

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

MARTIN T. BARLOW

**$L(B_t, t)$  is not a semimartingale**

*Séminaire de probabilités (Strasbourg)*, tome 16 (1982), p. 209-211

<[http://www.numdam.org/item?id=SPS\\_1982\\_\\_16\\_\\_209\\_0](http://www.numdam.org/item?id=SPS_1982__16__209_0)>

© Springer-Verlag, Berlin Heidelberg New York, 1982, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

$L(B_t, t)$  is not a semimartingale

by

M.T Barlow

Let  $B$  be a one-dimensional Brownian motion, with  $B_0 = 0$ , and let  $L(a, t)$ ,  $a \in \mathbf{R}$ ,  $t \geq 0$  be a continuous version of its local time. We shall show that the process  $Y$ , defined by  $Y_t = L(B_t, t)$ , is not a semimartingale. The essence of the proof is the remark that whereas the paths of a continuous semimartingale satisfy a Holder condition of order  $\frac{1}{2} - \varepsilon$  almost everywhere, for any  $\varepsilon > 0$ , the paths of  $Y$  just fail to satisfy a Holder condition of order  $\frac{1}{4}$ .

For a process or function  $X$  set

$$D^\alpha(X) = \{t \geq 0 : \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/\alpha} |X_{t+\varepsilon} - X_t| > 0\} .$$

LEMMA Let  $\alpha > 1$ , and  $f : \mathbf{R}_+ \rightarrow \mathbf{R}$  be a function such that

$D^\alpha(f) = \emptyset$ . Let  $\tau(t)$  be an increasing function, and  $g(t) = f(\tau(t))$ .

Then  $|D^\alpha(g)| = 0$ .

Proof By Lebesgue's density theorem,  $\tau'(t)$  exists and is finite almost everywhere. For such a  $t$

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/\alpha} |g(t+\varepsilon) - g(t)| \\ &= \lim_{\delta \rightarrow 0} (\tau'(t))^{1/\alpha} \delta^{-1/\alpha} |f(\tau(t) + \delta) - f(\tau(t))| \\ &= 0 , \end{aligned}$$

so that  $t \notin D^\alpha(g)$ .

**PROPOSITION** Let  $X$  be a continuous semimartingale. Then for  $\alpha > 2$ ,  $|D^\alpha(X)| = 0$  a.s.

**Proof** Let  $X = M + A^+ - A^-$  be the decomposition of  $X$  into the sum of a martingale and the difference of two increasing processes. It is plain that  $D^\alpha(X) \subset D^\alpha(M) \cup D^\alpha(A^+) \cup D^\alpha(A^-)$ . By the lemma, setting  $f(t) = t$  and  $\tau(t) = A_t^+$  or  $A_t^-$ , we have  $|D^\alpha(A^+)| = |D^\alpha(A^-)| = 0$ .

Now let  $\tau_t$  be the right-continuous inverse of  $\langle M \rangle$ , and  $U_t = M_{\tau_t}$ . Then  $U$  is a Brownian motion, and  $M_t = U_{\langle M \rangle_t}$ . By Lévy's Hölder condition on the variation of Brownian paths, for  $\alpha > 2$ ,  $D^\alpha(U) = \phi$  a.s., and thus, by the lemma,  $|D^\alpha(M)| = 0$  a.s.

**THEOREM (i)** For each  $t > 0$ ,  $B_t \in D^2(L(\cdot, t))$  a.s.

(ii)  $D^4(Y)$  is of full Lebesgue measure a.s.

(iii)  $Y$  is not a semimartingale.

**Proof** From the results of Ray [1] on Brownian local time,

$0 \in D^2(L(\cdot, t))$  a.s. Let  $t$  be fixed, and  $\tilde{B}_s = B_t - B_{t-s}$  for  $0 \leq s \leq t$ . Then  $\tilde{B}$  is a Brownian motion, and if  $\tilde{L}$  denotes its local time,  $\tilde{L}(a, t) = L(B_t - a, t)$ , so that  $B_t \in D^2(L(\cdot, t))$  whenever  $0 \in D^2(\tilde{L}(\cdot, t))$ , establishing (i).

We may restate (i) as follows: there exist  $\mathcal{B}_t$ -measurable random variables  $A_n$  and  $C$  with  $|A_n - B_t| < 1/n$ , and  $C > 0$  a.s., such that

$$|L(A_n, t) - L(B_t, t)| \geq |A_n - B_t|^{\frac{1}{2}} \cdot C \text{ for all } n.$$

If  $(a_n)$  is a sequence converging to 0, and  $T_n = \inf\{t \geq 0; B_t = a_n\}$ , then  $P(T_n < a_n^2) = k > 0$ , for some

constant  $k$ . Thus  $P(T_n < a_n^2 \text{ for infinitely many } n) = 1$  by the Borel-Cantelli lemmas, and the Blumenthal 01 law.

Now let  $S_n = \inf\{u > t: B_u = A_n\}$ . By the preceding argument, and the Markov property of  $B$  at  $t$ ,

$$S_n - t < (A_n - B_t)^2 \text{ for infinitely many } n, \text{ a.s.}$$

Thus

$$\begin{aligned} \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |Y_{S_n} - Y_t| \\ &= \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |L(A_n, t) - L(B_t, t)| \\ &\geq \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |A_n - B_t|^{\frac{1}{2}} C \\ &\geq C \qquad \qquad \qquad \text{a.s.} \\ &> 0 \qquad \qquad \qquad \text{a.s.} \end{aligned}$$

Therefore  $t \in D^2(Y)$  a.s., and (ii) follows by a Fubini argument.

(iii) is an immediate consequence of (ii) and the proposition.

#### Reference

1. D.B. Ray : Sojourn times of a diffusion process. Illinois J. Math. 7; 615-630. (1963).

Statistical Laboratory,  
16 Mill Lane,  
Cambridge, CB2 1SB  
England.