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Université de Strasbourg
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A SIMPLE REMARK ON THE CONDITIONED
SQUARE FUNCTIONS FOR MARTINGALE TRANSFORMS

by N. Kazamaki

1. Let $X=(X_n, \mathbb{F}_n)$ be a fixed uniformly integrable martingale defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and denote its difference sequence by $x=(x_n)$, $x_n = X_n - X_{n-1}, n \geq 1, X_0 = 0$. If $f=(f_n), f_n = \sum_{k=1}^n v_k x_k$, is a martingale transform of x , then the conditioned square function of f is $s(f) = \left\{ \sum_{k=1}^{\infty} v_k^2 E[x_k^2 | \mathbb{F}_{k-1}] \right\}^{1/2}$. Denote by \underline{M} the collection of all martingale transforms f of x . Let now $T_0 = 0 \leq T_1 \leq T_2 \leq \dots$ be \mathbb{F}_n -stopping times. Then (X_{T_n}) is a martingale over (\mathbb{F}_{T_n}) , because X is uniformly integrable. For simplicity we put $G_n = \mathbb{F}_{T_n}$. Since for each $k \geq 1$ v_{T_k+1} is G_{k-1} -mesurable, $\hat{f}_n = \sum_{k=1}^n v_{T_k+1} (X_{T_k} - X_{T_{k-1}})$ defines a new martingale transform. It should be noted that $\hat{f} = f$ if $T_k = k$, and that $\hat{f} = f^S$ if $T_k = k \wedge S$ for some stopping time S . Here f^S is the martingale transform stopped at S . Now we let

$$U(f) = \left\{ \sum_{k=1}^{\infty} v_{T_k+1}^2 E[(X_{T_k} - X_{T_{k-1}})^2 | G_{k-1}] \right\}^{1/2}$$

for $f_n = \sum_{k=1}^n v_k x_k$ in \underline{M} . This is none other than the conditioned square function $s(\hat{f})$. It follows at once that U is a symmetric and quasi-linear operator on \underline{M} . It seems to be interesting to investigate this operator, but to the best of our knowledge no papers on the subject have been published. In this paper we shall give an L^p -estimate between $U(f)$ and $s(f)$.

2. We start with these remarks: the operator U is not local, and $\|U(f)\|_p$ can not always be compared with $\|s(f)\|_p$. For example, let $w=(w_n)$ be an

independent sequence satisfying $P(w_k = -1) = P(w_k = 1) = 1/2, k \geq 1$, and W the martingale with difference sequence w . Define now $X_{2n+1} = X_{2n} = \sum_{k=1}^n \frac{1}{k} w_k, X_1 = X_0 = 0$. Then $x_{2n+1} = 0$ and $x_{2n} = \frac{1}{n} w_n$ so that $s(X) = (\sum_{n=1}^{\infty} \frac{1}{n^2})^{1/2}$. Thus X is an L^2 -bounded martingale. If $T_k = 2k, v_{2k+1} = 1$ and $v_{2k} = 0$ for each $k \geq 1$, then $s(f) = 0$ but $U(f) = (\sum_{n=1}^{\infty} \frac{1}{n^2})^{1/2}$. This implies that U is not local and that $\|U(f)\|_p \leq C_p \|s(f)\|_p$ does not hold in general. On the other hand, if $T_k = 2k, v_{2k+1} = 0$ and $v_{2k} = 1$ for each $k \geq 1$, then $U(f) = 0$ and $s(f) = (\sum_{k=1}^{\infty} \frac{1}{k^2})^{1/2}$. Moreover, in what follows we assume that the martingale X is locally square integrable.

PROPOSITION 1. Let $f_n = \sum_{k=1}^n v_k x_k, n \geq 1$, be a martingale transform in \underline{M} .

(1) If $|v_{T_{k-1}+1}| \leq |v_j|$ on $\{T_{k-1} < j \leq T_k\}$ for every j and k , then

$$\|U(f)\|_p \leq \sqrt{\frac{p}{2}} \|s(f)\|_p, \quad 2 \leq p < \infty$$

(2) If $|v_{T_{k-1}+1}| \geq |v_j|$ on $\{T_{k-1} < j \leq T_k\}$ for every j and k , then

$$\|U(f)\|_p \geq \sqrt{\frac{p}{2}} \|s(f)\|_p, \quad 0 < p \leq 2.$$

PROOF. We show only the part (1), the second part being proved similarly.

