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# A.S. APPROXIMATION RESULTS FOR MULTIPLICATIVE STOCHASTIC INTEGRALS

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This note is a contribution to the theory of the multiplicative integral for continuous semimartingales. Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $(\mathcal{F}_t)$ which satisfies the usual conditions, and let X be a continuous semimartingale with values in the space L(d) of all d×d matrices, and such that X(0)=0. The <u>multiplicative integral</u>

(1) $$Y(t) = \prod_0^t (I+dX)$$

can be defined as the only solution to the stochastic differential equation

(2) $$Y(t) = I + \int_0^t Y(s)dX(s)$$

and has been extensively studied ( see Ibero [6], Emery [4], [5] in the right continuous case, and see also in a different context Masani [11] ). We shall also call Y the <u>exponential</u> of X and denote it by $\varepsilon(X)$. If we replace in (1) dX by hdX , where h is a predictable, L(d)-valued process, then Y is called the ( left ) <u>multiplicative integral of</u> h <u>with respect to</u> X. More details will be given below.

Our first result will be an explicit formula for the inverse of Y, which we haven't found in the literature, though it is very simple, and its proof is easy.

In the second part of this paper, we shall deal with <u>a.s.</u> approximations to the multiplicative integral. These results are less general than those of Bichteler in [1], [2], but our proofs are so elementary ( they do not use anything deeper than Doob's maximal inequality[1]), that the editors of the Séminaire offered to publish them in this volume. The same method also gives a.s. convergence results for ordinary stochastic integrals.

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1. Prof. P.A. Meyer has pointed out to us that our main lemma is very close to the method of Métivier and Pellaumail, except that continuity simplifies things a great deal.

## I. A FORMULA FOR THE INVERSE OF Y

We first introduce some notation. Let $X$ be a continuous $L(d)$-valued semimartingale, and $H$ be a locally bounded predictable $L(d)$-valued process. Then we denote by $H{\cdot}X$ as usual the stochastic integral $\int HdX$. On the other hand, since $L(d)$ isn't commutative, we may consider right stochastic integrals $\int (dX)H$ . To avoid ambiguities, we denote them ( in this section only ) as $X{:}H$ . Obviously $(X{:}H) = (H'{\cdot}X')'$, where $'$ is the transpose operation.

Given two continuous semimartingales $U,V$ with values in $L(d)$, we denote by $\langle U,V\rangle$ the $L(d)$-valued process ( continuous, with finite variation paths ) defined by

$$\langle U,V\rangle^i_j = \Sigma_k \langle U^i_k, V^k_j \rangle$$

The following identities are trivially proved by looking at the entries

$$d(UV) = UdV + (dU)V + d\langle U,V\rangle \tag{3}$$

$$\langle H.U,V\rangle = H.\langle U,V\rangle \ , \ \langle U,V{:}H\rangle = \langle U,V\rangle{:}H \tag{4}$$

It is obvious that $\langle U,V\rangle=0$ if $U$ or $V$ is a finite variation process. We denote by $\varepsilon^{\bullet}(X)$ the <u>right exponential</u> of $X$ , i.e. the solution of the stochastic differential equation symmetric to (2)

$$Y^{\bullet}(t) = I + \int_0^t dX(s)Y^{\bullet}(s) \tag{5}$$

We have $\varepsilon^{\bullet}(X)=\varepsilon(X')'$. With these notations :

THEOREM 1. *The inverse of* $Y=\varepsilon(X)$ *is given by*

$$Y^{-1} = \varepsilon^{\bullet}(-X + \langle X,X\rangle\ )^{(1)} \tag{6}$$

PROOF. Set $U=\varepsilon(X)$, $V=\varepsilon^{\bullet}(-X+\langle X,X\rangle)$, so that $dU=UdX$, $dV=(-dX+d\langle X,X\rangle)V$. We apply (3) and compute $UdV=U((-dX+d\langle X,X\rangle)V)$, $(dU)V=(UdX)V$, and from the obvious associativity of the left and right stochastic integration, the position of the parentheses doesn't matter. On the other hand, from (4) $d\langle U,V\rangle = Ud\langle X,V\rangle = -U(d\langle X,X\rangle V)$. So finally $d(UV)=0$, and since $UV=I$ at time 0 it remains equal to $I$ for all $t$. The theorem is proved.

