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THERE EXISTS NO ULTIMATE SOLUTION TO SKOROKHOD'S
PROBLEM

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Abstract

Let (X, Y) be a mean zero martingale pair, i.e., X and Y possess mean zero and $E(Y|X) = X$ a.s.. It has been proved in various ways that (1) there exist stopping times τ on Brownian motion $\{B(t); t \geq 0\}$ such that $B(\tau)$ is distributed like X and $\{B(t \wedge \tau); t \geq 0\}$ is uniformly integrable; and (2) for any such τ there exist stopping times τ' such that $\tau \leq \tau'$ a.s., $(B(\tau), B(\tau'))$ is distributed like (X, Y) and $\{B(t \wedge \tau'); t \geq 0\}$ is uniformly integrable. In other words (to explain the role of uniform integrability), a martingale pair can be embedded in a piece of Brownian motion that is itself a martingale.

We will show that unless Y lives on one or two points, there can exist no stopping time τ' with $\{B(t \wedge \tau'); t \geq 0\}$ uniformly integrable and $B(\tau')$ distributed as Y , such that whenever (X, Y) is a martingale pair there exist τ with $\tau \leq \tau'$ a.s. and $B(\tau)$ distributed as X .

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1. INTRODUCTION

Let F be a distribution with mean zero and finite variance. It has been shown (Skorokhod [12], Dubins [3], Root [9], Chacon and Walsh [2], Azema and Yor [1] and others) that for some stopping time (st) τ of finite expectation on Brownian Motion starting at zero (BM) $\{B(t); t \geq 0\}$, $B(\tau)$ is distributed $(\sim) F$. This embeddability of F has been extended to more general Markov processes (Rost [10], Meyer [7]).

Monroe [8] has shown that if F has mean zero (but not necessarily finite variance), then $B(\tau) \sim F$ for some τ that makes $\{B(t\wedge\tau); t \geq 0\}$ uniformly integrable. Monroe proves this property of τ to be equivalent to τ being minimal, i.e. $B(\tau)$ has mean zero and for no $\tau' \leq \tau$ (except τ itself) is $B(\tau') \sim F$. An interesting property of minimality is that if $B(\tau)$ has mean zero and finite variance, then τ is minimal if and only if it has finite mean. The mean of a minimal stopping time is always equal to the variance of the variable it embeds.

A pair of distributions (F, G) admits a martingale (or, belongs to M), if F and G have expectations zero and for some two random variables X and Y on some space, $X \sim F$, $Y \sim G$ and $E(Y|X) = X$ a.s.. Equivalently, (see Meyer [6]), $(F, G) \in M$ if for all real x , $\phi_F(x) \geq \phi_G(x)$, where

$$(1) \quad \phi_{L(Z)}(x) = -E|Z-x|.$$

Equivalently, $EX = EY = 0$ and $E\psi(X) \leq E\psi(Y)$ for all nonnegative non-decreasing convex functions ψ . For more on ϕ , see Chacon and Walsh [2]. Further details on convex inequalities can be found in Meilijson and Nádas [5] and in Rost [1], and their relation to extremal martingales in Dubins and Gilat [4] and in Azema and Yor [1].

It is clear from the proofs of embeddability that if $(F, G) \in M$ then for every minimal st τ with $B(\tau) \sim F$ there exists a minimal st τ' with $B(\tau') \sim G$ and $\tau \leq \tau'$ a.s.. A minimal st is said to be ultimate if for every distribution F with $(F, L(B(\tau))) \in M$ there exists a st τ' with $B(\tau') \sim F$ and $\tau' \leq \tau$ a.s.

Let

$$(2) \quad \tau_A = \inf \{t \geq 0 \mid B(t) \in A\}.$$

THEOREM. A stopping time τ is ultimate if and only if for some $a \leq 0 \leq b$, $\tau = \tau_{\{a,b\}}$ a.s..

2. PROOF OF THE THEOREM.

A distribution F has atomic ends if for some $a < b$, (the "ends"),

$$(3) \quad 0 = F(a-) < F(a) \leq F(b-) < F(b) = 1.$$

If strict inequality occurs throughout (3), we will say that F has non exclusive atomic ends. We will prove in lemma 2 that if τ is an ultimate st and $B(\tau)$ is not supported by one or two points, then there exists an ultimate st τ' such that $L(B(\tau'))$ has non exclusive atomic ends. Lemma 4 will show this to be an impossibility.

