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SEMIMARTINGALES IN PREDICTABLE RANDOM OPEN SETS

ZHENG Wei An

INTRODUCTION

After L. Schwartz discussed in [1] the restriction of semimartingales to a random open set, P.A. Meyer [2] gave a definition of semimartingales in a random open set. This was discussed again by L. Schwartz [3] from the point of view of formal measures. A number of fine results, specially for the case of continuous semimartingales, have been proved since the definitions were given (see [1], [3], and the recent reference [7][1]).

The purpose of this paper is to extend to semimartingales in a predictable random open set as much of the classical theory of semimartingales as possible, following the lines of L. Schwartz and P.A. Meyer . The restriction of predictability is rather strong, but no restriction will be placed on the semimartingale itself. We follow systematically an idea due to L. Schwartz, which consists in neglecting locally constant processes in the random open set A. Then we can define pseudo-adapted processes in A, local pseudo-martingales, pseudo-semimartingales, etc. and prove for instance a decomposition result for pseudo-semimartingales, into a local pseudo-martingale part and a pseudo-adapted process of finite variation in A. This terminology is rather heavy, but if the reader is willing to make a slight effort of abstraction, and to deal with equivalence classes of processes in A modulo locally constant processes (these classes will be called pseudo-processes), then the prefix pseudo- will disappear from all other places, and all the definitions of adaptation, semimartingales, local martingales... for pseudo processes will appear as natural extensions of the classical ones.

The main application of this theory, and in fact the reason why it was developed by L. Schwartz, concerns processes in a differentiable manifold V : if X is a continuous process with values in V, and U is open, then $X^{-1}(U)$ is a predictable random open set. Therefore the usual localization procedures in V lead to localization on random open sets in $\mathbb{R}_+ \times \Omega$.

1. When the first version of this paper was written, we didn't know of this work by Meyer and Stricker, where the pseudo-semimartingales are introduced under the name of << semimartingale measures >>. This new version has been modified to take it into account. I thank P.A. Meyer for his help in preparing the final version.

§ 0 . NOTATION AND ELEMENTARY RESULTS

Throughout this paper, we work on a fixed probability space $(\Omega,\underline{\underline{F}},P)$ with a filtration $(\underline{\underline{F}}_t)$ which satisfies the usual conditions. We denote by $\underline{\underline{T}}$ ($\underline{\underline{PT}}$) the set of all $(\underline{\underline{F}}_t)$ stopping times (predictable times)

Let A be a random set, i.e. a measurable subset of $\mathbb{R}_+\times\Omega$. We say that A is a random open set if, for (almost) every ω, its section A(ω) is open in $\mathbb{R}_+$. This introduces a slight difficulty at 0 since $[0,a[$ and $]0,a[$ are both open sets in $\mathbb{R}_+$: rather than going into trivial discussions, we shall assume that A doesn't contain 0, leaving the easy extension to the reader. This will not prevent us from giving $\mathbb{R}_+\times\Omega$ as an example of a random open set !

In the whole paper, A will denote a random open set, predictable unless the contrary is specified.

A real valued process in A is a measurable mapping X from A to $\mathbb{R}$ (sometimes to the extended line $\overline{\mathbb{R}}$). As usual, we do not distinguish from A any set A', from X any process X', which are equal to A, X except on an evanescent set. A process X in A will be identified to XI_A , the process in $\mathbb{R}_+\times\Omega$ which is equal to X on A and to 0 on A^c.

The definition of a process X cadlag in A is obvious. We say that X is locally constant in A if (for a.e. ω) $X_\bullet(\omega)$ is constant in each one of the connected components of A(ω) - which are open intervals. We say that two processes X,Y are equivalent in A (notation : $X \sim_A Y$ or simply $X \sim Y$ if no confusion can arise) if X-Y is locally constant in A . An equivalence class of processes in A is called a pseudo-process.[1] For instance, if $A=\mathbb{R}_+\times\Omega$, pseudo-processes can be identified to processes X normalized to be 0 at the initial time, but already in $\mathbb{R}_+^*\times\Omega$ there is no natural way of << normalizing >> pseudo-processes.

