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ON A STOPPED BROWNIAN MOTION FORMULA OF H.M. TAYLOR

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1. This note illustrates the usefulness of local time theory and the Cameron-Martin formula by giving two very short proofs of a stopped Brownian motion formula obtained in H.M. Taylor's paper [8] and applied there to problems in process control and in playing the stock market. The reader will easily find in the literature other applied problems for which local time theory may be effectively employed.

Let Ω be the space of continuous functions $\omega : [0,\infty) \to \mathbb{R}$. For $t \geq 0$, let X_t be the t-th coordinate function so that $X_t(\omega) = \omega(t)$. Let $\mathcal{F}^0 = \sigma\{X_t : t \geq 0\}$ be the smallest σ-algebra on Ω which 'measures' each X_t. Define

$$M_t = \max_{0 \leq s \leq t} X_s .$$

(As usual, we suppress the ω's.) Fix $a > 0$ and define

$$T = \inf\{t : M_t - X_t = a\},$$

so that T is the first time that the process X drops a units below its maximum-to-date. We make the usual convention that $\inf\{\emptyset\} = \infty$. Note that $X_T = M_T - a$ on the set $\{T < \infty\}$.

On $(\Omega, \mathcal{F}^0)$ introduce the measures:

$\mathbf{W}$: Wiener measure, the law of standard Brownian motion starting at 0;

$\mathbf{D}_\mu$: the law of a Brownian motion starting at 0 with drift constant μ and variance coefficient 1, so that $\mathbf{D}_0 = \mathbf{W}$ and

$$\{X_t : t \geq 0 ; \mathbf{D}_\mu\} \sim \{X_t + \mu t ; t \geq 0, \mathbf{W}\},$$

"$\sim$" denoting "is identical in law to".

If $\mathbf{P}$ denotes any probability measure on $(\Omega, \mathcal{F}^0)$ and ξ is (say) a positive random variable on $(\Omega, \mathcal{F}^0)$, we write

$$\mathbf{P}\{\xi\} = \int \xi \, d\mathbf{P} = \int \xi(\omega) \mathbf{P}(d\omega)$$

for the $\mathbf{P}$-expectation of ξ. We use $\mathbf{E}$ for expection when the appropriate "$\mathbf{P}$" is obvious.

Taylor's formula characterises the joint $\underset{\sim}{D}_\mu$-distribution of X_T and T as follows:

(1.1i) $$\underset{\sim}{D}_\mu\{\exp(\alpha X_T - \beta T)\} = \frac{\delta \exp[-(\alpha+\mu)a]}{\delta \cosh(\delta a) - (\alpha+\mu)\sinh(\delta a)}$$

which holds for $\alpha > 0$ and $\beta < \theta$, where

(1.1ii) $$\delta = [\mu^2 + 2\beta]^{\frac{1}{2}}, \quad \theta = \delta \coth(\delta a) - \mu > 0.$$

(Note. The discussion in Section 3 of Taylor's paper misses the fact that $M_T = X_T + a$ is exactly exponentially distributed. This follows either by an obvious "lack of memory" argument or directly from 1.1(i).)

Notice that because of the Cameron-Martin formula:

(1.2) $$\underset{\sim}{D}_\mu\{\exp(\alpha X_T - \beta T)\} = \underset{\sim}{W}[\exp(\alpha X_T - \beta T)\exp(\mu X_T - \tfrac{1}{2}\mu^2 T)]$$

(see §3.7 of McKean [5]), it is enough to establish the "$\mu = 0$" case of (1.1) in the form:

(1.3i) $$\underset{\sim}{W}\{\exp(\alpha M_T - \beta T)\} = \delta[\delta \cosh(\delta a) - \alpha \sinh(\delta a)]^{-1}$$

for $\beta > 0$ and $\alpha < \theta$, where, from now on,

(1.3ii) $$\delta = (2\beta)^{\frac{1}{2}}, \quad \theta = \delta \coth(\delta a).$$

For rigorous justification of (1.2), it is necessary to check that

(1.4) $$\underset{\sim}{W}\{T < \infty\} = \underset{\sim}{D}_\mu\{T < \infty\} = 1.$$

The validity of (1.4) will become clear in a moment.

