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SOME REMARKS ON PROCESSES
WITH INDEPENDENT INCREMENTS

by WANG Jia-Gang

Let (Ω, \mathcal{F}, P) be a complete probability space. In this note we consider processes $X = (X_t, t \in \mathbb{R}_+)$ with (non homogeneous) independent increments, which have no fixed discontinuities :

- i. $P\{X_0=0\}=1$
- ii. For $0 \leq t_1 < t_2 \dots < t_n$, $X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}$, X_{t_1} are independent random variables
- iii. Every sample function of X is right continuous with left hand limits, and $P\{\Delta X_t \neq 0\}=0$ for every $t \in \mathbb{R}_+$.

\mathcal{F}_t^o is the σ -field $\sigma(X_s, s \leq t)$, and \mathcal{F}_t the σ -field generated by \mathcal{F}_t^o and all sets of measure 0 in \mathcal{F} .

1. In this section we are going to prove that the filtration (\mathcal{F}_t) satisfies the usual conditions and is quasi-left continuous, a fact which is essentially well-known, but difficult to find in the literature in the non homogeneous case. We also give some auxiliary results.

We need some notations concerning martingales. First of all, we denote by $M_t(u, r, s)$ the following martingale, for $u \in \mathbb{R}$, $r \leq s$

$$(1) \quad M_t(u, r, s) = E[e^{iu(X_s - X_r)} | \mathcal{F}_t^o] = \begin{cases} e^{iu(X_s - X_r)} & \text{if } t \leq s \\ e^{iu(X_t - X_r)} \varphi_{t,s}(u) & \text{if } r \leq t < s \\ \varphi_{r,s}(u) & \text{if } t \geq r \end{cases}$$

where $\varphi_{a,b}$ is the characteristic function of $X_b - X_a$ ($a \leq b$). This process is right continuous with left-hand limits, and jumps only at jump times of X . Next, consider the set H of all random variables

$$(2) \quad Z = e^{i(u_1 X_{t_1} + u_2 (X_{t_2} - X_{t_1}) + \dots + u_n (X_{t_n} - X_{t_{n-1}}))}$$

with $u_1, \dots, u_n \in \mathbb{R}$, $0 \leq t_1 < \dots < t_n$. Then the linear span of H is dense in L^1 and we have

$$(3) \quad Z_t = E[Z | \mathcal{F}_t^o] = M_t(u_1, 0, t_1) \dots M_t(u_n, t_{n-1}, t_n)$$

and therefore Z_t has the same continuity properties as M_t above.

PROPOSITION 1^[1]. For every $t \in \mathbb{R}_+$ we have $\mathcal{F}_t = \mathcal{F}_{t+}$ ($= \mathcal{F}_{t-}$ if $t > 0$).

Proof. Since Z_t is bounded and right continuous, $E[Z | \mathcal{F}_{t+}^o] = Z_{t+} = Z_t = E[Z | \mathcal{F}_t^o]$ a.s. This extends to all random variables in L^1 . In particular, any r.v.

in $L^1(\underline{F}_{\leq t+}^o) = L^1(\underline{F}_{\leq t+})$ is a.s. equal to a r.v. in $L^1(\underline{F}_{\leq t}^o)$. Hence $\underline{F}_{\leq t} = \underline{F}_{\leq t+}$.

The reasoning is the same on the left side.

PROPOSITION 2. If T is a stopping time of (\underline{F}_t) , then

$$\underline{F}_{T-} = \sigma(T, X_{T-}^T, \underline{N})$$

where X_{T-} is X stopped at $T-$, and \underline{N} is the class of all negligible sets.

Proof. This result is true for any process X which is continuous with left hand limits.

It is obvious that the σ -field \underline{K} on the right is contained in \underline{F}_{T-} . To prove the reverse inclusion it suffices to show that \underline{K} contains $A \cap \{t < T\}$ for $t \in \mathbb{R}_+$, $A \in \underline{F}_t^o = \sigma(X_s, s \leq t)$. Hence it suffices to show that, for $s \leq t$, any function $f(X_s) \mathbf{1}_{\{t < T\}}$ is \underline{K} -measurable. This is true since $X_s = X_{s-}^{T-}$ on $\{t < T\}$.

