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A CHARACTERIZATION OF BMO-MARTINGALES

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Let (Ω,F,P) be a probability space with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all P-null sets. <u>In this note we deal only with continuous martingales X over (F_t) such that $X_0=0$.</u> A martingale X belongs to the class BMO if $\|X\|^2_{BMO} = \sup_t \operatorname{ess.sup}_\omega E[\langle X\rangle_\infty - \langle X\rangle_t | F_t] < \infty$. Our aim is to prove the following.

<u>THEOREM.</u> Assume that M is an L^2-bounded martingale. Then M belongs to the class BMO if and only if $Z_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$ satisfies the condition :

$$(A_p) \qquad \sup_t \operatorname{ess.sup}_\omega Z_t E[(\frac{1}{Z_\infty})^{\frac{1}{p-1}} | F_t]^{p-1} < \infty$$

for some $p > 1$.

The condition (A_p) has already appeared many times in the literature in connection with several different questions (see B.Muckenhoupt [2]).

<u>PROOF.</u> Generally, if $\|X\|_{BMO} < 1$, then $E[e^{\langle X\rangle_\infty}] \leq \frac{1}{1-\|X\|^2_{BMO}}$ so that for each stopping time T $E[e^{\langle X\rangle_\infty - \langle X\rangle_T} | F_T] \leq \frac{1}{1-\|X\|^2_{BMO}}$. This is the John-Nierenberg type inequality (see R.K.Getoor and M.J.Sharpe [1]). It is clear that the process Z_t is a continuous local martingale. As M is L^2-bounded, we have $\langle M\rangle_\infty < \infty$ and so $Z_\infty > 0$.

Suppose firstly that $\|M\|_{BMO} < \infty$, and choose $p > 1$ such that $\|\frac{\sqrt{p+1}}{p-1} M\|_{BMO} < 1$. Then we get

$$Z_tE[(\frac{1}{Z_\infty})^{\frac{1}{p-1}}|F_t]^{p-1}$$

$$= E[\exp\Big(-\frac{1}{p-1}(M_\infty-M_t)+\frac{1}{2(p-1)}(\langle M\rangle_\infty-\langle M\rangle_t)\Big)|F_t]^{p-1}$$

$$\leqslant E[\exp\Big(-\frac{1}{p-1}(M_\infty-M_t)-\frac{1}{(p-1)^2}(\langle M\rangle_\infty-\langle M\rangle_t)\Big)$$

$$\times \exp\Big(\frac{p+1}{2(p-1)^2}(\langle M\rangle_\infty-\langle M\rangle_t)\Big)|F_t]^{p-1}$$

$$\leqslant E[\exp\Big(-\frac{2}{p-1}(M_\infty-M_t)-\frac{2}{(p-1)^2}(\langle M\rangle_\infty-\langle M\rangle_t)\Big)|F_t]^{\frac{p-1}{2}}$$

$$\times E[\exp\Big(\frac{p+1}{(p-1)^2}(\langle M\rangle_\infty-\langle M\rangle_t)\Big)|F_t]^{\frac{p-1}{2}}$$

$$\leqslant \frac{1}{\left\{1-\frac{p+1}{(p-1)^2}\|M\|_{BMO}^2\right\}^{\frac{p-1}{2}}}$$

by using the John-Nierenberg type inequality. Thus Z_t satisfies (A_p).

On the other hand, for every $p>1$, by the Jensen inequality

$$Z_tE[(\frac{1}{Z_\infty})^{\frac{1}{p-1}}|F_t]^{p-1}$$

$$= \Big\{\exp\Big(\frac{1}{p-1}M_t-\frac{1}{2(p-1)}\langle M\rangle_t\Big)E[\exp\Big(-\frac{1}{p-1}M_\infty+\frac{1}{2(p-1)}\langle M\rangle_\infty\Big)|F_t]\Big\}^{p-1}$$

$$\geqslant \Big\{\exp\Big(\frac{1}{p-1}M_t-\frac{1}{2(p-1)}\langle M\rangle_t-\frac{1}{p-1}M_t+\frac{1}{2(p-1)}E[\langle M\rangle_\infty|F_t]\Big)\Big\}^{p-1}$$

$$= \exp\big(\tfrac{1}{2}E[\langle M\rangle_\infty-\langle M\rangle_t|F_t]\big),$$

from which we get $\|M\|_{BMO}<\infty$ if Z_t satisfies the condition (A_p) for some p. This completes the proof.

In a forthcoming paper [3] we shall study another properties on the condition (A_p).

References

[1]. R.K.Getoor and M.J.Sharpe, Conformal martingales, Inventiones math. 16,(1972) pp.271-308

[2]. B.Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165,(1972) pp.207-226

[3]. M.Izumisawa and N.Kazamaki, Weighted norm inequalities for martingales, (to appear)

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