mathcal{E}_{(\alpha_{T_n}^{(2)})}(dx)I_{[a_{T_n} = 1]}$$

and to represent $\alpha_{T_n}^{(i)}$ as $f_n^{(i)}(\Delta_0, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n)$, $i = 1,2$.

Corollary 1 ([1]). If for all $n \geq 1$, $\Delta_n \neq 0 \Rightarrow \Delta_n = 1$, i.e. X is a simple point process, then X has the predictable representation property.

Corollary 2 ([4]). If $\underline{F}$ is quasi-left-continuous, then X has the predictable representation property if and only if for every $n \geq 1$, $\Delta_n = f_n(\Delta_0, T_1, T_2, \ldots, T_n)$ a.s., where f_n is Borel measurable.

Proof. Because of the quasi-left-contimuity of $\underline{F}$, for every $n \geq 1$, on the set $\{a_{T_n} > 0,\ T_n < \infty\}$ we have $\Delta_n = h_n(\Delta_0, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n)$ a.s., where h_n is Borel measurable (see [3] or [5]). Now the corollary can be deduced directly from the statement 2° in theorem 1.

Theorem 2. Let $(S_n)_{n\geq 1}$ be a sequence of predictable stopping times such that $D \subset \bigcup_{n=1}^{\infty} [\![S_n]\!]$ and the graphs $([\![S_n]\!])_{n\geq 1}$ are disjoint, i.e. X is accessible. Then X has the predictable representation property if and only if for every $n \geq 1$ there exist two $\underline{F}_{S_n-}$-measurable variables $\xi_n^{(i)}$, $i = 1,2$, such that on the set $\{ S_n < \infty \}$ ΔX_{S_n} equals to $\xi_n^{(1)}$ or $\xi_n^{(2)}$. In other words, on the set $\{ S_n < \infty \}$ the conditional distribution of ΔX_{S_n} with respect to $\underline{F}_{S_n-}$ is a two-valued discrete distribution.

The proof of theorem 2 is completely similar to that of theorem 1. It suffices to construct two predictable processes $\tilde{\alpha}^{(i)}$, $i = 1,2$, as follows.

$$\tilde{\alpha}^{(1)} = \sum_{n=1}^{\infty} \xi_n^{(1)} I_{[\![S_n]\!]} , \quad \tilde{\alpha}^{(2)} = \sum_{n=1}^{\infty} \xi_n^{(2)} I_{[\![S_n]\!]}$$

instead of (11). In reality, for each t and ω, either $t = S_n$ for some $n \geq 1$,

$$\Delta X_t = \Delta X_{S_n} \in \{ \xi_n^{(1)}, \xi_n^{(2)} \} = \{ \tilde{\alpha}_{S_n}^{(1)}, \tilde{\alpha}_{S_n}^{(2)} \} = \{ \tilde{\alpha}_t^{(1)}, \tilde{\alpha}_t^{(2)} \},$$

or $t \bar{\in} \bigcup_{n=1}^{\infty} [\![S_n]\!]$, $\Delta X_t = 0 = \tilde{\alpha}_t^{(2)}$. Hence, we still have

$$\Delta X_t \in \{ \tilde{\alpha}_t^{(1)}, \tilde{\alpha}_t^{(2)} \}.$$

Corollary. Let $X = (X_n)_{n\geq 0}$ be an arbitrary sequence of random variables. Then X has the predictable representation property if and only if for every $n \geq 1$, there exist two $(X_o, \ldots, X_{n-1})$-measurable variables $\xi_n^{(i)}$, $i = 1,2$, such that $X_n = \xi_n^{(1)}$ or $\xi_n^{(2)}$. In other words, the conditional distribution of X_n with respect to $(X_o, \ldots, X_{n-1})$ is a two-valued discrete distribution.

In addition, if $(X_n)_{n\geq 0}$ is an independent sequence, then X has the predictable representation property if and only if each of $(X_n)_{n\geq 1}$ has a two-valued discrete distribution.

Proof. Define a jump process

$$X_t = X_o + \sum_{n=1}^{\infty}(X_n - X_{n-1})I_{[\ n \leq t\ ]}$$

and take $S_n = n$. The conclusions follow immediately from theorem 2.

