SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

MICHEL MÉTIVIER

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Séminaire de probabilités (Strasbourg), tome 16 (1982), p. 490-502
<http://www.numdam.org/item?id=SPS_1982__16__490_0>

PATHWISE DIFFERENTIABILITY WITH RESPECT TO A PARAMETER OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

by

Michel METIVIER

Ecole Polytechnique - Palaiseau - France.

Abstract

We consider a stochastic differential equation

$$X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

where S *is a semimartingale and* q *a random measure and where the "coefficients" depend on a parameter* u . *We prove under suitable differentiability-conditions that the solution* $X^u(t,\omega)$ *can be choosen for each* u *in such a way that the mapping* $u \sim X^u(t,\omega)$ *is continuously differentiable for every* (t,ω) .

I - INTRODUCTION

The goal of this paper is to prove that under sufficient differentiability conditions on the coefficients, stochastic differential equations of the type

$$(1.1) \qquad X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

where S is a semimartingale, q a random measure with zero dual predictable projection and u a parameter taking its values in a bounded open subset G of $\mathbb{R}^d$, admit for each u a solution which can be determined in such a way that P.a.s. the functions $u \sim X^u(t,\omega)$ are for every t continuously differentiable.

This is a concept of differentiability different from the one considered by Gikhmann (see [3] and [4]),who studied the differentiability of the mapping $u \sim X^u_t(.)$ as a mapping from G into $L^p(\Omega)$ for some p and in the framework of Ito-equations. Recently Bichteler took the same point of view and considered equations of the type (1.1) with $q = u$ and S and X^u possibly infinite dimensional. J. Jacod in [6] considered differentiability "in probability".

Pathwise differentiability was considered by P. Malliavin and M. Bismut for the solutions of Ito-Stratonovitch equation as functions of the initial conditions (see [2] and [8]). In [7] H. Kunita proved pathwise differentiability with respect to the initial conditions for the solutions of an equation driven by a continuous martingale. In [11] P.A. Meyer proved the same result for equations driven by a semimartingale (equations of Doleans-Dade-Protter type).

We consider here equations of type (1.1) and of a more general type with coefficients depending on a parameter u.

In section II we recall a few facts on the type of equations which are studied here. In section III we give sufficient conditions for the continuity of solutions with respect to u and in section IV we deal with differentiability.

II - THE EQUATION UNDER CONSIDERATION

2.1. - Inequalities for stochastic integrals

We assume that the random measure q in (1.1) is of the form $\mu(\omega;ds;du) - \nu(\omega;ds;du)$ where $\mu(\omega;]0,t],du)$ is for each ω and t a borelian measure in an open subset $\mathbb{E}$ of $\mathbb{R}^m - \{0\}$ such that for some $\alpha > 0$ $\int \frac{|x|^\alpha}{1+|x|^\alpha} |\mu|(\omega;]0,t],du) < \infty$ ($|\mu|$ denotes the variation of μ and α does not depend on ω and t) and where ν is the dual predictable projection of μ).

$\mathbb{H}$ denotes a separable Hilbert space. We have shown in [9] (see also J. Jacod [5]) the existence of an increasing positive adapted process b and of a process $\{\overset{\circ}{q}(\omega,s,.) : (\omega,s) \in \Omega \times \mathbb{R}^+\}$ the values of which are measures on $\mathbb{E} \times \mathbb{E}$ such that :

i) For each $\mathbb{H}$-valued function h on $\mathbb{E}$ such that $<h(x),h(y)>_{\mathbb{H}}$ is $\overset{\circ}{q}(\omega,s,dx \otimes dy)$ integrable, the integral $\int <h(x),h(y)>_{\mathbb{H}} \overset{\circ}{q}(\omega,s,dx \otimes dy)$ defines a positive optional process ;

ii) If Y is an $\mathbb{H}$-valued $\mathscr{P}\otimes\mathscr{B}_{\mathbb{E}}$ measurable(*) function on $\mathbb{R}^+\times\Omega\times\mathbb{E}$ and if we denote by $\lambda_s(Y)$ the $\mathbb{H}$-valued positive random variable

$$\lambda_s(Y) := \int \langle Y(s,\ ,x),\ Y(s,\ ,y)\rangle_{\mathbb{H}}\ \overset{o}{q}(.,s,dx\otimes dy)$$