Let now $2 \leq p < \infty$, and suppose that for every $k, |v_{T_{k-1}+1}| \leq |v_j|$ for $T_{k-1} < j \leq T_k$.

An easy computation shows that $E[x_j^2 | F_{(j-1) \vee T_{k-1}}] = E[x_j^2 | F_{j-1}]$ on $\{T_{k-1} < j \leq T_k\}$

and $E[(X_{T_k} - X_{T_{k-1}})^2 | G_{k-1}] = \sum_{j=1}^{\infty} E[x_j^2 I_{\{T_{k-1} < j \leq T_k\}} | G_{k-1}]$. Therefore we have

$$\|U(f)\|_p = E \left\{ \sum_{k=1}^{\infty} v_{T_{k-1}+1}^2 E \left[\sum_{j=1}^{\infty} E[x_j^2 | F_{j-1}] I_{\{T_{k-1} < j \leq T_k\}} | G_{k-1} \right] \right\}^{p/2 - 1/p}$$

$$\begin{aligned}
&\leq E\left\{\sum_{k=1}^{\infty} E\left[\sum_{j=1}^{\infty} v_j^2 E[x_j^2 | F_{j-1}] \mathbb{1}_{\{T_{k-1} < j \leq T_k\}} | G_{k-1}\right]\right\}^{p/2}]^{1/p} \\
&= E\left\{\sum_{k=1}^{\infty} E[s_{T_k}(f)^2 - s_{T_{k-1}}(f)^2 | G_{k-1}]\right\}^{p/2}]^{1/p} \\
&\leq \sqrt{\frac{p}{2}} E\left[\left(\sum_{k=1}^{\infty} \{s_{T_k}(f)^2 - s_{T_{k-1}}(f)^2\}\right)^{p/2}\right]^{1/p} \\
&\leq \sqrt{\frac{p}{2}} \|s(f)\|_p.
\end{aligned}$$

We considered in [1] the special case $v=1$.

REMARK. Let f be any martingale transform in \underline{M} as above. Define the following stopping times: $k \geq 1$

$$\begin{aligned}
T_0 &= 0, \quad T_k = \text{Min} \left\{ j > T_{k-1}; |v_{j+1}| < |v_{T_{k-1}+1}| \right\} \\
S_0 &= 0, \quad S_k = \text{Min} \left\{ j > S_{k-1}; |v_{j+1}| > |v_{S_{k-1}+1}| \right\}
\end{aligned}$$

Then we get $|v_j| > |v_{T_{k-1}+1}|$ on $\{T_{k-1} < j \leq T_k\}$ and $|v_j| \leq |v_{S_{k-1}+1}|$ on $\{S_{k-1} < j \leq S_k\}$.

PROPOSITION 2. For any f in \underline{M} there exist martingale transforms $f^{(1)}$ and $f^{(2)}$ in M such that

$$1^\circ. \quad f = f^{(1)} + f^{(2)}$$

$$2^\circ. \quad \text{for each } i=1,2 \quad \|U(f^{(i)})\|_p \leq \sqrt{\frac{p}{2}} \|s(f^{(i)})\|_p, \quad 2 \leq p < \infty.$$

PROOF. Let $f_n = \sum_{k=1}^n v_k x_k, v_0 = 0$ and define

$$v_n^{(1)} = \sum_{k=1}^n (v_k - v_{k-1})^+, \quad v_n^{(2)} = -\sum_{k=1}^n (v_k - v_{k-1})^-.$$

Then each $v^{(i)}$ is a previsible process so that the martingale transform $f^{(i)}$ defined by $f_n^{(i)} = \sum_{k=1}^n v_k^{(i)} x_k$ belongs to \underline{M} . As $v_n = v_n^{(1)} + v_n^{(2)}$ for each n , we get $f = f^{(1)} + f^{(2)}$. It is clear that $|v_n^{(i)}| \leq |v_{n+1}^{(i)}|$ for each $i=1,2$. This completes the proof.

REFERENCE

- [1]. N.Kazamaki, An inequality for the conditioned square functions on martingales. Tohoku Math. Journ., (to appear).