Though the corollary of theorem 2 is rather banal, it motivated the following general result on the processes with independent increments (not necessarily stochastically continuous) (see [4]).

Theorem 3. Suppose that $X = (X_t)_{t\geq 0}$ is a process with independent increments, and with r.c.l.l. trajectories. Let (α, β, ν) be the local characterics of X. Then X has the predictable representation property if and only if

1) $\nu(dt,dx) = \{C_t^{(1)}\varepsilon_{(f_t^{(1)})}(dx) + C_t^{(2)}\varepsilon_{(f_t^{(2)})}(dx)I_{[\nu(\{t\}\times\mathbb{R})>0]}\}d\Lambda_t$,

where $C^{(i)}$, $f^{(i)}$, $i = 1,2$, are Borel measurable functions, with $C^{(1)} \geq 0$, $C^{(2)} \geq 0$, $C^{(1)} + C^{(2)} = 1$, and $d\Lambda_t$ is a σ-finite measure on $\mathbb{R}_+$;

2) $d\beta_t$ and $d\Lambda_t$ are mutually singular.

Note that $[\ \nu(\{t\}\times\mathbb{R}) > 0\ ]$ is the set of the fixed discontinuous points of X, only on this set the jumps of X can take two possible values.

3. Markov property. We turn to Markov property of jump processes and complete the demonstration of theorem 4 by proving that the statements 2° and 3° are equivalent.

2° ⇒ 3°. For $s \leq t$, put

$$q(s,x,t) = Q(s,x;]t,\infty] \times \mathbb{R}) \quad .$$

$q(s,x,.)$ is right-continuous and monotonely decreasing, and by (6) it satisfies the following functional equation:

$$q(s,x,t) = q(s,x,u)q(u,x,t) \qquad s \le u \le t$$
$$q(s,x,s) = 1 \tag{12}$$

Denote $\tau_s(x) = \inf\{ t > s: q(s,x,t) = 0 \}$. From (12) it is facile to get

1) $\tau_s(x) > s$;
2) $q(s,x,u) > 0,\ u \in [s, \tau_s(x)[$; (13)
3) $q(s,x,u) = 0,\ u \in [\tau_s(x), \infty[$.

We can decompose $\mathbb{R}_+$ into a series of disjoint intervals: $\mathbb{R}_+ = \bigcup_{n=1}^{\infty} [f_m(x),g_n(x)[$ such that for arbitrary two points s and t ($s < t$), $q(s,x,t) > 0$ if s and t belong to the same interval, and $q(s,x,t) = 0$ if s and t belong to different intervals. In fact, for x fixed we may classify the points of $\mathbb{R}_+$ as follows. For $s < t$, we stipulate that s and t belong to the same class $C_\alpha(x)$ if and only if $q(s,x,t) > 0$. Because of (12) there is no ambiguity. It suffices to prove that each class $C_\alpha(x)$ is an interval $[f_\alpha(x),g_\alpha(x)[$, since the number of disjoint intervals on $\mathbb{R}_+$ is at most denumerable. From (13) the proof is straightforward. We observe that if s and t belong to $C_\alpha(x)$ and $s < t$, then $[s,t] \subset C_\alpha(x)$. Set $f_\alpha(x) = \inf C_\alpha(x)$, $g_\alpha(x) = \sup C_\alpha(x)$, we get

$$]f_\alpha(x),g_\alpha(x)[\subset C_\alpha(x) \subset [f_\alpha(x),g_\alpha(x)].$$

It remains to show $f_\alpha(x) \in C_\alpha(x)$ and $g_\alpha(x) \bar{\in} C_\alpha(x)$ if $g_\alpha(x) < \infty$. Take $u \in]f_\alpha(x), g_\alpha(x)[$ such that $q(f_\alpha(x),x,u) > 0$. Then by (12) for every $t \in C_\alpha(x)$, $q(f_\alpha(x),x,t) > 0$, and this yields $f_\alpha(x) \in C_\alpha(x)$. Now suppose $g_\alpha(x) < \infty$. there exists $u > g_\alpha(x)$ such that $q(g_\alpha(x),x,u) > 0$. If $g_\alpha(x) \in C_\alpha(x)$, then $u \in C_\alpha(x)$. This contradicts to the fact that $g_\alpha(x)$ is the supremum of $C_\alpha(x)$.