## II. APPROXIMATION TO THE EXPONENTIAL BY ITERATION

Let again $X$ be a continuous $L(d)$-valued semimartingale with $X(0)=0$, and $Y$ be its exponential. We are going to prove in this section :

THEOREM 2. *The processes defined inductively by*

$$Y_0=I \ , \ Y_{n+1}(t) = I + \int_0^t Y_n(s)dX(s) \tag{7}$$

*converge a.s. to* $Y$ , *uniformly on compact intervals of* $\mathbb{R}_+$ .

If we replace a.s. convergence by convergence in probability, the

1. This is related to th. I.2 of Bismut, Sem. Prob. XII p. 194.

result is well known for general stochastic equations and without continuity hypothesis on X ( Emery [5], p. 290 ). In the same general setup, an a.s. convergence result is stated in Bichteler [2]. So theorem 2 isn't new, but its proof possibly is, and depends on quite elementary results. We are going to prove it first under the following auxiliary hypothesis on X :

($8_\beta$) $X^i_j = M^i_j + A^i_j$ ; $M^i_j$ is a local martingale with $M^i_j(0)=0$ ; $A^i_j$ is a process with finite variation paths ; each one of the increasing processes $\langle M^i_j, M^i_j \rangle$, $\int_0^t |dA^i_j(s)|$ satisfies a Lipschitz condition of order $\beta$ .

We then use a lemma from our paper [8] ( a sketch of the proof will be given at the end for the reader's convenience ). Here $\| \ \|$ denotes a norm on L(d) ( to make a definite choice, identify L(d) to $\mathbb{R}^{d^2}$ and use the euclidean norm ). Given a L(d)-valued process Z, set $\|Z\|^*_t = \sup_{s\leq t} \|Z_s\|$. Then :

LEMMA. If X satisfies ($8_\beta$) and H is left continuous and predictable, we have

(9) $$E[\,\|H\cdot X\|^{*2}_t] \leqq 8d^2\beta(1+t\beta)\int_0^t E[\,\|H_s\|^2]ds \ .$$

Taking this for granted, we prove theorem 2 under ($8_\beta$). We set $S_m = \inf\{t : \|Y\|_t \geqq m\}$. Replacing X by the stopped process $X^{S_m}$ amounts to stopping at $S_m$ all the processes concerned, in particular Y, $Y_n$ . We first assume that Y is bounded. Then set $\phi_n(t) = E[\,\|Y_n - Y\|^{*2}_t]$ ; since $Y_{n+1} - Y = (Y_n - Y)\cdot X$, the lemma gives us

$$\phi_{n+1}(t) \leqq C(1+\beta t)\int_0^t \phi_n(s)ds$$

Let M be a bound for $\phi_0$. Then an easy induction shows that $\phi_n(t)$ is dominated by $M(C(1+\beta t)t)^n/n!$, therefore the r.v. $\Sigma_n \|Y_n - Y\|^*_t$ has a finite expectation, and hence is a.s. finite. If Y isn't bounded, we apply this result to $X^{S_m}$ and let $m \to \infty$ , reaching the same conclusion. Hence $Y_n$ converges a.s. to Y, uniformly on finite intervals.

To end the proof, we just remark that we can reduce to ($8_1$) by a strictly increasing change of time $\sigma_t = \inf\{s : \lambda_s > t\}$, where $\lambda_s$ is the continuous, strictly increasing process

(10) $$\lambda_s = s + \Sigma_{ij} \langle M^i_j, M^i_j \rangle_s + \Sigma_{ij} \int_0^s |dA^i_j|$$

This step is certainly familiar to readers of this volume ( see for instance Kazamaki [9] ,[10] : these papers were pointed to us by Prof. P.A. Meyer ). So we omit the easy details.

## III. APPROXIMATION DY RIEMANN SUMS AND PRODUCTS

In this section, we consider a process g with values in L(d), adapted, right continuous with left limits, and define the semimartingale $Z = \int g_- dX$ ( since X is continuous, we might as well write g instead of $g_-$ , but we keep the standard notation ). We are going to express Z as the a.s. limit of Riemann sums, and $Y=\varepsilon(Z)$ as the a.s. limit of << Riemann products >> . Incidently, let us mention that this last result doesn't follow directly from Bichteler's theorems.