For any $S\in\underline{\underline{T}}$, define

(0.1) $$T_S = \inf\{t\geq S \ , \ (t,\omega)\in A\}$$

If S belongs to $\underline{\underline{PT}}$, so does T_S since $A^c\cap[S,\infty]$ is predictable and closed. We have (Q denoting as usual the set of rationals)

(0.2) $$A = \cup_{r\in Q_+} [r,T_r[$$

We denote by $\underline{\underline{T}}(A)$ ($\underline{\underline{PT}}(A)$) the set of all $S\in\underline{\underline{T}}$ ($\underline{\underline{PT}}$) such that $[S]\subset A$. Note that if $S\in\underline{\underline{PT}}$, the set $\{S\in A\}=\{S=T_S\}$ belongs to $\underline{\underline{F}}_{S-}$, and $S_{\{S\in A\}}$ belongs to $\underline{\underline{PT}}(A)$. A pair of times (S,T) is called a neighbouring pair in A (or T is called a neighbour of S in A) if we have : $S\in\underline{\underline{T}}(A)$, $T\in\underline{\underline{T}}(A)$, $S\leq T$, $S<T<\infty$ on $\{S<\infty\}$, and if the closed stochastic interval $[S,T]$ is contained in A. We denote by $\underline{\underline{NP}}(A)$ the set of neighbouring pairs in A.

1. As pointed out in section 3, a convenient notation for the equivalence class of X is dX.

We have the following easy lemma :

(0.3) LEMMA. 1) For every $S\in\underline{\underline{PT}}(A)$, the stochastic interval $[S,T_S[$ is the union of an increasing sequence of stochastic intervals $[S,T_n]$, where each T_n is a neighbour of S in A .

2) There exists a sequence (S_n,T_n) of neighbouring pairs[1] such that $A=\cup_n[S_n,T_n] = \cup_n\ ]S_n,T_n[$.

Proof. On $\Omega'=\{S<\infty\}$ consider the filtration $(\underline{\underline{F}}'_t)=(\underline{\underline{F}}_{S+t})$. Then it is well known that $T'=T_S-S$ is an $(\underline{\underline{F}}'_t)$-predictable time, and since A is open it is a.s. >0 on Ω' . Consider a sequence $T'_n>0$ foretelling T' and set $T_n=S+T'_n$ on Ω', $T_n=\infty$ on $(\Omega')^c$. It is obvious that 1) is satisfied.

As for ii), we set $S_r=r_{\{r\in A\}}$ for rational r, and apply 1) to S_r to construct a sequence (S_r,T_{rn}) of neighbouring pairs << filling >> $[S_r,T_{S_r}[=[r,T_r[$. Then we reorder these pairs into a single sequence (S_n,T_n) and apply (0.2).

The following result allows the construction of pseudo-processes by << pasting together of local data >> . It doesn't require a filtration, so we state explicitly that A, A^n aren't assumed to be predictable.

(0.4) LEMMA. Assume A is the union of a sequence A^n of random open sets, and that, for every n , a process X^n is given in A^n, right continuous in A^n, and the following compatibility condition is satisfied

$$\text{for every } (m,n)\ ,\ X^m \sim_{A^m\cap A^n} X^n$$

Then there exists in A a right continuous process X, unique up to equivalence, such that $X\sim_{A^n} X^n$ for every n .

Proof. First be begin with the deterministic case of the lemma : A is an open set in $\mathbb{R}_+$ (not containing 0 for simplicity), union of a sequence of open sets A^n, and the X^n are ordinary right continuous functions. Then we enumerate all open components $]u_k,v_k[$ of A, and choose in each one a point w_k : for instance, $w_k=(u_k+v_k)/2$ for a bounded component, $w_k = u_k+1$ for the unbounded component if it exists.

Let $\underline{\underline{I}}(A)$ be the smallest family of subsets of A, closed under finite operations $\cup,\cap,\setminus$, which contains all bounded intervals $]u,v]$ with $[u,v]\subset A$ ($\underline{\underline{I}}(A)$ simply consists of finite disjoint unions of such intervals). Any function X in A determines uniquely a finitely additive measure $\mu=\mu_X$ on $\underline{\underline{I}}(A)$ such that $\mu_X(]u,v])=X(v)-X(u)$, and $X\sim_A 0$ iff $\mu_X=0$. Conversely, given a finitely additive measure μ on $\underline{\underline{I}}(A)$, we may construct a function $\hat{X}$ such that $\mu=\mu_{\hat{X}}$ in the following way : given $u\in A$, we consider the component $]u_k,v_k[$ containing u and set

1. With a little more care, one can choose S_n,T_n predictable.

$\hat{X}(u) = \mu(]w_k,u])$ if $u \geqq w_k$, $-\mu(]u,w_k])$ if $u<w_k$

If we start from X, go to μ and then to $\hat{X}$, it is clear that we get a process equivalent to X, normalized to vanish at each w_k . We say that μ is right continuous if $\hat{X}$ is right continuous in A.

