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SOME REMARKABLE MARTINGALES

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0. INTRODUCTION :

In the second part (sections 5)-9)) of our previous paper [6], we discussed certain measurability problems which arise in the study of continuous martingales. In particular, we addressed the problem of determining when a continuous martingale is "pure" in the sense of Dubins and Schwarz [1]. That is, given a continuous martingale $M(\cdot)$ with $M(0) = 0$, one knows that $M(t) = B \circ \langle M,M\rangle_t$ where $B(\cdot)$ is a Brownian motion and $\langle M,M\rangle_\bullet$ is the increasing process in the Doob-Meyer decomposition of $M^2(\cdot)$. Assuming (as we do throughout) that $\langle M,M\rangle_\infty = \infty$, it is easily seen that $B(\cdot)$ is $M(\cdot)$-measurable. However, it is not true in general that $M(\cdot)$ is $B(\cdot)$-measurable.

In fact, if $M(\cdot)$ is $B(\cdot)$-measurable, then $M(\cdot)$ enjoys various special properties, of which the most interesting is that every $M(\cdot)$-adapted martingale admits a representation as a $dM(t)$-stochastic integral (cf. section 5) of [6]). Thus there is good reason for wanting to investigate when $M(\cdot)$ is $B(\cdot)$-measurable, and it is for this reason that Dubins and Schwarz assigned this property a name. The adjective which they chose is "pure".

The aim of our earlier work on this subject was to provide some insight into the property of "purity" and to relate it to questions about stochastic differential equations and martingale problems. Thus, for example, we pointed out that although a pure martingale is always extremal (cf. [1] or section (5) of [6]), a plentiful source of extremal martingales which are not pure comes from strictly weak (i.e. not strong) solutions to stochastic differential equations for which the associated martingale problem is well-posed (cf. *Theorem (6.2)* in [6]). Unfortunately, our results in [6] were far from being definitive and we are sorry to admit that even now this situation has not changed as much as we had hoped it might. Nonetheless, we present in sections 1) and 2) a few criteria which guarantee the purity of certain Brownian stochastic integrals.

In section 3) we take up a slightly different question about measurability relations between martingales which are intimately connected with one another. Here we look at a complex Brownian motion $\mathcal{Z}(t) = X(t) + iY(t)$ starting at $z_0 \in \mathbb{C}$ and the associated "Lévy area" process

$$\mathcal{A}(t) = \int_0^t (X(s)\, dY(s) - Y(s)\, dX(s)).$$

Obviously $\mathcal{A}(\cdot)$ is $\mathcal{Z}(\cdot)$-measurable. However, we show (cf. *Theorem (3.4)*) that $\mathcal{Z}(\cdot)$ is $\mathcal{A}(\cdot)$-measurable if and only if $z_0 \neq 0$. As we will see, what causes problem when $z_0 = 0$ is the impossibility of defining the phase of $\mathcal{Z}(t)$ as $t \downarrow 0$. We have included this example in the present paper not because we consider it to be closely related to the question of purity but because we believe that it provides another good example of the same sort of measurability questions coming from "naturally" connected martingales.

It remains our belief that there exist both a general formulation of such problems and a general method of attacking them. As yet, we are sorry to report that we

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ourselves have discovered neither.

1. PURITY AND CERTAIN STOCHASTIC INTEGRALS :

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space satisfying the usual completeness and continuity assumptions. Suppose that $\beta(\cdot)$ is an $(\mathcal{F}_\cdot)$-Brownian motion and let $X(\cdot)$ be an $(\mathcal{F}_\cdot)$-adapted solution to

$$(1.1) \qquad X(t) = x_0 + \int_0^t \sigma(X(s))d\beta(s) + \int_0^t b(X(s))ds, \ t \geq 0,$$

where σ and b are locally bounded measurable functions on R into itself. The main goal of this section is to prove the theorem whose statement we now give.

Theorem (1.2) : Let $\phi : R \to R$ be a measurable function and set

$$(1.3) \qquad M(t) = \int_0^t \phi(X(s))\, \sigma(X(s))d\beta(s), \quad t \geq 0.$$

If the following conditions hold :

i) $\phi(\cdot)$ and $\sigma^2(\cdot)$ are uniformly positive,

ii) $b(\cdot)$ is uniformly bounded,

iii) $\phi(\cdot)$ is a function of local bounded variation such that there is a bounded measurable function $f(\cdot)$ and a function $\xi(\cdot)$ of bounded variation for which $\phi(dx) = \phi^2(x)\, f(x)dx + \phi(x+)\, \xi(dx)$,

then $M(\cdot)$ is pure.

The proof of *Theorem (1.2)* will be accomplished in several steps. The first few of these steps relate the purity of $M(\cdot)$ to showing that all solutions of certain singular looking stochastic differential equations are strong solutions. To be precise, set $F(x) = \int_{x_0}^x \phi(y)dy$. Then by a generalization of Tanaka's variation on Itô's formula :

$$(1.4) \qquad F(X(t)) = M(t) + \int_0^t b(X(s))\, \phi(X(s))ds + {}^1/_2 \int L_t^a\, \phi(da),$$

where $(L_t^a)_{t>0}$ is the local time of $X(\cdot)$ at a as defined by Meyer in [2] via Tanaka's formula. Hence, if $\tau(\cdot)$ is the inverse of $\langle M,M\rangle_\cdot$, then

$$F(X(\tau(t))) = B(t) + \int_0^{\tau(t)} b(X(s))\, \phi(X(s))ds + {}^1/_2 \int L_{\tau(t)}^a\, \phi(da),$$

where $B(\cdot)$ is the Brownian motion appearing in the representation $M(t) = B \circ \langle M,M\rangle_t$. But

$$\langle M,M\rangle_t = \int_0^t \phi^2(X(s))\, \sigma^2(X(s))ds$$

and so

$$\tau(t) = \int_0^t 1\Big/\phi^2(X(\tau(s)))\, \sigma^2(X(\tau(s)))\, ds.$$

Thus

$$\int_0^{\tau(t)} b(X(s))\, \phi(X(s))ds = \int_0^t \frac{b}{\sigma^2\phi}(X(\tau(s))ds.$$