(set to be equal to $+\infty$ when the integral does not exist) and
iii) the following inequality holds for every stopping time τ

$$(2.1)\qquad E\Big(\sup_{t<\tau}\Big\|\int_{]0,t]\times\mathbb{E}} Y(s,.x)q(.,ds,dx)\Big\|^2\Big) \leqslant 4\ E\Big(\int_{[0,\tau[} \lambda_s(Y)db_s\Big)$$

where $\Big(\int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)\Big)_{t\geqslant 0}$ is the stochastic integral process of Y with respect to q which is defined as soon as the process $\Big(\int_{]0,t]}\lambda_s(Y)db(s)\Big)_{t\geqslant 0}$ is finite.

If S is a $\mathbb{K}$-valued ($\mathbb{K}$: separable Hilbert space) right continuous semimartingale we know that there exist two positive increasing adapted processes a and $\tilde{a}$ such that for every $\mathcal{L}(\mathbb{K};\mathbb{H})$-valued locally bounded predictable process $\{f(s,\omega);(s,\omega)\in\mathbb{R}^+\times\Omega\}$ and every stopping time τ :

$$(2.2)\qquad E\Big(\sup_{t<\tau}\Big\|\int f(s,.)dS_s\Big\|^2\Big) \leqslant E\Big(\tilde{a}_{\tau^-}.\int_{[0,\tau[}\|f(s)\|^2 da(s)\Big)$$

To simplify the writing we shall call Z_t the process $Z_t := (S_t,q(.,]0,t],dx))$ which takes its values in $(\mathcal{L}(\mathbb{K};\mathbb{H})\times\mathscr{M}^\alpha)$ where $\mathscr{M}^\alpha$ is the space of borelian measures ν on $\mathbb{E}$ such that $\int_{\mathbb{E}}\frac{|x|^\alpha}{1+|x|^\alpha}\,|\nu|\,(du)<\infty$.

Setting $A_t := b(t)+a(t)$ $\tilde{A}_t := 8+2\tilde{a}_t$ $\Phi := (f,Y)$

$$(2.3)\qquad \int_{]0,t]}\Phi(s)dZ_s := \int_{]0,t]} f(s,.)dS_s + \int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)$$

and

$$(2.4)\qquad \lambda_s(\Phi) := \|f(s,.)\|^2 + \lambda_s(Y)$$

the following inequality holds for every stopping time

$$(2.5)\qquad E\Big(\sup_{t<\tau}\Big\|\int_{]0,t]}\Phi(s)dZ_s\Big\|^2\Big) \leqslant E\Big(\tilde{A}_{\tau^-}.\int_{]0,\tau]}\lambda_s(\Phi)dA_s\Big)$$

(*) $\mathscr{P}$ is the σ-algebra of predictable subsets of $\mathbb{R}^+\times\Omega$ and $\mathscr{B}_{\mathbb{E}}$ of Borel subsets of $\mathbb{E}$.

Extending a classical argument on martingales (see [13]) it is also easy to see that for every $p \geq 2$ exists an increasing positive adapted process $(\widetilde{A}^p_t)_{t \geq 0}$ such that for every stopping τ

$$(2.6) \qquad E\left(\sup_{t<\tau} \left\| \int_{]0,t]} \Phi(s)\, dZ_s \right\|^p\right) \leq E\left(\widetilde{A}^p_{\tau^-} \cdot \int_{[0,\tau[} \left(\lambda_s(\Phi)\right)^{p/2} dA_s\right)$$

2.2. - Hypothesis on equation (1.1)

The space of parameters u is an open bounded subset G of $\mathbb{R}^d$.