Lévy tells us that under $\underset{\sim}{W}$, the process $Y = M - X$ is standard reflecting Brownian motion starting at 0 and M is the local time $L(\cdot,0)$ at 0 for Y:

$$M_t = L(t,0).$$

Recall that local time $L(t,y)$ at y before t for Y may be defined as follows:

$$L(t,y) = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \operatorname{meas}\{s \le t : Y_s \in [y, y+\varepsilon)\}.$$

(We are not bothering about "almost surely", etc..) See McKean [6] for a recent expository article on Brownian local time which contains all of the results which we shall need.

Since $M_T = L(T,0)$, we need only prove

(1.5) $$\underset{\sim}{W}\{\exp[\alpha L(T,0) - \beta T]\} = \delta[\delta \cosh(\delta a) - \alpha \sinh(\delta a)]^{-1}$$

for $\beta > 0$ and $\alpha < \theta$.

<u>Note</u>. Of course, under $\underset{\sim}{D}_\mu$, $M - X$ is a reflecting Brownian motion with drift constant $(-\mu)$ and M is again local time at 0. Result (1.4) is absolutely obvious if $\mu < 0$ while for $\mu \geq 0$ it is obvious on the grounds that "what can happen eventually will happen" (after an exponentially distributed <u>local</u> time M_T). But now to return to the case when $\mu = 0$, ...

2. <u>First proof of</u> (1.5). Let Y be a standard reflecting Brownian motion. In the case which concerns us, Y starts at 0 but we are entitled to consider other starting points. Suppose that $\alpha < 0$. Consider the function h on $[0,a]$ where

$$h(y) = \underset{\sim}{E}\{\exp[\alpha L(T,0) - \beta T] \mid Y_0 = y\} \qquad (0 \leq y \leq a)$$

where T is still defined by $T = \inf\{t : Y_t = a\}$. Then, according to the theory of <u>elastic Brownian motion</u> (sketched below - see §2.3 of Itô-McKean [2] for more information),

(2.1i) $h'' = \beta h$, so $h(y) = K_1 \cosh(\delta y) + K_2 \sinh(\delta y)$;

(2.1ii) $h^+(0) = -\alpha h(0)$, so $\delta K_2 = -\alpha K_1$ and $K_1 = K\delta$, $K_2 = -K\alpha$;

(2.1iii) $h(a) = 1$, so $K[\delta \cosh(\delta a) - \alpha \sinh(\delta a)] = 1$.

(We have written $h^+(0)$ for the right-hand derivative of h at 0.) Hence

$$\underset{\sim}{W}\{\exp[\alpha L(T,0) - \beta T]\} = h(0) = K_1 = K\delta$$
$$= \delta[\delta \cosh(\delta a) - \alpha \sinh(\delta a)]^{-1},$$

as required. That the formula extends to the range $\alpha < \theta$ is just (Abel-Dirichlet) analysis.

<u>Sketched proof of</u> (2.1). The formula (2.1i) is well-known and depends on the fact that for $0 \leq y - \varepsilon \leq y \leq y + \varepsilon \leq a$,

$$h(y) = \tfrac{1}{2}[h(y) - \varepsilon) + h(y+\varepsilon)] \operatorname{sech}(\delta\varepsilon),$$

the term $\operatorname{sech}(\delta\varepsilon)$ arising because (see Problem 2.3.2(a) of McKean [5] for instantaneous proof)

$$\operatorname{sech}(\delta\varepsilon) = \underset{\sim}{W}\{\exp(-\beta\tau_\varepsilon)\}, \quad \tau_\varepsilon = \inf\{t : |X_t| = \varepsilon\}.$$