Consider now two open sets A and A', two right continuous measures μ and μ' on $\underline{\underline{I}}(A)$, $\underline{\underline{I}}(A')$ respectively, which induce the same measure on $\underline{\underline{I}}(A \cap A')$. Then there is a unique measure λ on $\underline{\underline{I}}(A \cup A')$ which extends μ and μ', and λ is right continuous. We leave the trivial details to the reader, but indicate explicitly one point : let $[u,v]$ be an interval contained in $A \cup A'$, hence covered by the (open) components of A and A'. By compactness, there is a finite chain $u=t_0<\ldots<t_k<t_{k+1}=v$ with $t_1 \ldots t_k$ rational, such that each interval $[t_i,t_{i+1}]$ is contained in some component of either A or A' , and then $\lambda(]t_i,t_{i+1}])$ is unambiguously defined, due to the compatibility of μ and μ'. Since $]u,v]=\cup_i]t_i,t_{i+1}]$ we can deduce the value of $\lambda(]u,v])$.

Returning now to the statement of the lemma, we apply the procedure just described to construct by induction right continuous measures λ_n on $\underline{\underline{I}}(B_n)$, with $B_n=A_1 \cup A_2 \ldots \cup A_n$, such that λ_n induces μ_{X^k} on each $\underline{\underline{I}}(A^k)$ for $k=1,\ldots,n$. By compactness, every set $H \in \underline{\underline{I}}(A)$ belongs to some $\underline{\underline{I}}(B_n)$, and setting $\lambda(H)=\lambda_n(H)$ (which doesn't depend on n) we have the desired measure on $\underline{\underline{I}}(A)$, which is easily seen to be right continuous. From it we construct the desired function X as explained above.

Now we go to the general case. It is well known that the components of a measurable random open set A can be enumerated as stochastic intervals $]U_k,V_k[$, where the functions U_k,V_k are measurable (but they aren't stopping times in general, even if A is predictable). Let us say that $\omega \in \Omega$ is << good >> if all the a.s. properties are true at ω : $A(\omega)$ and $A^n(\omega)$ are open , $A(\omega)=\cup_n A^n(\omega)$, $X^n_\cdot(\omega)$ is right continuous in $A^n(\omega)$, the compatibility conditions are true... Since we work only up to evanescent sets, we may restrict ourselves to good ω's, and therefore the only condition we need to check is the following : if the components of A, A^n are measurably enumerated, the above deterministic constructions lead to a <u>measurable</u> process X in A . This is a little tedious, but easy, and we leave it to the reader.

§ 1 . LOCALIZATION OF PROPERTIES ON A

We are going first to give some generalities on ordinary processes in $\mathbb{R}_+ \times \Omega$. In this case, neighbouring pairs are simply pairs of stopping times (S,T) such that $S \leqq T$, $S<T<\infty$ on $S<\infty$. Given any process X, we denote by $X^{S,T}$ the process $(X^T_{t \vee S}-X_S)I_{\{S<\infty\}} = X^T-X^S$: formally, this is the stochastic integral $I_{]S,T]}\cdot X$ (we just demand here that $S\leq T$)

Consider now a property of stochastic processes, which we denote by $\mathcal{P}$. Let us say that $\mathcal{P}$ is localizable if the following statements are true.

1) If X has property $\mathcal{P}$, so does $X^{S,T}$ for any pair (S,T), $S\leq T$.
2) If (S,T) and (U,V) are pairs with $S\leq T\leq U\leq V$, and $X^{S,T}$, $X^{U,V}$ have property $\mathcal{P}$, and if $]S,T]\cup]U,V]=]S,W]$ is a stochastic interval, then $X^{S,W}$ has property $\mathcal{P}$.
3) If (S,T) and (U,V) are two pairs with $S\leq T$, $U\leq V$, S=U on $\{S<\infty ,U<\infty \}$, and if $X^{S,T}$, $X^{U,V}$ have property $\mathcal{P}$, then so does $X^{S\wedge U,T\vee V}$.