Furthermore, we can consider $f_n(x)$ and $g_n(x)$ to be measurable. In fact, we need only to arrange those intervals, whose lengths are more than $\frac{1}{n}$ and not more than $\frac{1}{n-1}$, and the number of such intervals in every finite time interval is finite. Set

$$\begin{aligned} &a_0^{(n)}(x) = b_0^{(n)}(x) = 0, \\ &a_m^{(n)}(x) = \inf\{ t > b_{m-1}^{(n)}(x): q(t,x,t+\tfrac{1}{n}) > 0,\ q(t,x,t+\tfrac{1}{n-1}) = 0 \}, \\ &b_m^{(n)}(x) = \inf\{ t > a_m^{(n)}(x): q(a_m^{(n)}(x),x,t) = 0 \}. \end{aligned} \tag{14}$$

Then $\mathbb{R}_+ = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} [a_m^{(n)}(x),b_m^{(n)}(x)[$. Becuse $q(t,x,t+\delta)$ ($\delta > 0$) and $q(a_m^{(n)}(x), x,t)$ are right-continuous in t, the infremums in (14) can be taken over the rational numbers. Hence, $a_m^{(n)}(x)$ and $b_m^{(n)}(x)$ are measurable. Taking away empty intervals and rearrange properly, we obtain the decomposition $\mathbb{R}_+ = \bigcup_{n=1}^{\infty}[f_n(x), g_n(x)[$ with measurable end point functions.

Put

$$A_n(x,dt) = \frac{q(f_n(x),x;dt)}{q(f_n(x),x;[t,\infty])}, \quad q(s,x;dt) = Q(s,x;dt,\mathbb{R})$$

$$A(x,dt) = \sum_{n=1}^{\infty} A_n(x,dt) .$$

Note that the support of $A_n(x,dt)$ is $]f_n(x),g_n(x)]$ and $A_n(x,\{t\}) \le 1$,

$$A_n(x,]f_n(x),u]) < \infty, \quad u \in]f_n(x),g_n(x)[.$$

So $A(x,dt)$ is well defined and satisfies the conditions demanded in the statement 2°

Take

$$Q_n(t,x;dy) = \frac{Q(f_n(x),x;dt,dy)}{q(f_n(x),x;dt)}$$

as the Radon-Nikodym derivative of $Q(f_n(x),x;dt,dy)$ with respect to $q(f_n(x),x,dt)$ such that it is a transition probability and vanishes for $t \in]f_n(x),g_n(x)]$. Similarly we define

$$Q(t,x;dy) = \sum_{n=1}^{\infty} Q_n(t,x;dy) ,$$

which is a transition probability from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$.

Now we verify the formula (7). Fix $n \ge 1$. On the set $\{ T_{n-1} \in [f_k(X_{T_{n-1}}),g_k(X_{T_{n-1}})[\}$ we have $q(T_{n-1},X_{T_{n-1}},]g_k(X_{T_{n-1}}),\infty]) = 0$, so $T_n \le g_k(X_{T_{n-1}})$, i.e.

$$]T_{n-1},T_n] \subset]f_k(X_{T_{n-1}}) , g_k(X_{T_{n-1}})] .$$

On the other hand, by (10) for any $u \in [f_n(x),g_n(x)[$ we have

$$\frac{q(u,x;dt)}{q(u,x;[t,\infty])} = A_n(x,dt), \quad t \ge u ,$$

particularly,

$$\frac{q(T_{n-1},X_{T_{n-1}};dt)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} = A_k(X_{T_{n-1}},dt) .$$

Hence,

$$\frac{Q(T_{n-1},X_{T_{n-1}};dt,X_{T_{n-1}} + dx)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} I_{[T_{n-1} < t \le T_n]}$$

$$= Q_k(t,X_{T_{n-1}};X_{T_{n-1}} + dx)A_k(X_{T_{n-1}},dt)I_{[T_{n-1} < t \le T_n]}$$

$$= Q(t,X_{t-};X_{t-}+dx)\Lambda(X_{t-},dt)I_{[T_{n-1}<t\leq T_n]} \cdot$$

According to (3) and utilizing the Markov property of $(T_n,X_{T_n})_{n\geq 0}$ we get