PROPOSITION 3. If T is a stopping time of (\underline{F}_t) , then

$$\underline{F}_T = \sigma(T, X_T^T, \underline{N})$$

Proof. It is obvious that the σ -field \underline{L} on the right is contained in \underline{F}_T .

To prove the reverse inclusion it suffices to show that for any random variable $Z \in L^1$, $E[Z | \underline{F}_T]$ is \underline{L} -measurable, and it suffices to prove it for the random variables (2). Because of (3) it suffices to prove that $M_T(u, r, s)$ is \underline{L} -measurable, which is obvious.

COROLLARY. If T is a.s. finite, $\underline{F}_T = \sigma(\underline{F}_{T-}, X_T)$.

THEOREM 1. The filtration (\underline{F}_t) is quasi-left continuous.

Proof. It is well known ([2], chap. 3) that all jump times of X are totally inaccessible. Therefore at any bounded predictable time T we have $X_T = X_{T-}$ a.s., and from the corollary above we have $\underline{F}_T = \underline{F}_{T-}$.

2. Assume now that X has only finitely many jumps in every finite interval, and is constant between jumps. Then we may consider X (or rather its jump process) as a multivariate point process, and it is natural to ask which kind of conditions on X , considered as a multivariate point process with values in \mathbb{R} , express that X has independent increments, with Lévy measure $v(dt, dx)$.

Let us introduce some notation : we denote by $T_1(\omega), T_2(\omega) \dots$ the successive jump times (if $T_m(\omega)$ is the last finite jump, we set $T_n(\omega) = +\infty$ for $n > m$) ; $\Delta_1(\omega), \Delta_2(\omega) \dots$ are the successive jump sizes (if $T_n(\omega) = +\infty$, we make the convention that $\Delta_n(\omega) = 0$). We denote by $n_t(\omega)$ the total number of jumps of $X(\omega)$ on $[0, t]$. Given the Lévy measure v , we set $v([0, t] \times \mathbb{R}) = \Lambda(t)$, a non decreasing, continuous function with $\Lambda(0) = 0$. Using the existence of regular conditional distributions, we may deduce that there exists a

transition probability $N(t, dx)$ from $]0, \infty[$ to \mathbb{R} such that $N(t, \{0\})=0$ and we have for any Borel set A

$$(4) \quad v([0, t] \times A) = \int_{[0, t]} N(u, A) d\Lambda(u).$$

We extend this definition to $t=+\infty$, setting then $N(+\infty, .) = \varepsilon_0$. We are going to prove (assuming always X is a multivariate point process).

THEOREM 2. If X is a process with independent increments, then it satisfies the following two properties,

- A) (n_t) is a Poisson process relative to (F_t^0) , with expectation $E[n_t] = \Lambda(t)$.
- B) Conditional to the event $T_1=t_1, T_2=t_2, \dots, T_n=t_n$ (i.e., conditional to the sample path of the process (n_t)), the random variables $\Delta_1, \Delta_2, \dots$ are independent, the law of Δ_n being $N(t_n, .)$.

Conversely, if X satisfies the slightly weaker properties

- A') (n_t) is a Poisson process (w.r. to its natural filtration) and $E[n_t] = \Lambda(t)$.

$$B') P\{\Delta_{n+1} e dx | T_1, \Delta_1, \dots, T_n, \Delta_n, T_{n+1}\} = N(T_{n+1}, dx)$$

Then X has independent increments, and its Lévy measure is given by (4).

REMARK. Like all statements concerning conditional distributions, it must be understood that property B) holds for almost every path of (n_t) . In particular, different choices of $N(u, .)$ in (4) will differ only for a set of values of u which has $d\Lambda$ -measure 0, and the conditional distributions will be the same for almost every path, but not for every path.