Setting $Y(t) = F(X(\tau(t)))$, we now have :

$$Y(t) = B(t) + \int_0^t \frac{b}{\sigma^2\phi} \circ F^{-1}(Y(s))ds + {}^1/_2 \int L^a_{\tau(t)}\ \phi(da).$$

Finally, if $(\mathcal{L}^b_t)_{t\geq 0}$ is the local time at b of $Y(\cdot)$, then by the "density of occupation formula" :

$$L^a_{\tau(t)} = \frac{1}{\phi(a+)} \mathcal{L}^{F(a)}_t .$$

Hence,

$$Y(t) = B(t) + \int_0^t \frac{b}{\sigma^2\phi} \circ F^{-1}(Y(s))ds + {}^1/_2 \int \mathcal{L}^{F(a)}_t \frac{\phi(da)}{\phi(a+)}.$$

Finally, if μ is the image of $\frac{\phi(da)}{\phi(a+)}$ under F and if we define

$\eta(da) = \frac{b}{\sigma^2\phi} \circ F^{-1}(a)da + {}^1/_2\ \mu(da)$, then we arrive at :

$$(1.5) \qquad Y(t) = B(t) + \int \mathcal{L}^a_t\ \eta(da).$$

Now suppose that we know that every solution of *(1.5)* is strong (ie. $B(\cdot)$-measurable). Then, since

$$\tau(t) = \int_0^t \frac{1}{\sigma^2\phi^2} \circ F^{-1}(Y(s))ds,$$

$\tau(\cdot)$ and therefore $\langle M,M\rangle_\bullet$ would be $B(\cdot)$-measurable. But $M(t) = B \circ \langle M,M\rangle_t$, and so we could conclude that $M(\cdot)$ is indeed pure. Thus we are led to the study of stochastic differential equations of the sort given in *(1.5)*. The key to our analysis is the following theorem due to S. Nakao [3] :

Theorem (1.6) : Let $(E,\mathcal{B},(\mathcal{B}_t),P)$ be a filtered probability space and let $B(\cdot)$ be a $(\mathcal{B}_\bullet)$-Brownian motion. Suppose that $a : R \to (0,\infty)$ is a bounded, uniformly positive function of local bounded variation and let $c : R \to R$ be a bounded measurable function. Then the equation

$$(1.7) \qquad \alpha(t) = \int_0^t a(\alpha(s))dB(s) + \int_0^t c(\alpha(s))ds$$

admits precisely one $(\mathcal{B}_\bullet)$-adapted solution and this solution is strong (ie. $B(\cdot)$-measurable).

Remark (1.8) : The existence part of *Theorem (1.6)* was not stated by Nakao, but it is an easy consequence of exercise (7.3.2) in [5]. Also as Yamada and Watanabe pointed out, the fact that $\alpha(\cdot)$ is $B(\cdot)$-measurable is a corollary of the uniqueness assertion (cf. Corollary 8.1.8 in [5]).

Using *Theorem (1.6)* we can now prove a result which will enable us to find out what we need to know about the solution of equation *(1.5)*.

Theorem (1.9) : Let $(E,\mathcal{B},(\mathcal{B}_t),P)$ and $B(\cdot)$ be as in *Theorem (1.6)*. Suppose that $m : R \to R$ is a function of local bounded variation such that $m(dx) = \psi(x)dx + \nu(dx)$, where ψ is a bounded measurable function and ν is a function of bounded variation satisfying $\nu(\{x\}) < {}^1/_2$, where $\nu(\{x\}) \equiv \nu(x+0)-\nu(x-0)$, for each $x \in R$. Then there is at most one $(\mathcal{B}_\cdot)$ continuous semi-martingale $\alpha(\cdot)$ which satisfies :

$$(1.10) \quad \alpha(t) = B(t) + \int L_t^a \, m(da)$$

where $(L_t^a)_{t>0}$ denotes the local time of $\alpha(\cdot)$ at a (we assume, as we may, according to [4], that $L^a_{s\wedge t}(\omega)$ is $\mathcal{B}_R \times \mathcal{B}_{[0,t]} \times \mathcal{B}$-measurable for each $t \geq 0$). Moreover, if it exists, $\alpha(\cdot)$ is $B(\cdot)$-measurable.

Proof : The idea is to introduce an increasing function H so that $H\circ\alpha(\cdot)$ satisfies an equation like *(1.7)*. To this end, define

$$h(x) = \left[\exp(-2\,\nu^c((-\infty,x]))\right]\left[\prod_{y\leq x}(1-2\nu(\{y\}))\right]$$

where ν^c denotes the continuous part of ν. Then h is a bounded, uniformly positive function of bounded variation and $h(dx) = -2h(x-)\,\nu(dx)$. Next set $H(x) = \int_0^x h(y)dy$. Then, by Itô's generalized formula :

$$H(\alpha(t)) = \int_0^t h(\alpha(s)-)\,d\alpha(s) + {}^1/_2 \int L_t^a\, h(da)$$

$$= \int_0^t h(\alpha(s))dB(s) + \int_0^t h\psi(\alpha(s))ds$$

$$+ \int L_t^a\, h(a-)\,\nu(da) + {}^1/_2 \int L_t^a\, h(da)$$

$$= \int_0^t h \circ H^{-1}(H(\alpha(s))dB(s) + \int_0^t (h\psi)\circ H^{-1}(H(\alpha(s)))ds.$$

Hence, by *Theorem (1.6)*, $H\circ\alpha(\cdot)$ is uniquely determined and is $B(\cdot)$-measurable

Q.E.D.