In equation (1.1) σ is a mapping from $(G \times \mathbb{R}^+ \times \Omega \times \mathbb{H})$ into $\mathcal{L}(\mathbb{K};\mathbb{H})$ which is continuous on $\mathbb{H}$ and such that for every $h \in \mathbb{H}$ and $u \in G$ the process $\{\sigma(u,s,\omega,h) : (s,\omega) \in \mathbb{R}^+ \times \Omega\}$ is predictable. f is a mapping of $(G \times \mathbb{R}^+ \times \Omega \times \mathbb{H}, \mathbb{E})$ into $\mathbb{H}$ which if continuous on $\mathbb{H}$ and such that for every $u \in G$, $h \in \mathbb{H}$ the mapping $(s,\omega,x) \rightsquigarrow f(u,s,\omega,h,x)$ is $\mathcal{P} \otimes \mathcal{B}_{\mathbb{E}}$ measurable

In the sequel we shall call g the couple (σ,f) and according to the notations of (2.1) the equation (1.1) will be written in the *abreviated form* :

$$(2.7) \qquad X^u(t) = V^u(t) + \int_0^t g(u,s,X^u_{s^-})\, dZ_s$$

Here V^u is for each $u \in G$ a given $\mathbb{H}$-valued adapted cad-lag process.

III - CONTINUITY OF THE SOLUTIONS WITH RESPECT TO u .

3.1. - Hypothesis

L is an increasing positive adapted process and p is a positive real number with $p \geq d + \varepsilon$ for some $\varepsilon > 0$.

If ξ is a cad-lag $\mathbb{H}$-valued adapted process we write $g(u,\xi)$ for the process $(t,\omega) \rightsquigarrow g(u,s,\omega,\xi_{s^-}(\omega))$ and $\lambda_s \circ g(u,\xi)$ for the positive functional of this process defined by formula (2.4).

With these notations we formulate the following hypotheses :

(H_1) $\sup_{s\leq t} \| V_s^u - V_s^v \| \leq L_t \| u-v \|$ for all t, u and $v \in G$

and

$$\sup_{u\in G}\ \sup_{s<t} \| V_t^u \| < \infty$$

(H_2) (Lipschitz hypotheses) :

$$\forall t\in\mathbb{R}^+ \quad \int_{]0,t]} [\lambda_s \circ (g(u,\xi)-g(u,\xi'))]^{p/2} dA_s \leq \int_{]0,t]} \sup_{r\leq s} \| \xi_r - \xi'_r \|^p dL_s$$

for every couple (ξ,ξ') of $\mathbb{H}$-valued adapted cad-lag processes, $\mathbb{P}$.a.s.

(H_3)
$$\int_{]0,t]} [\lambda_s \circ g(u,\xi)]^{p/2} dA_s \leq \int_{]0,t]} (1 + \sup_{r\leq s} \| \xi_s \|^p) dL_s$$

for every $u \in G$ every $\mathbb{H}$-valued adapted cal-lag ξ, P.a.s.
(Note that (H_3) is implied by (H_2) in most classical cases).

(H_4) Ψ being a given positive increasing (possibly constant) function on $\mathbb{R}^+$, for every stopping time τ the following inequality holds for every $\mathbb{H}$-valued cad-lag adapted ξ every u and v in G :

$$E\left(\sup_{t<\tau^-}\left[\lambda_t \circ [g(u,\xi)-g(v,\xi)]\right]^{p/2}\right) \leq \| u-v \|^{d+\varepsilon} \Psi\left(E(\sup_{t<\tau} \| \xi_t \|^p)\right)$$

3.2. - <u>Theorem</u>

1°) Under the above hypotheses (H_1) *to* (H_4) *, the equation* (2.7) *has for each* u *a unique strong solution* X^u *on* $\mathbb{R}^+$ *and the random function* $(t,\omega,u) \rightsquigarrow X_t^u(\omega)$ *can be determined in such a way that* $u \rightsquigarrow X_t^u(\omega)$ *is continuous on* G *for every* t *and* ω *while the mapping* $t \rightsquigarrow X_t^{(\cdot)}(\omega)$ *is for each* ω *cad-lag from* $\mathbb{R}^+$ *into the set* $C_b^{\mathbb{H}}(G)$ *of bounded continuous* $\mathbb{H}$*-valued functions on* G *endowed with the uniform topology.*