For each n, we consider a sequence of stopping times $(T^n_k)_{k=0,1,..}$ increasing with k, such that

$$(11)\qquad T^n_0 = 0 \quad , \quad \begin{array}{l} \|X(t)-X(T^n_k)\| \leq 2^{-n} \\ \|g(t-)-g(T^n_k)\| \leq 2^{-n} \end{array} \text{ for } t\in ]T^n_k, T^n_{k+1}]$$

Of course such a sequence exists, and can be explicitly constructed by induction, but our result depends on (11) only, not on the explicit construction. We define additive and multiplicative Riemann sums as follows

$$(12)\qquad \begin{array}{l} Z_n(t) = \Sigma_k\ g(T^n_k)(X(t\wedge T^n_{k+1})-X(t\wedge T^n_k)) \\ Y_n(t) = \Pi_k\ (I + g(T^n_k)(X(t\wedge T^n_{k+1})-X(t\wedge T^n_k))\ ) \end{array}$$

With these notation, we can state :

THEOREM 3. The processes $Z_n, Y_n$ a.s. converge uniformly on compact sets to the corresponding processes

$$Z_t = \int_0^t g_- dX \quad , \quad Y_t = \varepsilon(Z)_t = \prod_0^t ( I+g_- dX) \ .$$

Here again, we may reduce by a change of time to the case of a semimartingale X which satisfies hypothesis $(8_\beta)$ with $\beta=1$. It is necessary to remark here that the change of time transforms stopping times into stopping times, and preserves property (11). It will be convenient also to assume that all the processes g, Y, Z, $Y_n$, $Z_n$ are bounded ( by constants which may depend on n ). The construction of convenient times $S_m$ is a little more delicate here, and requires an application of the Borel-Cantelli lemma as in Dellacherie [3], p. 743, th. 4 . Remark that g is right continuous, and must be stopped at $S_m-$ to get boundedness. After these preliminary steps, the theorem will be deduced from the lemma, with a little more difficulty than in the preceding proof.

Given any process H, define $J_n H$ as the right continuous step process equal to $H(T^n_k)$ on the interval $[T^n_k, T^n_{k+1}[$. One checks easily that

$$(13)\qquad Y_n(t) = \Pi_k\ (\ I+ (Z_n(t\wedge T^n_{k+1})-Z_n(t\wedge T^n_k))\ ) = I+\int_0^t J_n Y_n(s-)dZ_n(s)$$

$$(14)\qquad Z_n(t) = \int_0^t J_n g(s-)dX(s)$$

Since $\|g\|$ is bounded by some constant $C_1$ and X satisfies $(8_1)$, $Z=g_-\cdot X$ satisfies $(8_\alpha)$ for a suitable constant $\alpha$ ( depending only on $C_1$ and the dimension d ), and the same is true for $Z_n$ according to (14). Applying the main lemma to the stochastic integral $Z_n-Z = (J_ng-g)_-\cdot X$ with X satisfying to $(8_\alpha)$ and $\|J_ng-g\| \leqq 2^{-n}$ gives an inequality

$$E[\,\|Z_n-Z\|_t^{*2}\,] \leqq 8d^2\alpha(1+t\alpha)t\; 2^{-2n}$$

from which the a.s. convergence of $Z_n$ to Z on compact intervals follows at once.

Let us study the convergence of $Y_n$ to Y. Since we aren't interested in the exact value of the constants, we assume t varies in a compact interval [0,M], and denote simply by a,b,c,... numbers which may change from place to place, with the only restriction that they shouldn't depend on n.

We write (13) in the following way

$$Y_n-Y = (J_nY_n-Y_n)_-\cdot Z_n + (Y_n-Y)_-\cdot Z_n + Y_-\cdot(Z_n-Z)$$

$$= \eta_1+\eta_2+\eta_3 \text{ (say )}$$

We recall that Y is assumed to be bounded by some constant C. Then the main lemma applied to the last term gives us as above

(15) $$E[\,\|\eta_3\|_t^{*2}] \leqq 8C^2d^2\alpha(1+t\alpha)t\; 2^{-2n} \leqq a\; 2^{-2n} \quad \text{if } t\in[0,M]$$

Also, $Y_n$ is bounded by $C_n$. Therefore

$$\phi_n(t) = E[\,\|Y_n-Y\|_t^{*2}]$$

is a bounded function, and the main lemma gives us

(16) $$E[\,\|\eta_2\|_t^{*2}] \leqq 8d^2\alpha(1+t\alpha)\int_0^t \phi_n(s)ds \leqq b\int_0^t \phi_n(s)ds$$