Examples of such properties are, of course, regularity properties of sample functions (continuity, cadlag property, finite variation...) ; many measurability properties (predictability, optionality, adaptation - understood in the strong sense, namely $\underline{\underline{F}}_T$-measurability at any stopping time T), and besides that the following properties, some of which obviously are not at all << local >> .

X is a martingale, a square integrable martingale, a super or submartingale, a process with integrable variation, a class (D) supermartingale, a local martingale, a semimartingale...

We say that $\mathcal{P}$ is local if it is localizable, and satisfies

4) If there exists a sequence $T_n\uparrow\infty$ of stopping times, such that X^{0,T_n} has property $\mathcal{P}$, then X has property $\mathcal{P}$.

Every localizable property $\mathcal{P}$ has a localization $\mathcal{P}_{loc}$, defined as follows : X has property $\mathcal{P}_{loc}$ iff there exists a sequence $T_n\uparrow\infty$ such that X^{0,T_n} has property $\mathcal{P}$. We leave it to the reader to check that $\mathcal{P}_{loc}$ is a local property. For instance, the localization of << X is a martingale >> is << $X-X_0$ is a local martingale >> . This is a rather unusual fact, since the constant process X_0 may not be adapted, but it is consistent with our point of view of neglecting altogether (locally) constant processes.

We now set a general principle : A being as always a predictable random open set, X a process in A, $\tilde{X}$ the equivalence class (pseudo-process) of X, we define :

(1.1.). DEFINITION. Let $\mathcal{P}$ be a local property. We say that X has the property pseudo-$\mathcal{P}$ in A (or that $\tilde{X}$ has property $\mathcal{P}$ in A) iff, for every neighbouring pair (S,T) in A, the ordinary process $X^{S,T}$ has property $\mathcal{P}$.

If X has the pseudo-$\mathcal{P}$ property in A, the same is true in any smaller open set. In particular, if X is the restriction to A of a process which has property $\mathcal{P}$ in $\mathbb{R}_+$, then it has property pseudo-$\mathcal{P}$ in A .

The following theorem shows that the word << local >> is used here in a sense consistent with its use in topology :

(1.2). THEOREM. Let ρ be a local property. Let A be the union of a sequence A^n of predictable random open sets. If X has the property pseudo-ρ in each A^n, then it has property pseudo-ρ in A .

<u>Proof</u>. We consider a neighbouring pair (S,T) in A ; we want to show that the ordinary process $X^{S,T}$ has property ρ .

Let $\aleph$ be the set of all stochastic intervals $]U,V[$, corresponding to neighbouring pairs (U,V) in A such that $[U]\subset[S]$, and $X^{U,V}$ has property ρ . According to property 3) of localizable properties, $\aleph$ is closed under finite unions. Therefore, there exists an increasing sequence $]U_n,V_n[$ of elements of $\aleph$, with union $]\bar{U},\bar{V}[$, such that $]\bar{U},\bar{V}[$ contains every element of $\aleph$ up to an evanescent set.

If we can prove that $\bar{U}=S$, $\bar{V}=T_S$ (see (0.1)) , then the theorem is true. Indeed, first fix m and for $n\geq m$ set $R_n=+\infty$ on $\{U_m=\infty\}$, $R_n=+\infty$ on $\{V_n>T\}$, $R_n=V_n$ on $\{U_m<\infty, V_n\leq T\}$. Then $R_n\uparrow+\infty$. On the other hand, set $T_m=T$ on $\{U_m<\infty\}$, $T_m=+\infty$ otherwise. We know that X^{U_n,V_n} has property ρ , hence the same is true for $(X^{U_n,V_n})^{U_m,R_n} = (X^{U_m,T_m})^{0,R_n}$ (statement 1) for localizable properties). Since ρ is local, we may apply statement 4), to deduce that X^{U_m,T_m} has property ρ . Setting $W_m=S$ if $S<\infty$,$U_m=\infty$, $W_m=\infty$ otherwise, this means that $(X^{S,T})^{0,W_m}$ has property ρ. Applying again statement 4), we get that $X^{S,T}$ has property ρ .

So let us prove that $\bar{U}=S$, $\bar{V}=T_S$. First of all, let S_m be the time $S_{\{S\in A^m\}}$, and consider a neighbouring pair (S_m,T_m) in A^m (lemma (0.3)). Since X has property pseudo-ρ in A^m, X^{S_m,T_m} has property ρ, hence $]S_m,T_m[$ belongs to $\aleph$, and therefore is contained in $]\bar{U},\bar{V}[$ up to an evanescent set. Hence $\{\bar{U}=S\}$ contains a.s. $\{S_m=S\}$, and finally $\bar{U}=S$.

