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A 'POTENTIAL-THEORETIC' NOTE ON THE QUADRATIC

WIENER-HOPF EQUATION FOR Q-MATRICES

by

David Williams

The only prerequisite for this note is Section 1 of Part A of the immediately-preceding Paper II by London, McKean, Rogers, and myself. The notation of that section will be used without further comment.

The quadratic matrix equation:

$$C + D\Pi^{+} + \Pi^{+}A + \Pi^{+}B\Pi^{+} = 0 \qquad (1)$$

follows immediately from equations (1.4) and (1.5) of Paper II. Equation (1) played an important part in Section 2 of Paper II.

The matrix Π^{+} is characterized by Lemma 1.8 of Paper II in the case when $V^{-1}Q$ is diagonalizable, and by an analogous result (see the Barlow-Rogers-Williams paper referred to in Paper II) in general. However, it is not clear from these characterizations that Π^{+} is even real, far less substochastic.

Even from the point of view of the <u>theory</u>, it therefore seems desirable to develop an alternative approach to the algebra of the problem. Moreover, for <u>practical</u> purposes, it is important to develop efficient algorithms for numerical computations which do not rely on calculation of (possibly complex) eigenvalues and eigenvectors. It is hoped to say more in this direction in future publications.

Here is a first step,

<u>THEOREM</u>. <u>Suppose that</u> Z <u>is a non-negative</u> $E^{-} \times E^{+}$ <u>matrix such that</u>

$$C + DZ + ZA + ZBZ \le 0. \qquad (2)$$

Then

$$Z \geq \Pi^+.$$

Remark. It will be seen that the system of equations (4) below gives an 'algorithm' for computing Π^+ as the limit of a monotone increasing sequence, and does at least exhibit Π^+ as a positive matrix. As a computational algorithm, the system (4) is almost unimaginably inefficient - for reasons which will be immediately apparent to the reader.

The argument around (6i) below shows that if (2) holds, so that

$$-(DZ + ZA) \geq C + ZBZ,$$

and if we define Y via the equation

$$-(DY + YA) = C + ZBZ,$$

then $Z \geq Y \geq \Pi^+$. Thus we might hope that in certain circumstances we might obtain a decreasing sequence converging to Π^+.

In general, it is not at all easy to find any solution of the inequality (2). I am grateful to Chris Rogers for correcting my ridiculous claim (in a first draft of this note) to have found an explicit solution of (2) in the general case.

PROOF OF THEOREM. Consider the effect of replacing

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ by } \begin{pmatrix} A & B \\ \rho C & D \end{pmatrix} \text{ where } 0 < \rho < 1.$$

We have:

$$\rho C + D\Pi^+(\rho) + \Pi^+(\rho)A + \Pi^+(\rho)B\Pi^+(\rho) = 0, \tag{3}$$

where

$$\Pi^+(\rho) = \sum_{n\geq 1} \rho^n \Gamma_n^+ ,$$

where for $n \in \mathbb{N}$, $i \in E^-$, and $j \in E^+$,

$$\Gamma_n^+(i,j) = P^i[X(\tau_0^+) = j, \; \tau_0^+ \text{ falls within the n-th excursion from } E^- \text{ into } E^+].$$

On comparing coefficients of powers of ρ in (3), we have

$$-(D\Gamma_1^+ + \Gamma_1^+ A) = C. \tag{4i}$$

$$-(D\Gamma_2^+ + \Gamma_2^+ A) = \Gamma_1^+ B \Gamma_1^+, \tag{4ii}$$

$$-(D\Gamma_3^+ + \Gamma_3^+ A) = \Gamma_1^+ B \Gamma_2^+ + \Gamma_2^+ B \Gamma_1^+ , \tag{4iii}$$

etc., etc..

Now, it is a strictly elementary exercise to prove that if R is any $E^- \times E^+$ matrix, then the equation

$$-(D\Gamma + \Gamma A) = R$$

has the <u>unique</u> solution:

$$\Gamma = \int_{t=0}^{\infty} e^{tD} R e^{tA} dt. \tag{5}$$

Suppose that Z is a non-negative $E^- \times E^+$ matrix such that

$$C + DZ + ZA + ZBZ \leq 0.$$

Then

$$-(DZ + ZA) \geq C + ZBZ \geq C. \tag{6i}$$

Since (5) exhibits Γ as an increasing function of R, we have (on comparing (6i) with (4i)):

$$Z \geq \Gamma_1^+ . \tag{7i}$$

Hence

$$-(DZ + ZA) \geq C + \Gamma_1^+ B \Gamma_1^+. \tag{6ii}$$

On comparing (6ii) with (4i) + (4ii), we see that

$$Z \geq \Gamma_1^+ + \Gamma_2^+. \tag{7ii}$$

The proof is concluded by the obvious induction argument.

<u>Remark</u>. Consider now the effect of replacing

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} A & B \\ C & D-\lambda I \end{pmatrix}.$$

We are led to the fact that the $E^- \times E^+$ matrix W defined by

$$W_{ij}(t) \;=\; P^i[\phi^-(\tau_0^+) \in dt;\;\; X(\tau_0^+) \;=\; j]/dt$$

satisfies the 'delayed Ricatti' equation:

$$W'(t) \;=\; DW(t) + W(t)A + \int_0^t W(s)BW(t-s)ds, \tag{8}$$

$$W(0) \;=\; C.$$

Equation (8) is, of course, the Kolmogorov 'backwards, forwards, and sideways' equation for our problem. It was equation (8) and certain analogues which motivated our various excursions into Wiener-Hopf theory.

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