2°) There exists an increasing sequence (σ_n) *of stopping times and constants* $K(\Psi,n,p,Z)$ *such that*

a) $\lim_n P\{\sigma_n < T\} = 0$ *for every* $T > 0$

b) $E\left(\sup_{t<\sigma_n} \| X^u(t) - X^v(t) \|^p\right) \leq K(Y,n,p,Z) \| u-v \|^p$

Proof.

The stopping times σ_n are defined as follows :

$$\sigma_n := \inf \{t : \widetilde{A}^p_t \vee L_t \vee \sup_{\substack{u\in G \\ s\leq t}} \|V^u_t\|^p \vee A_t > n\}$$

Next we have the following lemmas

3.3. - Lemma 1

$$E\left(\sup_{t<\sigma_n} \|X^u_t\|^p\right) \leq 2^p(n+n^2) \sum_{j=0}^{2^P n^2} (2^P n^2)^j$$

Proof of Lemma 1

We remark that $A^p_{\sigma_n^-} \leq n$, $L_{\sigma_n^-} \leq n$, $\sup_{t<\sigma_n} \sup_u \|V^u_t\|^p \leq n$

We then apply inequality (2.6) to the second member of (2.7) and get

$$E\left(\sup_{t<\sigma_n} \|\boldsymbol{X}^u_t\|^p\right) \leq 2^{(P-1)}\, n + 2^{(P-1)}\, E\left(\widetilde{A}^p_{\sigma_n^-} \int_{]0,\sigma_n[} [\lambda_s \circ g(u,\boldsymbol{X}^u)]^{P/2}\, dA_s\right)$$

and property (H_3) gives for every stopping time $\tau \leq \sigma_n$

$$E\left(\sup_{t<\sigma_n} \|\boldsymbol{X}^u_t\|^p\right) \leq 2^{(P-1)}(n+n^2) + 2^{(P-1)}\, n\, E\left(\int_{]0,\tau[} (\sup_{s<t} \|\boldsymbol{X}^u_s\|^P)\, dL_s\right)$$

Applying the "Gronwall stochastic lemma" as in [10] section 7.1 we get the inequality of the lemma.

3.4. - Lemma 2

There exist constants $K(\Psi,n,p,A,\widetilde{A}^p)$ such that

$$\forall\, u,v \quad E\left(\sup_{t<\sigma_n} \|X^u_t - X^v_t\|^p\right) \leq K(\Psi,n,p,A,\widetilde{A}^p)\, \|u-v\|^p$$

Proof of Lemma 2

Applying again inequality (2.6) to the stochastic integrals

$$\int_{]0,t]} (g_s(u,X^u_{s^-}) - g_s(v,X^u_{s^-}))dZ_s \quad \text{and}$$

$$\int_{]0,t]} [g_s(v,X^u_{s^-}) - g_s(v,X^v_{s^-})]dZ_s$$

and using properties (H_1), (H_2) and (H_4) we can write for every stopping time $\tau \leq \sigma_n$:

$$E\left(\sup_{s<\tau} \|X^u(s)-X^v(s)\|^p\right) \leq 3^{P-1}n^P\|u-v\|^p + 3^{(P-1)}n\,\Psi\left(E\left(\sup_{s<\tau}\|X^u_s\|^p\right)\right)$$
$$+ 3^{(P-1)}n\,E\left(\int_{]0,\tau[} (\sup_{t<s}\|X^u(s)-X^v(s)\|^P)\,dL_s\right)$$

Applying as above the same "Gronwall-inequality" we obtain the lemma.

