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Hermann Rost

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SKOROKHOD STOPPING TIMES OF MINIMAL VARIANCE

by H. Rost

1) Introduction.

One of the many possible ways of stopping one-dimensional Brownian motion in such a form that the stopped process has a desired distribution is due to ROOT([5]). In that article, the author introduces the notion of a barrier as a subset B of $R \times R_+$, for which $(x,t) \in B$, $t' > t$ implies $(x,t') \in B$, and establishes the following theorem :

Let $(X_t)_{t \geq 0}$ be Brownian motion on R, $X_0 = 0$; let ν be a probability measure on R satisfying $\int x\,\nu(dx) = 0$, $\int x^2 \nu(dx)$ Then there exists a closed barrier B such that the stopping time $T := \inf\{t: t \geq 0, (X_t,t) \in B\}$ has the following properties: X_T has distribution ν and $\mathcal{E}T = \int x^2 \nu(dx)$.

The question of uniqueness of the barrier B given ν has not been treated by ROOT, but a theorem of LOYNES([3]) says that at least the time T is uniquely determined by ν (with probability one, of course).

In a paper of KIEFER([2]) on Skorokhod embedding of a random walk into Brownian motion the conjecture is made that among all stopping times S satisfying

(1) $\quad X_S$ has distribution ν and $\mathcal{E}S = \int x^2 \nu(dx)$

the time T constructed by ROOT has minimal second moment and hence is the most appropriate candidate for Skorokhod embedding (provided $\mathcal{E}T^2 < \infty$ or, equivalently, $\int x^4 \nu(dx) < \infty$).

The aim of the present paper is to prove that conjecture and to state the theorem in a general, merely potential theoretic, form. The key notion thereby will be that of a <u>stopping time of minimal residual expectation</u>, which in the discrete time case has been introduced by DINGES([1]); the argument can be summarized as follows:

a) suppose one has a stopping time which in the class of all times satisfying (1) minimizes for each $t \in R_+$ the expectation

$$\mathcal{E} \int_{S\wedge t}^{S} du \quad = \quad \int_{t}^{\infty} P(S > u)\,du \ ;$$

then it minimizes for all $p > 1$ the moments

$$\mathcal{E}\, S^p \ = p(p-1) \int_0^\infty t^{p-2} dt \ \left(\int_t^\infty P(S > u)\,du \right) \ ; \qquad {}^{*)}$$

b) such a time exists; call it stopping time of minimal resi dual expectation with respect to ν . It is obtained by a construction like that of a réduite, from which it turnes out that it even minimizes all integrals of the form

$$\mathcal{E} \int_{S\wedge t}^{S} f\circ X_u\,du \ = \ \int_t^\infty (\mathcal{E}\, f\circ X_u \cdot 1_{\{S>u\}})\,du \ , \ t \in R_+, \ f \geq 0 \ \text{ on } R \ .$$

In this paper we will carry out the construction of those times in a quite general framework in Theorem 1. The proof of the theorem yields a possible potential theoretic interpretation of the family of distributions of $X_{T\wedge t}$, $t \geq 0$, for such a time T . The main result of this article is contained in Theorem 2 ,

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*) By the way, it <u>maximizes</u> for $0 < p < 1$ the moments

$$\mathcal{E}\, S^p \ = \ p(1-p) \cdot \int_0^\infty t^{p-2} dt \left(\mathcal{E}\, S - \int_t^\infty P(S > u)\,du \right) \ ,$$

in particular $\mathcal{E} S^{1/2}$, the expected quadratic variation of the martingale $X_{S\wedge t}$, $t \geq 0$.

which states that every first hitting time T to a barrier - in an obvious generalization of ROOT's definition - is of minimal residual expectation (with respect to the distribution of X_T) and hence of minimal second moment if it exists; any time S satisfying

$$\mathcal{E}\int_{S\wedge t}^{S} f\circ X_u du = \mathcal{E}\int_{T\wedge t}^{T} f\circ X_u du \text{ for all } t\in R_+,\ f\geq 0$$

is almost surely equal to T (Corollary to Th.2). This implies that under the assumptions of ROOT's theorem any stopping time S satisfying (1) and of the same variance as ROOT's time T is equal to T (if the variance is finite).

Conversely, any stopping time of minimal residual expectation with respect to some measure is essentially of the ROOT type : it can be included between the hitting times corresponding to two barriers which differ only by a "graph" , i.e. a set of the form $\{(x,t(x)): x \text{ in the state space}\}$ (Theorem 3). It is easy to see that in the Brownian motion case, more generally, if the one-point sets are regular for the process, these two hitting times coincide and hence any time of minimal residual expectation is the first hitting time to a barrier (Corollary to Th.3).

(Technical remark : in order to simplify notations we will formulate and prove the results only for transient processes; so, rigorously speaking, KIEFER's conjecture will only be proved, in the case of a measure ν of bounded support, because in this case we pass to Brownian motion killed after leaving some finite intervall. But it should be clear that all definitions and statements make still sense in the recurrent case if we limit ourselves to the class of stopping times T for which the measure $f \mapsto \mathcal{E}\int_0^T f\circ X_u du$ is σ-finite.)