Next, assume that $\bar{V}<T_S$ with positive probability. Then for some m $\{\bar{V}\in A^m\}$ has positive probability, and according to lemma (0.3), 2), we may find some neighbouring pair (J,K) in A^m such that $P\{J<\bar{V}<K\}>0$. Then for some n we would also have $P\{J<V_n<K\}>0$. Since $X^{J,K}$ has property ρ, so does $X^{L,M}$, where $L=V_n$, $M=K$ if $\{J<V_n<K\}$, $L=M=\infty$ otherwise. Applying property 2) for the first time, we get that X^{U_n,W_n} has property ρ, with $W_n=V_n$ if $L=\infty$, $W_n=M$ if $L<\infty$. Therefore $]U_n,W_n[\in\aleph$, which is absurd since K exceeds $\bar{V}$ with strictly positive probability.

(1.3). COROLLARY. Let A be the union of a sequence A^n of predictable open sets. In each A^n, let X^n be a process which has property pseudo-ρ. If the compatibility conditions of lemma (0.4) are satisfied, there exists in A a process X such that $X \underset{A^n}{\sim} X^n$ for every n , it is unique up to equivalence and possesses the pseudo-ρ property in A.

This amounts simply to a restatement of lemma (0.4) and theorem (1.2) together.

§ 2. PSEUDO-SEMIMARTINGALES AND DECOMPOSITIONS

Corollary (1.3) lends itself to the proof of a number of results. The following ones are examples, and it will be sufficient to prove one of them, the proofs being quite similar.

(2.1). THEOREM. Let X be a process of pseudo-locally integrable variation in A. There exists a pseudo-predictable process Y, of pseudo-finite variation in A, such that X-Y is a pseudo-local martingale in A, and Y is unique up to equivalence. Y is called the <u>compensator</u> of X in A.
REMARK. As usual, locally integrable variation includes adaptation. In expressions like << pseudo-local martingale >>, << pseudo-locally integrable >>, the word << local, locally >> could be suppressed without confusion, since pseudo-P must only refer to a <u>local</u> property. However, we prefer not to do it here.

<u>Proof</u>. Represent A as the union of a sequence of open sets $A^n =]S_n, T_n[$, where (S_n, T_n) are neighbouring pairs in A (lemma (0.3)). The process $X^n = X^{S_n, T_n}$ is a process of locally integrable variation in the usual sense, and therefore has a compensator Y^n. The intersection $A^n \cap A^m$ is an open stochastic interval, and the usual uniqueness of compensators implies that $Y^n - Y^m$ is constant in $A^n \cap A^m$. Then we apply corollary (1.3).

(2.2). THEOREM. Let X be a pseudo-supermartingale in A. Then X has a representation (unique up to equivalence) as a difference of a pseudo-local martingale in A and a pseudo-increasing, pseudo-predictable process in A.

(2.3). THEOREM. Let X be a pseudo-special semimartingale in A. Then X has a decomposition (unique up to equivalence) into a sum of a pseudo-local martingale in A and a pseudo-predictable process of pseudo-finite variation in A. This is called the <u>canonical decomposition</u> of X in A.

(2.4). THEOREM. Let X be a pseudo-semimartingale in A. Then X can be represented (without uniqueness) as a sum of a pseudo-local martingale in A and a process of pseudo-finite variation in A.
<u>Proof</u>. This theorem reduces to (2.3) after a first application of theorem (1.3), which consists in defining a process which, up to a locally constant process, represents the sum of the jumps of X whose size exceeds 1 in absolute value.

Other applications of theorem (1.3) concern the definition of $[X,X]$ for a pseudo-semimartingale, and then of $\langle X,X \rangle$ if compensation is legitimate (2.1). Similarly, pseudo-local martingales can be decomposed into their continuous and purely discontinuous parts, etc.

§ 3. THE MEASURE ASSOCIATED WITH A PSEUDO-SEMIMARTINGALE IN A

This section should be considered expository, since we had few results on this subject, and the principal result is borrowed from [7]. However, the reference [7] may not be easily available to the readers of this seminar, and therefore it may be useful to give a somewhat complete description of the subject.

