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A CHARACTERIZATION OF BMO-MARTINGALES

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Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a non-decreasing right continuous family  $(\mathcal{F}_t)$  of sub  $\sigma$ -fields of  $\mathcal{F}$  such that  $\mathcal{F}_0$  contains all  $P$ -null sets. In this note we deal only with continuous martingales  $X$  over  $(\mathcal{F}_t)$  such that  $X_0 = 0$ . A martingale  $X$  belongs to the class BMO if  $\|X\|_{\text{BMO}}^2 = \sup_t \text{ess. sup}_{\omega} E[\langle X \rangle_{\infty} - \langle X \rangle_t | \mathcal{F}_t] < \infty$ . Our aim is to prove the following.

THEOREM. Assume that  $M$  is an  $L^2$ -bounded martingale. Then  $M$  belongs to the class BMO if and only if  $Z_t = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$  satisfies the condition :

$$(A_p) \quad \sup_t \text{ess. sup}_{\omega} Z_t E[(\frac{1}{Z_{\infty}})^{p-1} | \mathcal{F}_t]^{p-1} < \infty$$

for some  $p > 1$ .

The condition  $(A_p)$  has already appeared many times in the literature in connection with several different questions (see B. Muckenhoupt [2]).

PROOF. Generally, if  $\|X\|_{\text{BMO}} < 1$ , then  $E[e^{\langle X \rangle_{\infty}}] \leq \frac{1}{1 - \|X\|_{\text{BMO}}^2}$  so that for each stopping time  $T$   $E[e^{\langle X \rangle_{\infty} - \langle X \rangle_T} | \mathcal{F}_T] \leq \frac{1}{1 - \|X\|_{\text{BMO}}^2}$ . This is the John-Nierenberg type inequality (see R.K. Gettoor and M.J. Sharpe [1]). It is clear that the process  $Z_t$  is a continuous local martingale. As  $M$  is  $L^2$ -bounded, we have  $\langle M \rangle_{\infty} < \infty$  and so  $Z_{\infty} > 0$ .

Suppose firstly that  $\|M\|_{\text{BMO}} < \infty$ , and choose  $p > 1$  such that  $\|\frac{\sqrt{p+1}}{p-1} M\|_{\text{BMO}} < 1$ . Then we get

$$\begin{aligned}
& Z_t E\left[\left(\frac{1}{Z_\infty}\right)^{\frac{1}{p-1}} \middle| \mathcal{F}_t\right]^{p-1} \\
&= E\left[\exp\left(-\frac{1}{p-1}(M_\infty - M_t) + \frac{1}{2(p-1)}(\langle M \rangle_\infty - \langle M \rangle_t)\right) \middle| \mathcal{F}_t\right]^{p-1} \\
&\leq E\left[\exp\left(-\frac{1}{p-1}(M_\infty - M_t) - \frac{1}{(p-1)^2}(\langle M \rangle_\infty - \langle M \rangle_t)\right)\right. \\
&\quad \times \exp\left(\frac{p+1}{2(p-1)^2}(\langle M \rangle_\infty - \langle M \rangle_t)\right) \middle| \mathcal{F}_t\right]^{p-1} \\
&\leq E\left[\exp\left(-\frac{2}{p-1}(M_\infty - M_t) - \frac{2}{(p-1)^2}(\langle M \rangle_\infty - \langle M \rangle_t)\right) \middle| \mathcal{F}_t\right]^{\frac{p-1}{2}} \\
&\quad \times E\left[\exp\left(\frac{p+1}{(p-1)^2}(\langle M \rangle_\infty - \langle M \rangle_t)\right) \middle| \mathcal{F}_t\right]^{\frac{p-1}{2}} \\
&\leq \frac{1}{\left\{1 - \frac{p+1}{(p-1)^2} \|M\|_{\text{BMO}}^2\right\}^{\frac{p-1}{2}}}
\end{aligned}$$

by using the John-Nirenberg type inequality. Thus  $Z_t$  satisfies  $(A_p)$ .

On the other hand, for every  $p > 1$ , by the Jensen inequality

$$\begin{aligned}
& Z_t E\left[\left(\frac{1}{Z_\infty}\right)^{\frac{1}{p-1}} \middle| \mathcal{F}_t\right]^{p-1} \\
&= \left\{ \exp\left(\frac{1}{p-1} M_t - \frac{1}{2(p-1)} \langle M \rangle_t\right) E\left[\exp\left(-\frac{1}{p-1} M_\infty + \frac{1}{2(p-1)} \langle M \rangle_\infty\right) \middle| \mathcal{F}_t\right] \right\}^{p-1} \\
&\geq \left\{ \exp\left(\frac{1}{p-1} M_t - \frac{1}{2(p-1)} \langle M \rangle_t - \frac{1}{p-1} M_t + \frac{1}{2(p-1)} E[\langle M \rangle_\infty | \mathcal{F}_t]\right) \right\}^{p-1} \\
&= \exp\left(\frac{1}{2} E[\langle M \rangle_\infty - \langle M \rangle_t | \mathcal{F}_t]\right),
\end{aligned}$$

from which we get  $\|M\|_{\text{BMO}} < \infty$  if  $Z_t$  satisfies the condition  $(A_p)$  for some  $p$ .

This completes the proof.

In a forthcoming paper [3] we shall study another properties on the condition  $(A_p)$ .

## References

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