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SOME REMARKS ON SINGLE JUMP PROCESSES

by S.W. HE

Let $(\Omega,\underline{\underline{F}},P)$ be a complete probability space, and T be a strictly positive random variable. We denote by $\underline{\underline{F}} = (\underline{\underline{F}}_t)_{t\geq 0}$ the natural filtration of the single jump process $X = (X_t)_{t\geq 0} = I_{[\![T,\infty[\![}$, i.e.

$$\underline{\underline{F}}_t = \sigma(X_s\ ,\ s\leq t\ )$$

(we make the convention that all sets of measure 0 in $\underline{\underline{F}}$ are implicitly added to all σ-fields). This filtration has been much studied, starting with Dellacherie [2], and the literature concerning it is extensive. However, we couldn't find in it the following simple remarks (from the article [4] in Chinese).

We begin with the following proposition (which is closely related t Dellacherie-Meyer [3], chapter VII, n^{os} 105-106) .

PROPOSITION 1. Let $S\in\sigma(T)$ be a non-negative random variable.
a) S is a stopping time if and only if there exists some constant $c\leq+\infty$ such that a.s.

(1) $\quad S\geq T$ on $\{T<c\}$, $S\geq c$ on $\{T=c\}$, $S=c$ on $\{T>c\}$

b) S is a predictable stopping time if and only if there exists some $c\leq+\infty$ such that a.s.

(2) $\quad S>T$ on $\{T<c\}$, $S=c$ on $\{T=c\}$, $S=c$ on $\{T>c\}$

c) S is totally inaccessible if and only if there exists some set $A\in\sigma(T$ such that $T<\infty$ on A, the distribution of T on A is diffuse, and $S=T_A$ (i.e. S=T on A, $S=+\infty$ on A^c , $P\{A, T=t\} = 0$ for all t).

Let us also recall a few facts about the uniqueness of c : if $S\geq T$ a.s., we may choose for c in (1) any constant which a.s. dominates T (recall that $+\infty$ is allowed). If $P\{S<T\}>0$, S is a.s. constant on $\{S<T\}$ and its a.s. value is the <u>only</u> possible value of c in (1) and in (2). Similarly, if $P\{S<T\}=0$ but $P\{S=T\}>0$, there may be several values of c satisfying (1), but <u>at most</u> one satisfying (2), namely the a.s. constant value of S on $\{S=T\}$.

Our first remark concerns predictability : the condition $P\{S=T<\infty\}$ is sufficient for predictability if the distribution of T has no atom on $[0,\infty[$, but <u>not sufficient</u> otherwise — contrary to a statement in [1]. Here is an example. We assume that the distribution of T is given by $\frac{1}{2}\varepsilon_1 + \frac{1}{2}\mu(dt)$, where the support of μ is the whole of $\mathbb{R}_+$.

We take

$$S=2T \text{ on } \{T<1\}\ ,\quad S=2 \text{ on } \{T=1\}\ ,\quad S=1 \text{ on } \{T>1\}$$

Since $P\{S<T\}>0$, $c=1$ is the only constant satisfying (1), and hence the only possible candidate for (2). Since the middle condition of (2) isn't a.s. satisfied, S cannot be predictable.

Our second remark is a necessary and sufficient condition for quasi-left-continuity, much easier to check than those given in [6], for example.

PROPOSITION 2. The filtration $\underline{F}$ is quasi-left-continuous if and only if there exists a constant $\alpha\leqq+\infty$ such that $P\{T>\alpha\}=0$, and the distribution of T has no atom in $[0,\alpha[$ (otherwise stated : the law λ of T has at most one atom, which then is the last point in the support of λ).

PROOF. Assume the distribution of T has an atom c such that $P\{T>c\}>0$. Then the stopping time S defined by

$$S=+\infty \text{ on } \{T<c\}\ ,\quad S=2c \text{ on } \{T=c\}\ ,\quad S=c \text{ on } \{T>c\}$$

is accessible and by the same reasoning as above isn't predictable. So $\underline{F}$ isn't quasi-left-continuous.

Conversely, assume the properties in the statement, and prove that any accessible stopping time (represented by (1)) is predictable. We may assume $P\{S\leqq T\}>0$, otherwise the result is trivial. We must only check the first two properties in (2), the third one being obvious.

If $P\{S<T\}>0$, then $P\{S=c<T\}>0$ from (1), and therefore $c<\alpha$, and $P\{T=c\}=0$ (so the middle condition is true). From Proposition 1 c) applied to $A=\{T\neq\alpha\}$ we get that T_A is totally inaccessible, so $P\{S=T_A<\infty\}=0$, and the first property in (2) follows from (1).

If $P\{S<T\}=0$, then we must have $c\geqq\alpha$ a.s., and (1) is satisfied with $c=\alpha$. Then we have the first property (2) for the same reason as above. On the other hand, $P\{S\leqq T\}>0$, hence $P\{S=T\}>0$, which in turn implies $P\{S=T=\alpha\}>0$ since $P\{S=T<\alpha\}=0$. Now $S\in\sigma(T)$, so S is a.s. constant on $\{T=\alpha\}$, and the middle condition is also satisfied. The proposition is proved.

REMARK. We have proved in [5] that if $\underline{F}$ is quasi-left-continuous, then $\underline{\underline{F}}_S=\underline{\underline{F}}_{S-}$ for any stopping time S.

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