We are at last ready to complete the proof of *Theorem (1.2)*. As we have already seen, we need only show that every solution to *(1.5)* is $B(\cdot)$-measurable. In view of the preceding, this will be done once we check that the $\eta(\cdot)$ appearing on the right side of *(1.5)* satisfies the conditions put on $m(\cdot)$ in *Theorem (1.9)*. Since $\frac{b}{\sigma^2\phi}$ is bounded, this boils down to checking that $1-2\eta(\{x\}) > 0$ for each $x \in R$. But $2\eta(\{x\}) = \frac{\phi(y+)-\phi(y-)}{\phi(y+)} = 1 - \frac{\phi(y-)}{\phi(y+)}$ where $x = F(y)$. Hence $1 - 2\eta(\{x\}) = \frac{\phi(y-)}{\phi(y+)} > 0$.

We therefore know that $Y(\cdot)$ in *(1.5)* must be $B(\cdot)$-measurable. The proof of *Theorem (1.2)* is now complete.

Corollary (1.11) : Let $X(\cdot)$ satisfy *(1.1)* where $\sigma^2(\cdot)$ and $b(\cdot)$ are measurable functions, $b(\cdot)$ is bounded, and $\sigma^2(\cdot)$ is locally bounded and uniformly positive.
Let ϕ be a uniformly positive measurable function which satisfies one of the following conditions :

a) ϕ is a polynomial,

b) ϕ is bounded and non-decreasing,

c) ϕ is simple (ie. piecewise constant and takes only a finite number of values).

Then the $M(\cdot)$ given in *(1.3)* is pure.

Proof : The only case which is not obviously covered by *Theorem (1.2)* is b). However, in this case, simply take $\xi(dx)=\frac{\phi(dx)}{\phi(x+)}$ and notice that

$$\int_{-\infty}^{\infty} \frac{\phi(dx)}{\phi(x+)} \leq \int_{-\infty}^{\infty} \frac{\phi(dx)}{\phi(x-)} = \mathrm{Log}\ \phi(\infty) - \mathrm{Log}\ \phi(-\infty) < \infty.$$

Q.E.D.

Remark (1.12) : Of course *Theorem (1.2)* and *Corollary (1.11)* admit generalizations in various directions. However, they seem to us to cover reasonably well the situations to which the given techniques apply. The essential characteristic which all these situations share in common is the non-degeneracy of the function $\phi(\cdot)$. Indeed, as our results indicate, the smoothness of $\phi(\cdot)$ does not appear to be of great importance so long as $\phi(\cdot)$ stays away from 0.
This observation forces one to ask to what extent one can handle situations in which $\phi(\cdot)$ is allowed to vanish. We will present in the next section what little information we have on this subject.

2. QUESTIONS OF PURITY FOR DEGENERATE MARTINGALES :

In this section, we will be looking at martingales of the form :

$$(2.1) \qquad M(t) = \int_0^t \phi(\beta(s))d\beta(s)$$

where $\beta(\cdot)$ is a Brownian motion and $\phi : R \to R$ is locally bounded and measurable. As a consequence of *Corollary (1.11)*, we know that $M(\cdot)$ will be pure if ϕ is bounded, uniformly positive and non-decreasing. We now want to know what can happen if $\phi(\cdot)$ is allowed to vanish.

To see how quickly the situation can change when $\phi(\cdot)$ is permitted to vanish, consider the martingale :

$$(2.2) \qquad M_+(t) = \int_0^t 1_{(0,\infty)}(\beta(s))d\beta(s).$$

Obviously, the only condition which $\phi(\cdot)$ violates is that $\phi(\cdot)$ can vanish. As we now show, this one violation is fatal. In fact, we will show that $M_+(\cdot)$ is not even extremal and therefore certainly is not pure. To see that $M_+(\cdot)$ is not extremal, let $\{\mathcal{F}_t : t \geq 0\}$ be the completed right-continuous filtration determined by $M_+(\cdot)$. Then, since $\int_0^t 1_{(0,\infty)}(\beta(s)ds = \langle M_+,M_+\rangle_t$, $\int_0^t 1_{(0,\infty)}(\beta(s))ds$ is $(\mathcal{F}_\cdot)$-adapted. Now define $\sigma = \inf\{t \geq 1 : \int_{t-1}^t 1_{(0,\infty)}(\beta(s))ds = 0\}$ and $\tau = \inf\{t \geq \sigma : \int_\sigma^t 1_{(0,\infty)}(\beta(s))ds > 0\}$. Then σ and τ are finite $(\mathcal{F}_\cdot)$-stopping times. Furthermore, it is easy to check that τ cannot be $\mathcal{F}_\sigma$-measurable (this can be seen from $P(\beta(\sigma) < 0) > 0$). Now suppose $M_+(\cdot)$ is extremal. Then we could find an $(\mathcal{F}_\cdot)$-adapted $\theta(\cdot)$ such that $E\left[\int_0^\infty \theta_s^2 ds\right] < \infty$ and $e^{-\tau} = c + \int_0^\infty \theta(s)dM_+(s)$, where $c = E\left[e^{-\tau}\right]$. Since $e^{-\tau}$ is $\mathcal{F}_\tau$-measurable, we would necessarily have, $e^{-\tau} = c + \int_0^\tau \theta(s)dM_+(s)$. But

$$E\left[\left(\int_0^\tau \theta(s)dM_+(s) - \int_0^\sigma \theta(s)dM_+(s)\right)^2\right] = E\left[\int_\sigma^\tau \theta^2(s)\chi_{(0,\infty)}(\beta(s))ds\right] = 0$$

since $\beta(s) \leq 0$ for $s \in (\sigma,\tau)$. Hence, we would have : $e^{-\tau} = c + \int_0^\sigma \theta(s)dM(s)$. Because, τ is not $\mathcal{F}_\sigma$-measurable, this is impossible.

Remark (2.3) : With a more refined analysis one can prove more about the structure of $(\mathcal{F}_\cdot)$-martingales. In fact, one can show that there are purely discontinuous $(\mathcal{F}_\cdot)$-martingales, and certainly none of these could be $dM_+(t)$-stochastic integrals.