Theorem 3.2 is now a direct consequence of the following lemma which is a straight-forward extension of a lemma as stated by Neveu in [12] (see also P. Priouret [13] chap. 3. lemme 13 :

3.5 - Lemma 3

Let $\{Y(t,\omega,u) : t \in \mathbb{R}^+, \omega \in \Omega, u \in G\}$ *an* $\mathbb{H}$*-valued random function such that for every* $u : t \rightsquigarrow Y(t,\omega,u)$ *is a.s. cad-lag and such that for every* t *:*

$$E\left(\sup_{s\leq t}\|Y_{s,u} - Y_{s,v}\|^p\right) \leq a_{t,p}\|u-v\|^{d+\varepsilon}$$

Then there exists a mapping $Y^* : (t,\omega,u) \rightsquigarrow Y^*(t,\omega,u) \in \mathbb{H}$ *such that*

a) $u \rightsquigarrow Y^*(t,\omega,u)$ *is continuous*

b) $\forall u \in G$, $Y(t,u,.) = Y^*(t,u,.)$ *for all* t *a.s.*

b) $t \rightsquigarrow Y^*(t,.,\omega)$ *is for* P*-almost all* ω *a cad-lag mapping from* $\mathbb{R}^+$ *into* $C_b^{\mathbb{H}}(G)$ *endowed with the uniform topology.*

Proof.

We omit the proof which is pretty similar to the one given in [13].

This finishes the proof of theorem 3.2. ∎

IV - PATHWISE DIFFERENTIABILITY

4.1. - Hypothesis

We consider the same equation (1.1) or in abreviated notation : (2.7).

For a couple $g := (\sigma, f)$ of "coefficients" as in (1.1) we write to simplify :

$$\| g(u,s,\omega,h,.)\|_{\Lambda} := \Big[\| \sigma(u,s,\omega,h)\|^2_{\mathcal{L}(\mathbb{K}\,;\,\mathbb{H})} + \int_{\mathbb{E}\times\mathbb{E}} \langle f(u,s,\omega,h,x), f(u,s,\omega,h,y)\rangle_{\mathbb{H}} \, \overset{o}{q}(\omega,s,dx \otimes dy) \Big]^{\frac{1}{2}}$$

We set $v^*_t := \sup_{u\in G} \sup_{s<t} \| D_u V^u_s \| + \| V^u_s \| + \| D^2_{u^2} V^u_s \|$

were $D_u \Phi$ denotes the first order derivative and $D^2_{u^2}\Phi$ the second order derivative with respect to u of a function Φ on u.

In the hypothesis below C is a constant and $(K_t)_{t\geq 0}$ is an increasing positive process.

[D_1] For all t and ω the derivatives $D_u V^u(t,\omega)$ and $D^2_{u^2} V^u(t,\omega)$ exist and $v^*_t < \infty$

[D_2] The derivatives $D_u g(s,u,x)$ $D_u\, g(s,u,x)$ $D_{u,x} g(s,u,x)$ and $D_x g(s,u,x)$ exist and

$$\sup_{u,s,x} (\| D_u g(s,u,x)\|_{\Lambda} + \| D^2_{u^2} g(s,u,x)\|_{\Lambda} + \| D^2_{ux} g(s,u,x)\|_{\Lambda} + \| D_x g(s,u,x)\|_{\Lambda}) \leq C$$

[D_3] For all x, y u and v :

$$\| D_x g(s,u,x) - D_x g(s,v,y)\|_{\Lambda} \leq C(\| y-x\| + \| u-v\|)$$

4.2. - Theorem

Under the above hypothesis [D_1] *to* [D_3] *equation* (2.7) *has a unique (up to indistinguability) solution* X^u *on* $\mathbb{R}^+$ *and there exists a version* $(\omega,t,u) \rightsquigarrow X^u_t(\omega)$ *of this random function such that for* P*-almost all* ω *:*

a) $u \rightsquigarrow X^u_t(\omega)$ *is continuously differentiable for every* t

