PETRUS JOHANNES HOLEWIJN
ISAAC MEILIJSON

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NOTE ON THE CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

by

P.J. Holewijn [1)] and I. Meilijson [2)].

SUMMARY

A very simple proof, using a Skorokhod type embedding, of Billingsley's and Ibragimov's central limit theorem for sums of stationary ergodic martingale differences is presented.

INTRODUCTION

Various invariance and central limit theorems for sums of stationary ergodic processes have been obtained by showing the process to be homologous (see Gordin [4] or Bowen [2]) to a stationary ergodic martingale difference process. The central limit theorem of Billingsley (and Ibragimov, see [1]) for such processes can then be applied. This theorem is proved by showing that stationary ergodic martingale differences in L_2 satisfy the Lindeberg-Lévy conditions for asymptotic normality of martingales (see Scott [5]). Skorokhod's representation of a martingale as optionally sampled standard Brownian motion plays an important role in some of the proofs, but any such a representation is as good as any other.

In the present note we will present a particular Skorokhod representation that will make the incremental stopping times stationary ergodic in L_1. This will provide a simple direct proof of Billingley's theorem.

THEOREM

Let $(X_1, X_2, \dots, X_n, \dots)$ be a stationary and ergodic process such that $EX_1 = 0$, $EX_1^2 \in (0,\infty)$ and $E(X_n | X_1, X_2, \dots, X_{n-1}) = 0$ a.s., $n = 2,3,\dots$. Then there exists a sequence of (randomized) stopping times $0 \le \tau_1 \le \tau_2 \le \dots$ $\dots \le \tau_n \le \dots$ in Brownian motion $B(t)$, $t \ge 0$, such that

(i) $(B(\tau_1), B(\tau_2), \dots, B(\tau_n), \dots)$ is distributed as $(X_1, X_2, \dots, X_n, \dots)$;

(ii) The process of pairs $((B(\tau_1), \tau_1), (B(\tau_2)-B(\tau_1), \tau_2-\tau_1), \dots, (B(\tau_n)-B(\tau_{n-1}), \tau_n-\tau_{n-1}), \dots)$ is stationary and ergodic;

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1) The Free University of Amsterdam

2) University of Tel-Aviv, temporarily the Free University of Amsterdam.

and

(iii) $E(\tau_1) < \infty$.

PROOF

Extend $(X_1, X_2, \ldots, X_n, \ldots)$ to a doubly infinite sequence. The martingale difference property carries over to infinite pasts since $E(X_0 | X_{-n}, X_{-n+1}, \ldots, X_{-1}) = 0$ a.s. and converges a.s. as $n \to \infty$ to $E(X_0 | \ldots, X_{-2}, X_{-1})$. Let the probabilityspace contain random variables $(\ldots, X_{-2}, X_{-1}, X_0)$ with the correct joint distribution, and a standard Brownian motion $B(t)$, $t \geq 0$, starting at zero, and independent of $(\ldots, X_{-2}, X_{-1}, X_0)$. Fix any of the methods to embed in Brownian motion, in finite expected time, distributions with mean zero and finite variance. Let $\tau_0 = 0$ and suppose, inductively, that stopping times $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_{n-1}$ on B have been defined. For $1 \leq i \leq n-1$, let X_i denote $B(\tau_i) - B(\tau_{i-1})$. On the Brownian motion $B^*(t) = B(\tau_{n-1} + t) - B(\tau_{n-1})$, $t \geq 0$, use the rule fixed above to embed the conditional distribution of X_n given $(\ldots, X_{n-3}, X_{n-2}, X_{n-1})$. If τ is the embedding stopping time, let $\tau_n = \tau_{n-1} + \tau$, $X_n = B^*(\tau) = B(\tau_n) - B(\tau_{n-1})$.

By construction,

(iv) $((X_1, \tau_1), (X_2, \tau_2 - \tau_1), \ldots, (X_n, \tau_n - \tau_{n-1}), \ldots)$ is stationary;

(v) $(\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \ldots)$ are conditionally independent given $(\ldots, X_{-1}, X_0, X_1, \ldots)$; and

(vi) $E(\tau_1) < \infty$.

By (iv), any L_1-function of the process of pairs depending on finitely many coordinates has an almost surely convergent average of its shifts. This (limiting) average is a tail function in $(\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \ldots)$ given $(\ldots, X_{-1}, X_0, X_1, \ldots)$. Hence, by Kolmogorov's 0-1 law (because of (v)), the average is measurable $(\ldots, X_{-1}, X_0, X_1, \ldots)$. But as such, it is an invariant function because the shifted X sequence can be realized as the above construction is read from the second step onwards; the average will thus be unchanged. Since X is ergodic, the average is a.s. constant. This implies the ergodicity of the sequence of pairs. □

COROLLARY

(Billingsley, Ibragimov). Under the conditions of the theorem,

$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i$, $t \geq 0$, converges in distribution to standard Brownian motion.

PROOF

Let $B(t)$, $t \geq 0$ be standard Brownian motion and consider for each n Brownian motion $\sqrt{n}B(t/n)$, $t \geq 0$, in which $W_n(t)$ is embedded at time $\tau_{[nt]}$. Now $\tau_{[nt]}$ converges a.s. to t by the theorem and following Breiman [3] pp. 279-281 we can conclude that

$\sup_{0 \leq t \leq 1} |W_{n_k}(t) - B(t)| \to 0$ a.s. as $k \to \infty$ for subsequences $\{n_k\}$ that increase fast enough. But then, if f is a bounded continuous function on the space $D[0,1]$ endowed with the sup.norm metric of paths that are rights continuous and have left hand limits it follows by the bounded convergence theorem along the same subsequence $\{n_k\}$ that $Ef(W_{n_k}(\cdot)) \to Ef(B(\cdot))$, which implies the convergence of the full sequence and therefore the convergence in distribution. □

REFERENCES

[1] Billingley, P: Convergence of probability measures, Wiley New York.

[2] Bowen, R: Equilibrium States and the ergodic theory of Anosov diffeomorphisms; Lecture Notes in Mathematics (470), Springer, Berlin.

[3] Breiman, L: Probability; Addison-Wesley, London.

[4] Gordin, M.J.: The central limit theorem for stationary processes; Soviet Math. Dokl. Vol.10 (1969), No 5, pp. 1174-1176.

[5] Scott, D.J: Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach; Adv. Appl. Prob. 5, 119-137 (1973).