The example $M_+(\cdot)$ shows that we cannot afford to drop the positivity condition on $\phi(\cdot)$ when the only regularity hypothesis which we make is that $\phi(\cdot)$ is bounded and non-decreasing.

It is now reasonable to ask what happens if $\phi(\cdot)$ is a polynomial which is allowed to vanish. In particular, which of the martingales

$$(2.4) \qquad M_n(t) = \int_0^t \beta^n(s)d\beta(s), \qquad n \geq 1,$$

are pure ?

It is embarrassing for us to have to admit that we can only give a partial answer to this seemingly elementary question. What we will show is that for all $n \geq 1$ $M_n(\cdot)$ is extremal and that for odd $n \geq 1$ it is pure. Whether or not $M_{2n}(\cdot)$ (even for $n = 1$) is pure remains an open question. Exactly what is underlying the distinction between the odd and even cases we are unable to say, but the next proposition provides a hint.

Proposition (2.5) : <u>If</u> n <u>is even, then the filtrations determined by</u> $M_n(\cdot)$ <u>and</u> $\beta(\cdot)$ <u>are a.s. equal. If</u> n <u>is odd, then the filtrations determined by</u> $M_n(\cdot)$ <u>and</u> $|\beta(\cdot)|$ <u>are a.s. equal. For all</u> $n \geq 1$, $M_n(\cdot)$ <u>is extremal</u>.

Proof : First suppose that n is even. Since $M_n(\cdot)$ is necessarily $\beta(\cdot)$ measurable, we need only show that $\beta(\cdot)$ is a.s. $M_n(\cdot)$-adapted to conclude that the two filtrations are a.s. equal. But $\langle M_n,M_n\rangle_t = \int_0^t \beta^{2n}(s)ds$ and so $|\beta(\cdot)|$ is a.s. $M_n(\cdot)$-adapted. Hence $\int_0^t \frac{\beta^n(s)}{\beta^n(s)+\varepsilon^2} d\beta(s) = \int_0^t \frac{1}{\beta^n(s)+\varepsilon^2} dM_n(s)$ is a.s. $M_n(\cdot)$-adapted. Upon letting $\varepsilon \downarrow 0$, we see that $\beta(\cdot)$ is $M_n(\cdot)$-measurable. Now that we know that $\beta(\cdot)$ and $M_n(\cdot)$ have the same completed filtrations, it is clear that $M_n(\cdot)$ is extremal. Indeed, since $\beta(\cdot)$ is extremal, every square-integrable $\beta(\cdot)$-measurable random variable X can be represented as

$$X = E[X] + \int_0^\infty \theta(s)d\beta(s),$$

where $\theta(\cdot)$ is $\beta(\cdot)$-adapted and $E[\int_0^\infty \theta(s)^2 ds] < \infty$.

Thus, $X = E[X] + \int_0^\infty \frac{\theta(s)}{\beta^n(s)} dM_n(s)$; $\theta(\cdot)/\beta^n(\cdot)$ is $M_n(\cdot)$-adapted and satisfies

$$E[\int_0^\infty (\frac{\theta(s)}{\beta^n(s)})^2 d\langle M_n,M_n\rangle_s] < \infty.$$

Since every $M_n(\cdot)$-measurable random variable is $\beta(\cdot)$-measurable we see that $M_n(\cdot)$ has the representation property, which is equivalent to extremality.

If n is odd, then again $\langle M_n,M_n\rangle_t = \int_0^t \beta^{2n}(s)ds$ and so $|\beta(\cdot)|$ is a.s. $M_n(\cdot)$-adapted. On the other hand, $M_n(\cdot) = \int_0^t |\beta|^n(s)\,\mathrm{sgn}\,\beta(s)d\beta(s)$; and by Tanaka's formula :

$$|\beta(t)| = \int_0^t \mathrm{sgn}\,\beta(s)d\beta(s) + L_t^0$$

where $(L_t^0)_{t\geq 0}$ is the local time at 0 of $\beta(\cdot)$. From Tanaka's formula it is easy to conclude that if $S(t) = \int_0^t \mathrm{sgn}\,\beta(s)d\beta(s)$, then $S(\cdot)$ is a.s. $|\beta(\cdot)|$-adapted. Hence, $M_n(\cdot)$ is also a.s. $|\beta(\cdot)|$-adapted, and we see that $M_n(\cdot)$ and $|\beta(\cdot)|$ have a.s. the same filtrations. Finally, to show that $M_n(\cdot)$ is extremal, it suffices to show that $M_n(\cdot)$ and $S(\cdot)$ have a.s. the same filtrations, since $S(\cdot)$, being a Brownian motion, is extremal and therefore the same argument as we used above would apply. Hence we only have to check that $|\beta(\cdot)|$ is a.s. $S(\cdot)$-adapted. But

$$|\beta(t)|^2 = 2\int_0^t |\beta(s)|dS(s) + t$$

and so $|\beta(\cdot)|$ is a.s. $S(\cdot)$-adapted by the well-known results of T. Yamada and S. Watanabe [7].