Formula (2.1iii) is obvious. Formula (2.1ii) is true because (for $\varepsilon < -\alpha^{-1}$),

$$\begin{aligned} h(0) &= h(\varepsilon)\underset{\sim}{W}\{\exp[\alpha L(T_\varepsilon, 0) - \beta T_\varepsilon]\} \\ &= h(\varepsilon)\underset{\sim}{W}\{1 + \alpha L(T_\varepsilon, 0) - \beta T_\varepsilon + \dots\} \\ &= h(\varepsilon)\{1 + \alpha\varepsilon - \beta\varepsilon^2 + o(\varepsilon^2)\} \\ &= h(0) + \varepsilon[\alpha h(0) + h^+(0)] + o(\varepsilon^2), \end{aligned}$$

because $\underset{\sim}{W}\{L(T_\varepsilon, 0)\} = \varepsilon$. See Theorem 4.2 of Williams [11] for a simple proof that under $\underset{\sim}{W}$, $L(T_\varepsilon, 0)$ is exponentially distributed with mean ε. (In particular, under $\underset{\sim}{W}$, M_T is exponentially distributed with mean a.)

3. <u>Second proof of</u> (1.5). The most profound results on Brownian local time are the Markov properties discovered by Ray ([7]) and Knight ([3]). The basic Markov property which they discovered is the following:

(3.1) $$\{L(T, a-y) : 0 \le y \le a ; \underset{\sim}{W}\} \sim \{\tfrac{1}{2}R_y^2 : 0 \le y \le a\},$$

where R is a "2-dimensional" Bessel process starting at 0. Thus R is a continuous process identical in law to the radial part of standard 2-dimensional Brownian motion starting at 0. Williams ([9]) gave a simple proof of (3.1) and ([10,11]) showed that all the other Ray-Knight properties follow from (3.1) via time-reversal, time-substitution, etc.. See also McKean [6].

We now calculate

$$\underset{\sim}{W}\{\exp[\alpha L(T,0) - \beta T]\} = \underset{\sim}{W}\{\exp[\alpha L(T,0) - 2\beta\int_0^a L(T, a-y)dy]\}$$

$$= \underset{\sim}{E}\{\exp[\tfrac{1}{2}\alpha R_a^2 - \beta\int_0^a R_y^2\, dy]\} \quad \text{(by (3.1))}$$

$$= \underset{\sim}{W}\{\exp[\tfrac{1}{2}\alpha X_a^2 - \beta\int_0^a X_y^2\, dy]\}^2 \quad \text{(Pythagoras)}$$

$$= \underset{\sim}{E}\{\exp[\tfrac{1}{2}(\alpha+\delta)U_a^2 - \tfrac{1}{2}\delta]\}^2 \quad \text{(Cameron-Martin)}$$

where U is an Ornstein-Uhlenbeck process with generator $\tfrac{1}{2}D^2 - \delta x D$ $(D = d/dx)$ and starting at 0. Now (see Chapter 16 of Breiman [1]) U_a is Gaussian with mean 0 and variance $(2\delta)^{-1}[1 - e^{-2\delta a}]$. Let V_a be a variable independent of U_a and with the same distribution as U_a. Then

$$\underset{\sim}{W}\{\exp[\alpha L(T,0) - \beta T]\} = \underset{\sim}{E}\{\exp[\tfrac{1}{2}(\alpha+\delta)(U_a^2 + V_a^2) - \delta a]\}$$
$$= \delta[\delta \cosh(\delta a) - a \sinh(\delta a)]^{-1}$$

because $\frac{1}{2}(U_a^2 + V_a^2)$ is exponentially distributed with mean $(2\delta)^{-1}[1 - e^{-2\delta a}]$.
That the range of validity of (1.5) extends to $\alpha < \theta$ is now obvious.

Notes (i). The Ray-Knight theorems and Cameron-Martin formula have been used in conjunction before. See for example Knight [4] , Williams [10] .

(ii) It is no accident that the proof of the Ray-Knight theorm in [9] depends on the theory of elastic Brownian motion.

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