b) $t \rightsquigarrow X^{(.)}_t(\omega)$ *and* $t \rightsquigarrow D_u X^{(.)}_t(\omega)$ *are cad-lag for the uniform norm on* $C_b(G\,;\,\mathbb{H})$ *and* $C_b(G\,;\,\mathcal{L}(G\,;\,\mathbb{H}))$ *respectively.*

c) For every u *the stochastic process* $(D_u X^u_t)_{t\geq 0}$ *is a strong solution of the following stochastic equation (where* X^u *is the process solution of* 2.7 *as in theorem* 3.2*) :*

$$(4.1) \quad Y^u(t) = D_u V^u_t + \int_{]0,t]} \Big(D_u g(s,u,X^u_{s-}) + D_x g(s,u,X^u_{s-}) \circ Y^u_s \Big) dZ_s$$

Proof.

The proof is in several steps corresponding to lemmas 4 and 5 and section 4.5 bellow :

4.3. - Lemma 4

Under hypothesis $[D_1]$, $[D_2]$, $[D_3]$, *equations* (2.7) *and* (4.1) *satisfy the conditions* $[H_1]$ *to* $[H_4]$ *of section* 3.1 *for every* $p \geq 2$ *on any interval* $]0,\sigma_n]$ *as defined in theorem* 1.

Proof.

Let us first consider equation (2.7). (H_1) is trivially implied by $[D_1]$. $[D_2]$ implies also the Lipschitz property (H_2) and conditions (H_3) and (H_4) which is here expressed in the much stronger form $\|g(s,u,x)-g(s,v,x)\|_\Lambda \leq C\,\|u-v\|$.

We turn now to equation (4.1). The only condition (H_i) which is not immediately implied by the hypothesis of the lemma is condition (H_4). We write

$$\|D_u g(s,v,X^v_{t-}) - D_u g(s,u,X^u_{t-}) + D_x g(s,v,X^v_{t-}) \circ \xi_t - D_x g(s,u,X^u_{t-}) \circ \xi_{t-}\|^p_\Lambda$$

$$\leq 4^{p-1}\,\{\|D_u g(s,v,X^v_{t-}) - D_u g(s,u,X^v_{t-})\|^p_\Lambda\} +$$

$$+\, 4^{p-1}\,\{\|D_u g(s,u,X^v_{t-}) - D_u g(s,u,X^u_{t-})\|^p_\Lambda\}$$

$$+\, 4^{p-1}\,\{\|[D_x g(s,v,X^v_{t-}) - D_x g(s,u,X^v_{t-})] \circ \xi_{t-}\|^p_\Lambda\}$$

$$+\, 4^{p-1}\,\{\|[D_x g(s,u,X^v_{t-}) - D_x g(s,u,X^u_{t-})] \circ \xi_{t-}\|^p_\Lambda\}$$

$$\leq 4^{p-1}\,C^p(\|u-v\|^p + \|X^v_{t-} - X^u_{t-}\|^p +$$

$$+\, 4^{p-1}\,C^p\|u-v\|^p\,\|\xi_{t-}\|^p + 4^{p-1}\,C^p\|(X^v_{t-} - X^u_{t-}) \circ \xi_{t-}\|^p$$

One knows from proposition 2 that there exists an increasing sequence (σ_n) of stopping times and constants C_n such that

$$E \sup_{s<\sigma_n} \|Y^u(s) - Y^v(s)\|^{2p} \leq C_n\|u-v\|^{2p}$$

If we write for every stopping time τ

$$E\Big(\sup_{t<\tau\wedge\sigma_n} \|(X^v_t - X^u_t) \circ \xi_{t-}\|^p\Big) \leq$$

$$\left[E\left(\sup_{t<\tau\wedge\sigma_n} \|X_t^v - X_t^u\|^{2p}\right)\right]^{\frac{1}{2}} \left[E\left(\sup_{t<\tau\wedge\sigma_n} \|\xi_t\|^{\frac{2p}{2p-1}}\right)\right]^{\frac{2p-1}{2}}$$