We now turn to the proof that $M_n(\cdot)$ is pure when n is odd. The first step is precisely the same as the first step in the proof of *Theorem (1.2)* :

$$\beta^{n+1}(t) = (n+1)\, M_n(t) + \frac{n(n+1)}{2}\int_0^t \beta^{n-1}(s)ds.$$

Thus, if $\tau_n(\cdot)$ is the inverse of $\langle M_n,M_n\rangle_\cdot = \int_0^\cdot \beta^{2n}(s)ds$, if $\gamma_n(\cdot) = \beta^{n+1}(\tau_n(\cdot))$ and $B_n(\cdot) = M_n(\tau(\cdot))$, then $\gamma_n(t) = (n+1)B_n(t) + \frac{n(n+1)}{2}\int_0^t \gamma_n(s)^{-1}ds$.
Hence

$$(2.6)\qquad \gamma_n^2(t) = 2(n+1)\int_0^t \gamma_n(s)dB_n(s) + \frac{(n+1)\ (3n+1)}{2}\, t.$$

Up to this point we have not used the parity of n.
However, if we wish to conclude from *(2.6)* that $\gamma_n^2(\cdot)$ is $B_n(\cdot)$-adapted, then we must be able to write $\gamma_n(\cdot) = (\gamma_n^2(\cdot))^{1/2}$. In other words, we need to know that $\gamma_n(\cdot) \geq 0$, and obviously this will be the case if and only if n is odd. Assuming that n is odd and therefore that $\gamma_n(\cdot) = (\gamma_n^2(\cdot))^{1/2}$, we can apply the previously mentioned theorem due to Yamada and Watanabe and therefore show that $\gamma_n^2(\cdot)$ is indeed a.s. $B_n(\cdot)$-adapted. But this implies that $\beta^2(\tau_n(\cdot))$ is a.s. $B_n(\cdot)$-adapted, and therefore, since $\tau_n(t) = \langle\beta(\tau_n(\cdot)),\beta(\tau_n(\cdot))\rangle_t$, $\tau_n(\cdot)$ is a.s. $(B_n(\cdot)$-adapted. From here it is clear that $\langle M_n,M_n\rangle_\cdot$ is a.s. $B_n(\cdot)$-measurable and so is $M_n(\cdot) = B_n \circ \langle M_n,M_n\rangle_\cdot$. In other words :

Proposition (2.7) : $M_n(\cdot)$ is pure if n is odd.

Remark (2.8) : The argument just given to prove *Proposition (2.7)* can be used to prove the purity of certain martingales which come from the so called Bessel processes. To be precise, let $q > 1$ be given and let $\rho(\cdot)$ be the unique non-negative solution to :

$$\rho(t) = \rho_0 + \beta(t) + \frac{q-1}{2}\int_0^t {}^1/_{\rho(s)}\, ds,$$

where $\rho_0 \in [0,\infty)$. Then for any $\lambda > 1$, the martingale $M_\lambda(t) = \int_0^t \rho^{\lambda-1}(s)d\beta(s)$ is pure. The ideas underlying the proof are exactly the same as those presented above. Furthermore, the same reasoning applies to $\rho(\cdot)$ defined by

$$\rho(t) = \rho_0 + \beta(t) + L_t^0$$

where $(L_t^0)_{t\geq 0}$ is the local time of $\rho(\cdot)$ at 0.

3. COMPLEX BROWNIAN MOTION :

As mentioned in the introduction, this section deals with a slightly different topic. For those few readers who have born with us to this point, we are sure that the change of pace will come as a relief.

Let $X(\cdot)$ and $Y(\cdot)$ be independent 1-dimensional Brownian motions starting from 0 and let $(\mathfrak{J}_\cdot)$ be the completed filtration determined by $(X(\cdot),Y(\cdot))$. Given $z_0 \in \mathbb{C}$, set $\mathfrak{z}(\cdot) = z_0 + X(\cdot) + iY(\cdot)$. $\mathfrak{z}(\cdot)$ is called a complex Brownian motion starting from z_0. Associated with $\mathfrak{z}(\cdot)$ is Lévy's area process

$$\mathcal{A}(t) = \mathcal{A}_{\mathfrak{z}}(t) \equiv \int_0^t (X(s)dY(s) - Y(s)dX(s))$$

and the two processes :

$$\beta(t) = \beta_{\mathfrak{z}}(t) \equiv \int_0^t \frac{X(s)dX(s)+Y(s)dY(s)}{\rho_{\mathfrak{z}}(s)}$$

and

$$\gamma(t) = \gamma_{\mathfrak{z}}(t) \equiv \int_0^t \frac{X(s)dY(s)-Y(s)dX(s)}{\rho_{\mathfrak{z}}(s)}$$

where

$$\rho(t) = \rho_{\mathfrak{z}}(t) \equiv |\mathfrak{z}(t)|.$$

Let $(\mathfrak{F}^{\mathcal{A}}_\cdot)$, $(\mathfrak{F}^{\beta}_\cdot)$, $(\mathfrak{F}^{\gamma}_\cdot)$, and $(\mathfrak{F}^{\rho}_\cdot)$ denote the completed filtrations determined, respectively, by $\mathcal{A}(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\rho(\cdot)$; and let $(\mathfrak{F}^{(\beta,\gamma)}_\cdot)$ be the completed filtration determined by $(\beta(\cdot),\gamma(\cdot))$.

Proposition (3.1) : The processes $\beta(\cdot)$ and $\gamma(\cdot)$ are independent $(\mathfrak{J}_\cdot)$-Brownian motions. Furthermore, $(\mathfrak{F}^{\rho}_\cdot) = (\mathfrak{F}^{\beta}_\cdot)$ and $(\mathfrak{F}^{(\beta,\gamma)}_\cdot) = (\mathfrak{F}^{\mathcal{A}}_\cdot)$. Finally, if $z_0 = \rho_0 e^{i\theta_0} \neq 0$, then

$$(3.2) \qquad \mathfrak{z}(t) = \rho(t) \exp i(\theta_0 + \int_0^t \frac{d\gamma(s)}{\rho(s)}), \quad t \geq 0 ;$$

and so, in this case, $(\mathfrak{J}_\cdot) = (\mathfrak{F}^{(\beta,\gamma)}_\cdot) = (\mathfrak{F}^{\mathcal{A}}_\cdot)$.