LEMMA 1. For a distribution function G with mean zero for which there exist real numbers a and b with $\text{ess inf}(G) < a < 0 < b < \text{ess sup}(G)$, let $0 < u < v < 1$ be defined by

$$(4) \quad E(G^{-1}(U) \mid U \leq u) = a, \quad E(G^{-1}(U) \mid U \geq v) = b,$$

where $U \sim U[0,1]$, and let $G_{(a,b)}$ be the distribution of a random variable X with

$$(5) \quad X = \begin{cases} a & U \leq u \\ b & U \geq v \\ G^{-1}(U) & \text{otherwise} \end{cases}$$

$$= E(G^{-1}(U) \mid \mathcal{I}_{\{U \leq u\}}, \mathcal{I}_{\{U \leq v\}}, \max(u, \min(U, v)))$$

Then $(G_{(a,b)}, G) \in M$ and whenever F is supported by $[a, b]$ and $(F, G) \in M$, then $(F, G_{(a,b)}) \in M$.

Proof. As mentioned in the introduction, the test for belonging to M is a pointwise inequality of the functions ϕ : This function ϕ (see Chacon and Walsh [2]) is concave and is asymptotic to $-|x|$ as $|x| \rightarrow \infty$. It agrees with $-|x|$ outside any interval supporting the distribution. $\phi_{G_{(a,b)}}$ agrees with ϕ_G on $[G^{-1}(u), G^{-1}(v)]$ and is linear on $[a, G^{-1}(u)]$ and on $[G^{-1}(v), b]$. These two linear pieces are tangents to ϕ_G , and thus $\phi_{G_{(a,b)}}$ is the minimal concave function that exceeds ϕ_G and agrees with $-|x|$ outside $[a, b]$.

□

LEMMA 2.

- (i) For every $a \leq 0 \leq b$, $\tau_{\{a,b\}}$ is ultimate.
- (ii) If τ is ultimate, so is $\min(\tau, \tau_{\{a,b\}})$.
- (iii) If τ is ultimate and $B(\tau)$ is not supported by one or two points, then there exists an ultimate τ' such that $B(\tau')$ has non exclusive atomic ends.

Proof. (i) If $a \cdot b = 0$, the result is clear.

If $a \cdot b > 0$ then every F with $(F, \mathcal{L}(B(\tau_{\{a,b\}}))) \in M$ is supported by $[a, b]$ and every minimal st τ for which $B(\tau) \sim F$ satisfies $\tau \leq \tau_{\{a,b\}}$ a.s. (ii) Let τ be ultimate.

Denote $G = \mathcal{L}(B(\tau))$. Then every F with $(F, G) \in M$ that is supported by $[a, b]$ must be embeddable before time τ and also before time $\tau_{\{a,b\}}$. In particular, this is the case for $F = \hat{G}_{(a,b)}$. This implies that

$$(6) \quad (G_{(a,b)}, \mathcal{L}(B(\min(\tau, \tau_{\{a,b\}})))) \in M.$$

But, by lemma 1, (for, if its conditions are not met, the statement of (ii) is trivial),

$$(7) \quad (\mathcal{L}(B(\min(\tau, \tau_{\{a,b\}}))), G_{(a,b)}) \in M.$$

Combine (6) and (7) to obtain that $\mathcal{L}(B(\min(\tau, \tau_{\{a,b\}}))) = G_{(a,b)}$ and that $\min(\tau, \tau_{\{a,b\}})$ is ultimate.

(iii) The conditions of (iii) imply the conditions of lemma 1. Take a proper pair (a, b) and apply (ii).