$$\leqslant C_n^{\frac{1}{2}}\|u-v\|^p \, E\left(\sup_{t<\tau\wedge\sigma_n} \|\xi_t\|^{\alpha}\right)^{p/\alpha}$$

with $\alpha = \frac{2p}{2p-1}$

Therefore

$$E\left(\sup_{s<\tau\wedge\sigma_n} \|g(s,u,\xi_{s^-}) - g(s,v,\xi_{s^-})\|_{\Lambda}^{p}\right) \leqslant 4^{p,1}\, C^p\|u-v\|^p[1+C_n + E(\sup_{s<\tau\wedge\sigma_n} \|\xi_s\|^p)]$$

$$+ C_n^{\frac{1}{2}}[E(\sup_{t<\tau\wedge\sigma_n} \|\xi_t\|^{\alpha})]^{p/\alpha}$$

If we remark that $E(\sup_{t<\tau\wedge\sigma_n} \|\xi_{t^-}\|^p) \geqslant [E(\sup_{t<\tau\wedge\sigma_n} \|\xi_{t^-}\|^{\alpha})]^{p/\alpha}$

we see that property (H_4) holds with

$$\Psi(\rho) = 1 + C_n + (1 + C_n^{\frac{1}{2}})\rho$$

4.4. - Lemma 2

If we define

$$\Phi_t(e,u,\lambda) = \frac{1}{\lambda}[X_t^{u+\lambda e} - X_t^u - \lambda Y_t^u \circ e]$$

there exists an increasing sequence (τ_n) *of stopping times such that* $\lim_n P\{\tau_u < T\} = 0$ *and a sequence* C_n *of constants such that*

$$E\left\{\sup_{t<\tau_n} \|\Phi_t(e,.,\lambda)\|^2_{L^2(G)}\right\} \leqslant C_n\,\lambda^2$$

Proof.

For each u the process $(\Phi_t(e,u,\lambda))_{t\leqslant T}$ is solution of

$$(4.2)\quad \Phi_t(e,u,\lambda) = \frac{1}{\lambda}(V_t^{u+\lambda e} - V_t^u - \lambda D_e V_t^u) +$$

$$+ \int_{]0,t]}\frac{1}{\lambda}\Big[g(s,u+\lambda e,X_{s^-}^{u+\lambda e}) - g(s,u,X_{s^-}^u) -$$

$$\lambda D_e g(s,u,X_{s^-}^u) - \lambda D_x g(s,u,X_{s^-}^u) \circ Y_{s^-}^u \circ e\Big]\, dS_s$$

We may write for $x,y \in \mathbb{H}$ and $\eta \in \mathcal{L}(\mathbb{H};\mathbb{H})$

$$(4.3) \quad g(s,u+\lambda e,y) - g(s,u,x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e =$$

$$\lambda D_e g(s,u,y) + D_x g(s,u,x) \circ (y-x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e +$$

$$+ h(s,u,x,y,\eta,\lambda,e)$$

$$= D_x g(s,u,x) \circ (y-x-\lambda\eta \circ e) + \tilde{h}(s,u,x,y,\eta,\lambda)$$

with

$$(4.4) \quad \|\tilde{h}(s,u,x,y,\eta,\lambda)\|_\Lambda \leq |\lambda| \, K \, (\|y-x\| + |\lambda|)$$

for some constant K

The equation (4.2) can therefore be written

$$(4.5) \quad \Phi_t(e,u\lambda) = H_t(u,\lambda,e) + \int_{]0,t]} D_x g(s,u,X^u_{s^-}) \circ \Phi_{s^-}(e,u,\lambda) \, dZ_s$$

where the process $H(u,\lambda,e)$ satisfies

$$(4.6) \quad \|H_t(u,\lambda,e)\|_{\mathbb{H}} \leq |\lambda| \quad v^k_t + \left\| \int_{]0,t]} \frac{1}{\lambda} \, h(s,u,X^{u+\lambda e}_{s^-}, X^u_{s^-}, Y^u_{s^-} \circ e) \, dZ_s \right\|$$