Proof : Since $\langle\beta,\beta\rangle_t = \langle\gamma,\gamma\rangle_t = t$ and $\langle\beta,\gamma\rangle_t = 0$, the first assertion is obvious. To prove that $(\mathfrak{F}^{\rho}_\cdot) = (\mathfrak{F}^{\beta}_\cdot)$, note that

$$(3.3) \qquad \rho^2(t) - \rho_0^2 = 2\int_0^t \rho(s)d\beta(s) + 2t.$$

From *(3.3)* it is clear that $\int_0^\cdot \rho(s)d\beta(s)$ is $(\mathfrak{F}^{\rho}_\cdot)$-adapted and therefore that $\beta(\cdot)$ is also. Hence $(\mathfrak{F}^{\beta}_\cdot) \subseteq (\mathfrak{F}^{\rho}_\cdot)$. At the same time, *(3.3)* plus the theorem of Yamada and Watanabe imply that $(\mathfrak{F}^{\rho}_\cdot) \subseteq (\mathfrak{F}^{\beta}_\cdot)$. That is, $(\mathfrak{F}^{\rho}_\cdot) = (\mathfrak{F}^{\beta}_\cdot)$.

To see that $(\mathfrak{F}_{\bullet}^{(\beta,\gamma)}) = (\mathfrak{F}_{\bullet}^{\mathcal{A}})$, first note that

$$\mathcal{A}(t) = \int_0^t \rho(s)d\gamma(s).$$

Since $\rho(\cdot)$ is $(\mathfrak{F}_{\bullet}^{\beta})$-adapted, this proves that $(\mathfrak{F}_{\bullet}^{\mathcal{A}}) \subseteq (\mathfrak{F}_{\bullet}^{\beta})$. On the other hand :

$$\gamma(t) = \int_0^t \frac{d\mathcal{A}(s)}{\rho(s)}$$

and so we will have $(\mathfrak{F}_{\bullet}^{(\beta,\gamma)}) \subseteq (\mathfrak{F}_{\bullet}^{\mathcal{A}})$ once we have shown that $(\mathfrak{F}_{\bullet}^{\rho}) \subseteq (\mathfrak{F}_{\bullet}^{\mathcal{A}})$. But $\int_0^t \rho^2(s)ds = \langle\mathcal{A},\mathcal{A}\rangle_t$, and so $\rho(\cdot)$ is $(\mathfrak{F}_{\bullet}^{\mathcal{A}})$-adapted.

Finally, if $z_o = \rho_o e^{i\theta_o} \neq 0$, then (since $P((\exists\, t \geq 0)\ \mathfrak{z}(t) = 0) = 0$) we can a.s. make a unique continuous determination of the phase (ie. argument) $\theta(\cdot)$ of $\mathfrak{z}(\cdot)$ so that $\theta(0) = \theta_o$. Moreover, $d\theta(t) = \operatorname{Im}(\frac{d\mathfrak{z}(t)}{\mathfrak{z}(t)}) = \frac{X(t)dY(t)-Y(t)dX(t)}{\rho^2(t)} = \frac{d\gamma(t)}{\rho(t)}$. Hence the representation in *(3.2)* is proved. Clearly $(\mathfrak{F}_{\bullet}^{\mathcal{A}}) = (\mathfrak{F}_{\bullet}^{(\beta,\gamma)}) = (\mathfrak{J}_{\bullet})$ follows from this plus the preceding considerations.

Q.E.D.

In order to explain what happens to the equality $(\mathfrak{F}_{\bullet}^{\mathcal{A}}) = (\mathfrak{J}_{\bullet})$ when $z_o = 0$, we assume that the sample space of $\mathfrak{z}(\cdot)$ is $C([0,\infty),\mathbb{C})$ and $\mathfrak{z}(t)$ is the evaluation at time t. For $\theta \in [0,2\pi)$, define $R_\theta : C([0,\infty),\mathbb{C}) \to C([0,\infty),\mathbb{C})$ by $R_\theta \mathfrak{z}(\cdot) = e^{i\theta}Z(\cdot)$. We next define $\mathcal{R}_t = \sigma$ (H : H is a $\mathfrak{J}_t$-measurable random variable and $H = H \circ R_\theta$ a.s. for each $\theta \in [0,2\pi)$).

<u>*Theorem (3.4)*</u> : <u>If</u> $z_o = 0$, <u>then</u> $(\mathfrak{F}_{\bullet}^{\mathcal{A}_{\mathfrak{z}}}) = (\mathcal{R}_{\bullet})$.
<u>Moreover for each</u> $t > 0$, $m_t \equiv \mathfrak{z}(t)/|\mathfrak{z}(t)|$ <u>is a uniformly distributed random variable on</u> $S \equiv \{z \in \mathbb{C} : |z| = 1\}$ <u>and</u> m_t <u>is independent of</u> $\mathcal{R}_\infty$. <u>In particular</u> $\mathfrak{F}_t^{\mathcal{A}_{\mathfrak{z}}} \subsetneqq \mathfrak{J}_t$ <u>for each</u> $t \in (0,\infty]$.

<u>*Proof*</u> : We first prove that $(\mathfrak{F}_{\bullet}^{\mathcal{A}}) \subseteq (\mathcal{R}_{\bullet})$. To this end, note that $\rho(\cdot)$ is obviously $(\mathcal{R}_{\bullet})$-adapted.

Next, for $s > 0$ we can a.s. define a unique continuous determination $\theta_s(\cdot)$ of $\arg(\mathfrak{z}(\cdot\vee s)/\mathfrak{z}(s))$ such that $\theta_s(s) = 0$. Moreover, just as in the preceding

$$\theta_s(t) = \int_s^t \frac{d\gamma(u)}{\rho(u)}, \quad t \geq s.$$

Since $\theta_s(t)$ is clearly $\mathcal{R}_t$-measurable, we now see that $\int_s^{s\vee\cdot} \frac{d\gamma(u)}{\rho(u)}$ is $(\mathcal{R}_\cdot)$-adapted, and therefore that $\gamma(\cdot)$ is $(\mathcal{R}_\cdot)$-adapted. Thus, since $(\mathcal{F}^\beta_\cdot) = (\mathcal{F}^\rho_\cdot)$, $(\mathcal{F}^{\mathcal{a}}_\cdot) = (\mathcal{F}^{(\beta,\gamma)}_\cdot) \subseteq (\mathcal{R}_\cdot)$.