The first term is a little more delicate. We remark that it can also be written as $J_nY_n(J_nZ_n-Z_n)$ - a product, not a stochastic integral - and that we have on $[T_k^n,T_{k+1}^n[$ $(J_nZ_n-Z_n)_t = \int_{T_k^n}^t J_ng(s-)dX_s = g(T_k^n-)(X_t-X_{T_k^n})$, which is dominated in absolute value by $C_1 2^{-n}$. On the other hand, we may apply the main lemma to formula (13) to get a << Gronwall type formula >> for $E[\,\|Y_n\|_t^{*2}\,]$, from which we get

$$E[\,\|Y_n\|_t^{*2}\,] \leqq Ke^{Ht} \text{ , bounded for } t\in[0,M]$$

from which we deduce

(17) $$E[\,\|\eta_1\|_t^{*2}] \leqq c2^{-2n} \quad \text{for } t\in[0,M]$$

Adding these inequalities ( with a little care, because of the exponent 2 ) and recalling that $\phi_n(t) = E[\|Y_n - Y\|_t^{*2}]$ ), we get ( with new constants a,b )

$$\phi_n(t) \leqq a2^{-2n} + b\int_0^t \phi_n(s)ds \quad , \; t\in[0,M]$$

Therefore $\phi_n(t) \leq c(M)2^{-2n}$ from Gronwall's inequality, on the compact interval $[0,M]$. The Borel-Cantelli lemma now implies the a.s. convergence of $Y_n$ to $Y$ .

IV. SOME OTHER MULTIPLICATIVE INTEGRALS

Emery has studied in [4] multiplicative integrals of the following kind

$$(18) \qquad Y_t = \prod_0^t h(dX)$$

where h is a $C^3$ mapping from L(d) to L(d) such that h(0)=I. The most important among them concerns the matrix exponential, which turns out to be the stochastic exponential in the Stratonovitch sense

$$(19) \qquad Y_t = \prod_0^t e^{dX} = \prod_0^t ( I + dX + \tfrac{1}{2}d\langle X,X\rangle )$$

In a similar way, all the multiplicative integrals (18) can be reduced to ordinary multiplicative integrals relative to a semimartingale $X'_t = \int_0^t f(dX_s)$, and Emery shows that the obvious Riemann products for (18) converge uniformly <u>in probability</u> to the multiplicative integral. An adaptation of our method shows that, for continuous semimartingales, Riemann products relative to random partitions satisfying (11) will converge uniformly <u>a.s.</u> to the multiplicative integral. The principle of the proof remains exactly the same, but the computations are a little more cumbersome.

V. ON THE MAIN LEMMA

$H\cdot X$ is a $d\times d$ matrix. It will be sufficient to prove the following inequality for fixed i,j and to sum it over i,j

$$E[\ |\Sigma_k H_k^i\cdot X_j^k|_t^{*2}] \leqq 8d\beta(1+t\beta)\Sigma_k \int_0^t E[(H_{ks}^i)^2]ds$$

We split $X_j^k$ into $M_j^k$ and $A_j^k$ ; it is sufficient to prove that

$$E[\ |\Sigma_k H_k^i\cdot M_j^k|_t^{*2}] \leqq 4d\beta\ \Sigma_k \int_0^t E[(H_{ks}^i)^2]ds$$

$$E[\ |\Sigma_k H_k^i\cdot A_j^k|_t^{*2}] \leqq d\beta^2 t\ \Sigma_k \int_0^t E[(H_{ks}^i)^2]ds$$

First inequality : $|\Sigma_k \int_0^t H_k^i dM_j^k|^2 \leqq d\Sigma_k (\int_0^t H_k^i dM_j^k)^2$, hence the same inequality for the corresponding sup. From Doob's maximal inequality and the isometry of the $L^2$ stochastic integral, we get on the right $4d\Sigma_k E[\int_0^t H_k^i d\langle M_j^k, M_j^k\rangle \leqq 4d\beta E[\Sigma_k \int_0^t E[H_{ks}^{i2}]ds$.

Second inequality : We start in the same way and replace $(H_k^i \cdot A_j^k)_t^*$ by $(\int_0^t |H_{ks}^i||dA_s^k|)^2 \leqq \beta^2(\int_0^t |H_{ks}^i|ds)^2 \leqq \beta^2 t\int_0^t (H_{ks}^i)^2 ds$. Then we integrate.

As we mentioned in the footnote to the introduction, this lemma is a Métivier-Pellaumail inequality ( see for instance Emery's report in the preceding volume of the seminar, vol XIV p. 118 ), with dt as the << controlling process >>.

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