Let $\underline{\underline{I}}(A)$ be the smallest family of subsets of A, closed under finitely many operations $\cup,\cap,\setminus$, such that for any neighbouring pair (S,T) in A the stochastic interval $]S,T]$ belongs to $\underline{\underline{I}}(A)$. It is easily seen that any element of $\underline{\underline{I}}(A)$ can be explicitly written as a disjoint union $]S_1,T_1]\cup\ldots]S_n,T_n]$, with n non random, S_i,T_i being stopping times such that $S_i\leq T_i\leq S_{i+1}$, $S_i<T_i$ on $\{S_i<\infty\}$, $T_i<S_{i+1}$ on $\{T_i<\infty\}$. Given any right continuous process X in A, we associate with X the only finitely additive mapping μ from $\underline{\underline{I}}(A)$ to random variables such that

$$\mu(]S,T]) = (X_T-X_S)I_{\{S<\infty\}} \text{ for any neighbouring pair } (S,T)$$

The knowledge of μ doesn't determine uniquely X, but it characterizes the corresponding pseudo-process, as may be easily seen by looking at neighbouring pairs $(s_{\{s\in A\}},t_{\{s\in A\}})$ with s,t rationals. We already used this << measure >> μ in the deterministic case, in the proof of lemma (0.4). In this section, we shall use for μ the stochastic integral notation $\mu=dX$, $\mu(B)=\int_B dX$ for $B\in\underline{\underline{I}}(A)$.

The main result of [7] on pseudo-semimartingales in A can now be expressed as follows (theorem 8, p. 591).

(3.1). THEOREM. Let X be a pseudo-semimartingale in A. There exist a true semimartingale Y in $\mathbb{R}_+$ and a predictable process H in $\mathbb{R}_+$, strictly positive in A, such that :

for any neighbouring pair (S,T), the stochastic integral (of a predictable, not locally bounded process) $HI_{]S,T]}\cdot Y$ is defined in the usual sense, and equal to $X^{S,T}$.

We shall write simply : $dX = H.dY$.

A version of this theorem is also proved for optional random open sets, which we didn't consider here.

We are not going to prove this in detail, but we use the same proof to get a result on processes of finite variation. The proof for semimartingales just uses the Banach space $\underline{\underline{H}}^1$ instead of $\underline{\underline{V}}^1$. Note that we prove for H something a little better than strict positivity in A .

(3.2). THEOREM. Let X be a process of pseudo-finite variation in A. There exist a true process of finite variation Y in $\mathbb{R}_+$, a predictable process H in $\mathbb{R}_+$, strictly positive in A, such that $dX=H.dY$ in A. If X is predictable, Y can be chosen predictable. If X is pseudo-locally integrable, Y can be chosen integrable.

Proof. According to lemma (0.3), we may consider a sequence of neighbouring pairs (S_n,T_n) such that $A=\cup_n]S_n,T_n[$. It is easily seen that, if X is pseudo-locally integrable, we may choose these pairs such that X^{S_n,T_n} has integrable variation.

Set $X^n = X^{S_n,T_n}$, $V^n= \int_{]S_n,T_n]} |dX_s|$. Since these random variables are finite valued, and there are at most countably many of them, there exists a law Q equivalent to P such that they are all Q-integrable. Otherwise stated, X^n belongs to the space $\underline{\underline{V}}^1(Q)$ of processes of integrable variation relative to Q. Choose a sequence of constants $a_n>0$ such that $\Sigma_n a_n\|X^n\|_{\underline{\underline{V}}^1(Q)} < \infty$. Set $C_1=]S_1,T_1]$, $C_n=]S_n,T_n]\setminus \bigcup_{k<n}]S_k,T_k]$ and define processes Y, K by

$$Y_t = \Sigma_n \int_0^t a_n I_{C_n}(s)dX^n_s \quad , \quad H = \Sigma_n a_n^{-1} I_{C_n}$$

It is clear that Y belongs to $\underline{\underline{V}}^1(Q)$ (predictable if the X^n are predictable), that H is predictable and >0 in A - it can even be said that for a.e. ω, $H_\bullet(\omega)$ is bounded below on every compact set of $A(\omega)$ - and that $H.dY = dX$ in each one of the C_n, hence in A.

If X is pseudo-locally integrable, the change of law isn't necessary, and Y belongs to $\underline{\underline{V}}^1(P)$.

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