$$\nu(dt,dx) = \sum_{n=1}^{\infty} \frac{Q(T_{n-1},X_{T_{n-1}};dt,X_{T_{n-1}}+dx)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} I_{[T_{n-1}<t\leq T_n]}$$

$$= Q(t,X_{t-};X_{t-}+dx)\Lambda(X_{t-},dt).$$

Remark. If $(X_t)_{t\geq 0}$ is a homogeneous Markovian process, the functions $q(s,x,t)$ are only dependent of $t-s$: $q(s,x,t) = q(t-s,x)$, and equation (12) becomes

$$q(s+t,x) = q(s,x)q(t,x) , \quad s,t \geq 0 .$$

Immediately, we have $q(t,x) = e^{-q(x)t}$, hence $\Lambda(x,dt) = q(x)dt$, and $Q(t,x;dy)$ is independent of t. At the same time, since $\Lambda(x,dt)$ is continuous in t, X is quasi-left-continuous, i.e. all $(T_n)_{n\geq 1}$ are totally inaccessible.

3° ⇒ 2°. According to Doleans-Dade's exponential formula, we define

$$q(s,x,t) = e^{-\Lambda^c(x,]s,t\wedge g_n(x)])} \prod_{s<u\leq t\wedge g_n(x)}(1-\Lambda(x,\{u\})), \qquad s\leq t ,$$

$$Q(s,x;dt,dy) = Q(u,x;dy)q(s,x;du)I_{]s,g_n(x)]}(u), \qquad s\in[f_n(x),g_n(x)[\tag{15}$$

where $\Lambda^c(x,dt)$ is the continuous part of $\Lambda(x,dt)$. It is facile to verify that $Q(s,x;dt,dy)$ defined in (15) together with (7), (8) constitutes a transition probability and satisfies the condition (6).

Now we can construct a jump process $\bar{X}$ such that the corresponding chain $(\bar{T}_n,\bar{X}_{\bar{T}_n})_{n\geq 0}$ is homogeneous Markovian with $Q(s,x;dt,dy)$ as its transition probability, and $\bar{X}_0$ has the same law as X_0. Then from the proof 2° ⇒ 3°, the corresponding predictable dual projection $\bar{\nu}$ has the same form as ν

$$\bar{\nu}(dt,dx) = Q(t,\bar{X}_{t-};\bar{X}_{t-}+dx)\Lambda(\bar{X}_{t-},dt).$$

Therefore, $\bar{X}$ has the same law as X. This implies that $(T_n,X_{T_n})_{n\geq 0}$ has the same law as $(\bar{T}_n,\bar{X}_{\bar{T}_n})_{n\geq 0}$. Hencz, $(T_n,X_{T_n})_{n\geq 0}$ is a homogeneous Markovian chain.

References

[1] Chow,C.S.,Meyer,P.A.: Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels. Sém. Prob. IX. Lecture Notes in Math. n° 465, 1975, 226–236.

[2] Gihman,I.I., Skorohod,A.N.: The theory of Stochastic Processes II. 1975, Springer-Verlag.

[3] He,S.W.: Necessary and sufficient conditions for quasi-left-continuity of natural σ-fields of jump processes. Journal of East China Normal University, 1981, 24-30.

[4] He,S.W., Wang,J.G.: The total continuity of natural filtrations and the strong property of predictable representation for jump processes and processes with independent increments. Sém. Prob. XVI. Lecture Notes in Math. n° 920, 1982, 348-354.

[5] Itmi,M.: Processus ponctuels marqués stochastiques. Representation des martingales et filtration naturelle quasi-continue à gauche. Sém. Prob. XV. Lecture Notes in Math. n° 850, 1981, 618-626.

[6] Jacobsen,M.: A characterization of minimal Markov jump processes. Z. Warhscheinlichkeitstheorie verw. Geb. 23, 1972, 32-46.

[7] Jacod,J.: Multivariate point processes, predictable projection, Radon-Nikodym derivatives, representation of martingales. Z. Warhscheinlichkeitstheorie verw. Geb. 31, 1975, 235-253.

[8] Jacod,J.: Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math. n° 714, 1979. Springer-Verlag.

[9] Wang,J.G.: Some remarks on processes with independent increments. Sém. Prob. XV. Lecture Notes in Math. n° 850, 1981, 627-631.