[]

LEMMA 3. Let $a < x < 0 < y < b$ and let

$$(8) \quad \tau^{(1)} = \min(\tau_{\{a,b\}}, \max(\tau_{\{x\}}, \tau_{\{y\}})).$$

Let X_a have values a and y and mean zero, let X_b have values x and b and mean zero. Then

$$(9) \quad E(\tau^{(1)}) = xy - ay - xb = \\ = \frac{a^2 P(X_a = a)}{1 - P(X_a = a)} + \frac{b^2 P(X_b = b)}{1 - P(X_b = b)} + \frac{ab P(X_a = a) P(X_b = b)}{(1 - P(X_a = a))(1 - P(X_b = b))}.$$

$$(10) \quad P(\tau^{(1)} = \tau_{\{a\}}) = y/(v-a) = P(X_a = a)$$

$$(11) \quad P(\tau^{(1)} = \tau_{\{b\}}) = -x/(b-x) = P(X_b = b).$$

Proof. Express $\tau^{(1)}$ as the sum of two terms. The first is the hitting time $\min(\tau_{\{x\}}, \tau_{\{y\}})$. If $\tau_{\{x\}} \leq \tau_{\{y\}}$ ($\tau_{\{y\}} < \tau_{\{x\}}$), the second is the hitting time $\min(\tau_{\{y\}}, \tau_{\{a\}})$ ($\min(\tau_{\{x\}}, \tau_{\{b\}})$). Apply repeatedly the well known fact that if $u < 0 < v$, then $E(\tau_{\{u,v\}}) = -uv$ and $P(\tau_{\{u\}} < \tau_{\{v\}}) = v/(v-u)$.

□

LEMMA 4. If F has non exclusive atomic ends, then there is no ultimate st to embed F .

Proof. For $X \sim F$, $a = \text{ess inf}(F)$, $b = \text{ess sup}(F)$:

Let $X_a = E(X \mid 1_{\{X=a\}})$, $X_b = E(X \mid 1_{\{X=b\}})$.

Let $y = E(X \mid X > a)$, $x = E(X \mid X < b)$.

If τ is ultimate and $B_\tau \sim F$, then $\{a < B_\tau < b\} = \{\tau < \tau_{\{a,b\}}\}$ and on $\{a < B_\tau < b\}$,

BM must have visited both x and y up to time τ , since X_a and X_b must have been embedded. Hence, $\tau \geq \tau^{(1)}$ of lemma 3.

In view of (10) and (11), on $\{\tau^{(1)} < \tau_{\{a,b\}}\}$ (which equals a.s.

$(\tau < \tau_{\{a,b\}})$, the st $\tau - \tau^{(1)}$, defined on $B \circ \tau^{(1)}$, must embed the conditional distribution of X given that $a < X < b$. Following Monroe [8], if τ is minimal and $B(\tau)$ is bounded, then the expectation of τ is the variance of $B(\tau)$, which is finite. We thus obtain (upon substituting P_a for $P(X = a)$ and P_b for $P(X = b)$) that

$$(12) \quad \text{Var}(X) = (1 - P_a - P_b)\text{Var}(X \mid a < X < b) + E(\tau^{(1)}).$$

But, on the other hand,

$$(13) \quad \begin{aligned} \text{Var}(X) &= E(X^2) = (1 - P_a - P_b)E(X^2 \mid a < X < b) + a^2P_a + b^2P_b = \\ &= (1 - P_a - P_b)\text{Var}(X \mid a < X < b) + \frac{(aP_a + bP_b)^2}{1 - P_a - P_b} + a^2P_a + b^2P_b. \end{aligned}$$

Compare (12) and (13):

$$(14) \quad \begin{aligned} E(\tau^{(1)}) &= a^2P_a + b^2P_b + \frac{(aP_a + bP_b)^2}{1 - P_a - P_b} = \\ &= \frac{a^2P_a}{1 - P_a} + \frac{b^2P_b}{1 - P_b} + \frac{abP_aP_b}{(1 - P_a)(1 - P_b)} \\ &\quad - \frac{P_aP_b(b-a)^2\left(\frac{b}{b-a} - P_a\right)\left(\frac{-a}{b-a} - P_b\right)}{(1 - P_a)(1 - P_b)(1 - P_a - P_b)} \end{aligned}$$

But (14) conflicts with (9), since $b/(b-a)$ and $-a/(b-a)$ are strictly bigger than P_a and P_b respectively.

Hence, τ is not ultimate.

□

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