Proof. We first show that A') and B') imply X has independent increments. We begin by assuming that $\Lambda(t)=t$. Then (n_t) is a homogeneous Poisson process with parameter 1, so all differences $T_{n+1}-T_n$ are independent exponential random variables. We then can compute

$$\begin{aligned} G_n(dt, dx) &= P\{T_{n+1} \in dt, \Delta_{n+1} \in dx | T_1, \Delta_1, \dots, T_n, \Delta_n\} \\ &= 1_{\{t > T_n\}} e^{-(t-T_n)} N(t, dx) dt \end{aligned}$$

$$\text{and also } H_n(]t, \infty[) = P\{T_{n+1} > t | T_1, \Delta_1, \dots, T_n, \Delta_n\} = e^{-(t-T_n)} \text{ for } t > T_n$$

According to [2], p. 86, prop. 3.41, we can compute the predictable compensating measure of the process X as

$$v(dt, dx) = \sum_{n=0}^{\infty} \frac{G_n(dt, dx)}{H_n(]t, \infty[)} 1_{\{T_n < t \leq T_{n+1}\}}$$

According to the above computations, this is a deterministic measure, which implies ([2], p. 91, theorem 3.51) that X has independent increments. The Lévy measure of this process is $v(dt, dx) = N(t, dx)dt$.

The case of arbitrary $\Lambda(t)$ reduces to the preceding one by a deterministic change of time. If Λ is unbounded the reduction is trivial, while if $\Lambda(\infty)=a$, we are reduced to a homogeneous Poisson process on the finite interval $[0,a[$. But then we may extend X by an independent Poisson process on $[a,\infty[$, and apply the preceding reasoning. So finally A') and B') are sufficient conditions.

Conversely, we want to show that the process X with independent increments and Lévy measure ν satisfies A) and B). Property A) is well known. On the other hand, it is simple to construct a multivariate point process \bar{X} such that 1) the corresponding process (\bar{n}_t) is Poisson with expectation $E[\bar{n}_t]=\Lambda(t)$ and 2) the conditional law of the jump sizes given (\bar{n}_t) is given by B). Since this process satisfies A') and B), it also satisfies A') and B'), hence from the first part of the proof it is a process with independent increments and Lévy measure ν , and finally it has the same law as X . This implies that, inversely, X has the same conditional law given (n_t) as \bar{X} given (\bar{n}_t) . That is, X satisfies B).

3. We have used martingale theory to prove theorem 2, but we may interpret it in a way which doesn't use the order structure of the time set.

Let us denote by (E, \underline{E}) the measurable space $(\mathbb{R}_+, \underline{B}(\mathbb{R}_+))$, and by (F, \underline{F}) the space $(\mathbb{R} \setminus \{0\}, \underline{B}(\mathbb{R} \setminus \{0\}))$. We are given a measure $d\Lambda$ on (E, \underline{E}) , which ascribes finite mass to the sets $A_n = [n, n+1[$ whose union in E , and we construct the Poisson random measure $d\eta_t$ with expectation measure $d\Lambda$.

On the other hand, we are given a transition probability $N(t, du)$ from E to F , and construct a new random measure on $E \times F$ as follows : for each sample $\eta = \sum_n \varepsilon_{t_n}$ of the measure $d\eta$, we choose independently elements Δ_n in F , each one according to the law $N(t_n, \cdot)$, and set $\xi = \sum_n \varepsilon_{t_n, \Delta_n}$.

Then theorem 2 asserts that ξ is again a Poisson random measure, with expectation measure $\nu(dt, dx) = N(t, dx)\Lambda(dt)$.

Of course, we have proved it only for particular spaces, but on the other hand, if we start from the above description, assuming just that (E, \underline{E}) and (F, \underline{F}) are Lusin measurable spaces, we may identify A_n to a Borel subset of $[n, n+1[$ ([4], chapter III, 20), F to a Borel subset of $\mathbb{R} \setminus \{0\}$, and it is very easy to see that theorem 2 for \mathbb{R}_+ , $\mathbb{R} \setminus \{0\}$ will imply that the above statement applies to E and F .

This statement is well known in the theory of Poisson random measures, the oldest version being probably that of Doob's book ([3], chapter VIII, § 5, p. 404-406), and in this form it is true for general measurable spaces, without the Lusin restriction. But still it is interesting to see that a theorem like theorem 2 which seems very special contains fairly general "abstract" results.

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