We next show that m_t is uniformly distributed on S and that m_t is independent of $\mathcal{R}_\infty$. But if H is a bounded $\mathcal{R}_\infty$-measurable random variable then for any $f \in B(S)$ and $\theta \in [0,2\pi)$:

$$E\left[f(m_t)H\right] = E\left[f(m_t)\ H \circ R_\theta\right] = E\left[f(e^{-i\theta}\, m_t \circ R_\theta)\ H \circ R_\theta\right]$$

$$= E\left[f(e^{-i\theta}\, m_t)\ H\right],$$

where the last equality results from the rotation invariance of the distribution of $z(\cdot)$. Hence

$$E\left[f(m_t)H\right] = E\left[\frac{1}{2\pi}\int_0^{2\pi} f(e^{-i\theta}\, m_t)d\theta\ H\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{-i\theta})d\theta\ E[H].$$

Finally, to prove that $(\mathcal{R}_\cdot) \subseteq (\mathcal{F}^{\mathcal{a}}_\cdot)$, let $t > 0$ be fixed and note that

$$z_s = \rho(s)\, e^{-i\theta_s(t)}\, m_t, \quad 0 \le s \le t.$$

Hence $\mathcal{R}_t \subseteq \mathcal{J}_t \overset{a.s.}{\subseteq} \mathcal{F}^{\mathcal{a}}_t \vee (\sigma(m_t))$. Since $\mathcal{F}^{\mathcal{a}}_t \subseteq \mathcal{R}_t$ and $\mathcal{R}_t$ is independent of $\sigma(m_t)$, it follows that $\mathcal{R}_t \subseteq \mathcal{F}^{\mathcal{a}}_t$.

<u>*Remark (3.5)*</u> : It follows easily from *(3.2)* that when $z_0 \neq 0$ we can write

$$\text{(3.6)} \qquad z(t) = \rho(t)\ \omega\left(\int_0^t ds/\rho^2(s)\right)$$

where $\omega(\cdot)$ is independent of $\rho(\cdot)$ and has the distribution of the Brownian motion on S starting from $z_0/|z_0|$. The analogue of *(3.6)* when $z_0 = 0$ is

$$\text{(3.7)} \qquad z(t) = \rho(t)\ \omega\left(\int_1^t 1/\rho(s)^2\, ds\right), \quad t > 0$$

where $\omega(\cdot)$ is independent of $\rho(\cdot)$ and is the stationary Brownian motion (defined for all $t \in R$) on S such that $\omega(t)$ is uniformly distributed for each $t \in R$. The proof of *(3.7)* is not difficult and is left to the reader.

<u>*Remark (3.8)*:</u> The situation described in *Proposition (3.1)* and *Theorem (3.4)* should be compared to the situation in one-dimension. To be precise, let $B(\cdot)$ be a one-dimensional Brownian motion starting at 0 and set $X(\cdot) = x_0 + B(\cdot)$, where $x_0 \in R$. Then, the analogue of $\beta_z(\cdot)$ is clearly $\beta_X(t) \equiv \int_0^t \mathrm{sgn}(X(s))dB(s)$.

It is not so clear what should be taken as the analogue of $\gamma_{\mathcal{Z}}(\cdot)$. The most intuitively appealing choice would be a process which counts the "number of times" that $X(\cdot)$ passes through 0. But the only candidate for that role is $(L_t^0)_{t\geq 0}$ (the local time of $X(\cdot)$ at 0) and, since $(L_t^0)_{t>0}$ is already $\beta_X(\cdot)$-adapted, nothing new is going to be gained by considering its filtration. Hence, the analogue of $(\beta_{\mathcal{Z}}(\cdot), \gamma_{\mathcal{Z}}(\cdot))$ is just $\beta_X(\cdot)$. Since it is well-known that the filtration of $\beta_X(\cdot)$ is a.s. equal to that of $|X(\cdot)|$, we now see that in one-dimension the analogue of the second part of *Proposition (3.1)* fails for <u>all</u> $x_o \in R$, not just for $x_o = 0$. Obviously, the fact which underlies this difference is the inability of a complex Brownian motion to hit 0 at a positive time.

Before closing this section, we want to reinterpret our results in terms of stochastic differential equations. To this end, let $\mathcal{Z}(\cdot)$, starting at $z_o \in \mathbb{C}$ be given and define $\mathcal{O}(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\rho(\cdot)$ accordingly. Then it is easy to check that

$$\mathcal{Z}(t) = z_o + \int_0^t \frac{\mathcal{Z}(s)}{|\mathcal{Z}(s)|} d(\beta(s) + i\gamma(s)),$$

(where we take $\mathcal{Z}(s)/|\mathcal{Z}(s)| \equiv 1$ if $\mathcal{Z}(s) = 0$). Since we know that $(\mathcal{J}_\cdot) = (\mathfrak{F}_\cdot^{(\beta,\gamma)})$ when $z_o \neq 0$, we should expect that this equation uniquely determines $\mathcal{Z}(\cdot)$ so long as $z_o \neq 0$. That is, we expect that $\mathcal{Z}(\cdot)$ is the one and only solution to

$$(3.9) \quad \Xi(t) = z_o + \int_0^t \frac{\Xi(s)}{|\Xi(s)|} d(\beta(s) + i\gamma(s))$$

when $z_o \neq 0$. We can verify this expectation in various ways. In the first place, it is easy to check that any solution $\Xi(\cdot)$ is a complex Brownian motion starting from z_o. Also, using the Picard contraction argument introduced by Itô long ago, it is easy to see that for each $\varepsilon > 0$ $\Xi(\cdot)$ is uniquely determined by *(3.9)* up until $\tau_\varepsilon = \inf\{t \geq 0 : |\Xi(t)| \leq \varepsilon\}$. Since $\tau_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$, $\Xi(\cdot)$ is unique for all time. In particular, $\Xi(\cdot) = \mathcal{Z}(\cdot)$ a.s..