Using (4.5) we obtain from (4.6) for every stopping time σ :

$$E\left(\sup_{t<\sigma} \|H_t(u,\lambda,e)\|^2\right) \leq 2\,\lambda^2\, v^*_{\sigma^-} + E\left(\tilde{A}_{\tau^-} \cdot \int_{]0,\tau[} [\lambda^2 + C^2 \|X^{u+\lambda e}_{s^-} - X^u_s\|^2] \, dA_s\right)$$

Using then theorem we see that there exists a sequence (σ_n) of stopping times and a sequence of constants (K_n) such that

$$(4.7) \quad \sup_{s<\sigma_n} (\tilde{A}_s \vee A_s) \leq n \quad \text{and}$$

$$(4.8) \quad E(\sup_{t<\sigma_n} \|H_t(u,\lambda,e)\|^2) \leq K_n \, \lambda^2$$ (use a standard stopping procedure for processes $v^*, \tilde{A}$ and A).

This implies

$$(4.9) \quad E\Big(\sup_{t<\sigma_n} \int_G \| H_t(u,\lambda,e)\|^2 \, du\Big) \leq \int_G K_n \lambda^2 \, du \leq \widetilde{K}_n \lambda^2$$

We next consider the $L^2(G)$-valued process $(\Phi_t(e,.,\lambda))_{t\leq T}$

As $D_x g$ is bounded by some constant C, inequality (4.6) shows that the $L^2(G)$-valued process Φ_t satisfies an inequality of the following type for every stopping time $\tau \leq \sigma_n$

$$E\Big\{\sup_{t<\tau} \|\Phi_t(e,.,\lambda)\|_{L^2(G)}\Big\} \leq 2\widetilde{K}_n \lambda^2 + 2\,E\Big(\widetilde{A}_\tau - \int_{[0,\tau[} C^2 \sup_{s<t} \|\Phi_s(e,.,\lambda)\|^2_{L^2(G)} \, dA_s\Big)$$

$$\leq 2\widetilde{K}_n \lambda^2 + 2n\,C^2 \int_{[0,\tau[} \sup_{s<t} \|\Phi_s(e,.,\lambda)\|_{L^2(G)} \, dA_s$$

The already used "Gronwall inequality" of [10] shows immediately the existence of a constant C_n as in the lemma.

4.5. - End of the proof of the theorem

We make use of the following easily proved property : let $f \in L^2_{\mathbb{H}}(\overline{G})$ let $f \in L^2(\mathbb{G};\mathbb{H}) \cap C(G;\mathbb{H})$ and $\overline{f} \in L^2(G;\mathcal{L}(\mathbb{H};\mathbb{H}) \cap C(G;\mathcal{L}(\mathbb{H};\mathbb{H}))$ such that for all $e \in \mathbb{R}^d$, all $u \in \mathbb{R}^d$ and some decreasing sequence $\lambda_k \downarrow 0$:

$$\lim_{k\to\infty} \| f(u+\lambda_k e) - f(u) - \lambda_k \overline{f}(u) \circ e\|_{L^2(G;\mathbb{H})} = 0$$

Then $\overline{f}$ is the derivative of f in the sense of distributions and therefore in the ordinary sense in every point $u \in G$. Let us consider for each ω and n a P-negligeable set Ω_n and a sequence λ_k such that $\lambda_k \downarrow 0$ and

$$\lim_{k\to\infty} \sup_{t<\tau_n(\omega)} \|\Phi_t(e,.,\omega,\lambda_k)\|_{L^2(G)} = 0 \quad \text{for every } \omega \notin \Omega_n$$

The above property shows that for every $\omega \notin \Omega_n$ and $t < \tau_n(\omega)$ $Y^u_t(\omega)$ is the derivative of $u \leadsto X^u_t(\omega)$ at point u. Therefore $Y^u_t(\omega)$ is the derivative of $u \leadsto X^u_t(\omega)$ for all $t<\tau_n(\omega)$ and $\omega \notin (\bigcup_n \Omega_n)$.

This proves the theorem. ■

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