Another approach is the following. Knowing that $\Xi(\cdot)$ is a complex Brownian motion starting at $z_o \neq 0$, we can choose a unique continuous version of $\text{Log}\,\Xi(\cdot)$ so that $\text{Log}\,\Xi(0) = \text{Log}\,\rho_o + i\theta_o$, where $z_o = \rho_o e^{i\theta_o}$. Furthermore,

$d\,\text{Log}\,\ (t) = \frac{d\,\Xi(t)}{\Xi(t)}$ and so

$$\Xi(t) = z_o \exp\Big[\int_0^t \frac{d\,\Xi(s)}{\Xi(s)}\Big]$$

$$= z_o \exp\Big[\int_0^t \frac{d(\beta(s)+i\gamma(s))}{\rho_{\Xi}(s)}\Big].$$

But using *(3.9)*, it is easy to derive

$$\rho_{\Xi}^2(t) - \rho_o^2 = 2\int_0^t \rho_{\Xi}(s)\, d\beta(s) + 2t.$$

Comparing this equation for $\rho_{\Xi}(\cdot)$ with equation *(3.3)* and using the theorem of Yamada and Watanabe, we conclude that $\rho(\cdot) = \rho_{\Xi}(\cdot)$ and so

$$\Xi(t) = z_o \exp\Big[\int_0^t \frac{d(\beta_s + i\gamma(s))}{\rho_{\Xi}(s)}\Big].$$

Now suppose that $z_o = 0$ and that $\Xi(\cdot)$ satisfies *(3.9)*. Again $\Xi(\cdot)$ is a complex Brownian motion starting from 0 and again one can show in the same manner that $\rho_{\Xi}(\cdot) = \rho(\cdot)$ and that

$$\text{(3.10)}\quad \frac{\Xi(t)}{\Xi(s)} = \exp\Big[\int_s^t \frac{d(\beta(u)+i\gamma(u))}{\rho(u)}\Big],\quad 0 < s \le t.$$

Hence, if $\mu_t = \frac{\Xi(t)}{\mathcal{Z}(t)}$, then $|\mu_t| = 1$ and

$$\mu_s = \frac{\Xi(s)}{\mathcal{Z}(s)} = \frac{\Xi(s)}{\Xi(t)}\,\frac{\mathcal{Z}(t)}{\mathcal{Z}(s)}\,\frac{\Xi(t)}{\mathcal{Z}(t)} = \mu_t$$

since *(3.10)* holds for any $\Xi(\cdot)$ satisfying *(3.9)* and therefore it holds for $\mathcal{Z}(\cdot)$ also. In other words if $\Xi(\cdot)$ satisfies *(3.9)* with $z_o = 0$, then

$$\text{(3.11)}\quad \Xi(\cdot) = \mu\, \mathcal{Z}(\cdot),$$

where μ is a random variable with values on S and μ is independent of $(\mathfrak{F}_{\cdot}^{(\beta,\gamma)})$. Conversely, if $\Xi(\cdot)$ satisfies *(3.11)* with a μ of this sort, then it is easy to see that $\Xi(\cdot)$ satisfies *(3.9)*.

Theorem (3.12) : Let $\mathcal{Z}(\cdot)$ be a complex Brownian motion starting from $z_o \in \mathbb{C}$, and let $\mathcal{A}(\cdot) = \mathcal{A}_{\mathcal{Z}}(\cdot)$, $\beta(\cdot) = \beta_{\mathcal{Z}}(\cdot)$, $\gamma(\cdot) = \gamma_{\mathcal{Z}}(\cdot)$, and $\rho(\cdot) = \rho_{\mathcal{Z}}(\cdot)$ be defined accordingly. Then $\mathcal{Z}(\cdot)$ satisfies *(3.9)*. Moreover, if $\Xi(\cdot)$ is any solution of *(3.9)*, then $\Xi(\cdot)$ is a complex Brownian motion starting at z_o and $\mathcal{A}_{\Xi}(\cdot) = \mathcal{A}(\cdot)$, $\beta_{\Xi}(\cdot) = \beta(\cdot)$, $\gamma_{\Xi}(\cdot) = \gamma(\cdot)$, and $\rho_{\Xi}(\cdot) = \rho(\cdot)$. In fact, if $z_o \neq 0$, then $\Xi(\cdot) = \mathcal{Z}(\cdot)$. Finally, if $z_o = 0$, then the set of solutions $\Xi(\cdot)$ to *(3.9)* is precisely the set of processes $\mu\mathcal{Z}(\cdot)$ where μ is a random variable with values in S and μ is independent of $\mathfrak{F}_{\infty}^{\mathcal{A}}$.

Proof : The only assertions which we have not already proved are the equalities $\mathcal{A}_{\Xi}(\cdot) = \mathcal{A}(\cdot)$, $\beta_{\Xi}(\cdot) = \beta(\cdot)$, and $\gamma_{\Xi}(\cdot) = \gamma(\cdot)$. But

$$\frac{d\Xi}{\Xi} = \frac{d(\beta_{\Xi} + i\gamma_{\Xi})}{\rho_{\Xi}}$$

and from *(3.9)* :

$$\frac{d\Xi}{\Xi} = \frac{d(\beta+i\gamma)}{\rho}.$$

Since $\rho_{\equiv}(\cdot) = \rho(\cdot)$, this proves that $\beta_{\equiv}(\cdot) = \beta(\cdot)$ and $\gamma_{\equiv}(\cdot) = \gamma(\cdot)$.
Finally, $d\mathcal{A}_{\equiv} = \rho_{\equiv}\, d\gamma$, and so $\mathcal{A}(\cdot) = \mathcal{A}_{\equiv}(\cdot)$.

Q.E.D.

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2) J. Pitman kindly showed us that the content of Remark (3.5) is already (in fact, in a more general setting !) in Itô-McKean's book, p. 276.

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