2) Basic assumptions and notations.

We consider a Borel set E in a compact metric space; denote by $\mathcal{E}_+$ the positive Borel measurable functions on E. $(X_t)_{t\geq 0}$ is a Markov process on E which we assume to satisfy the "right hypotheses" (right continuous paths, strong Markov property); we denote by (P_t) its transition semigroup and by $U = \int_0^\infty P_t dt$ its potential kernel. On (the Borel sets of) E we are given a probability measure μ with σ-finite potential μU.

The process (X_t) is defined on a fixed probability space $(\Omega, \mathcal{F}, P^\mu)$ and Markovian with respect to a family $(\mathcal{F}_t)$ of σ-fields; the distribution of X_0 is μ. We assume that $\mathcal{F}$ admits a random variable with atomfree distribution and independent of the process. The notion of stopping time is always understood with respect to $(\mathcal{F}_t)$; all stopping times T are normalized so that $T = \infty$ on the set $\{T \geq \zeta\}$, where ζ is the lifetime of the process.

The measure μP_T on E is defined as usual by

$$\langle \mu P_T, f \rangle = \mathcal{E}^\mu f\circ X_T \cdot 1_{\{T<\infty\}}, \quad f \in \mathcal{E}_+,$$

$(= \mathcal{E}^\mu f\circ X_T$ if we make the convention $f\circ X_t = 0$ for $t = \infty$).

If A is an almost Borel set in E we denote by μH_A the measure μP_{D_A}, where $D_A = \inf\{t: t\geq 0,\ X_t \in A\}$.

We will use the following characterization of balayage order, which holds under these assumptions (see e.g. [4]):

Every finite measure ν on E with νU σ-finite admits a decomposition $\nu = \bar{\nu} + \nu_\infty$, where $\nu_\infty U$ is the rêduite of $(\nu - \mu)U$ and

$\bar{\nu}$ is of the form μP_T , T a stopping time. Further, there exists a finely closed set A which carries ν_∞ and for which $\nu_\infty = (\nu - \mu)H_A$ or $(\bar{\nu}-\mu)H_A = 0$. In the special case $\nu U \le \mu U$ one has $\nu_\infty = 0$ and $\nu = \mu P_T$ for some T .

3) Stopping times of minimal residual expectation.

Definition 1. Let ν be a measure on E with $\nu U \le \mu U$. We say that a stopping time T is of minimal residual expectation (m. r.e.) with respect to ν , if $\mu P_T = \nu$ and if for all S such that $\mu P_S = \nu$ one has

$$\mathcal{E}^{\mu}\int_{T\wedge t}^{T} f\circ X_u du \le \mathcal{E}^{\mu}\int_{S\wedge t}^{S} f\circ X_u du \quad \text{for all } f \in \mathcal{E}_+,\ t \in R_+$$

(or, equivalently, $\mu P_{T\wedge t}U \le \mu P_{S\wedge t}U$ for all $t \in R_+$).

Theorem 1. If a measure ν satisfies $\nu U \le \mu U$ then there exists a stopping time of minimal residual expectation with respect to ν.

Proof. 1) One introduces in $E \times R$ the semigroup of the space-time process $(\bar{P}_t)_{t\geq 0}$:

$$\bar{P}_t(x,r;\ A\times B) = P_t(x,A)\cdot 1_{\{r+t\in B\}},\ A\subset E,\ B\subset R ;$$

on $E\times R$ one defines the measure M by

$$M(A\times B) = \int_B M_t(A)\,dt \quad \text{where} \quad M_t = \mu U\cdot 1_{\{t<0\}} + \nu U\cdot 1_{\{t\geq 0\}} .$$

Let S be a stopping time with $\mu P_S = \nu$ (such a time exists under our assumptions); then the measure L on $E\times R$ is defined by

$$L(A\times B) = \int_B L_t(A)\,dt \quad \text{where} \quad L_t = \mu U\cdot 1_{\{t<0\}} + \mu P_{S\wedge t}U\cdot 1_{\{t\geq 0\}} .$$

It is easy to see that $L \geq M$ holds and that L is $(\bar{P}_t)$-excessive. If one writes the $(\bar{P}_t)$-réduite $\hat{M}$ of M in the form

$$\hat{M}(A\times B) = \int_B \hat{M}_t(A)\,dt \quad \text{with } \hat{M}_\cdot \text{ decreasing, right continuous}$$

the theorem will be proved, because of $\hat{M}_t \le L_t$, if we can show that there exists a stopping time T satisfying

(*) $\mu P_T = \nu$ and $\mu P_{T\wedge t} U = \hat{M}_t$ for all $t \in R_+$.

2) As in [6] one sees that the second condition in (*) can be satisfied if there exists a family μ_t, $t \in R_+$, of measures with

$$\mu_0 \le \mu \; ; \; \mu_s P_t \geqslant \mu_{t+s} \; , \; t,s \geqslant 0 \; ;$$

$$\mu U - \hat{M}_t = \int_0^t \mu_s ds \; , \; t \geqslant 0.$$

For then one chooses a T such that $\langle \mu_t, f \rangle = \mathcal{E}^{\mu} f \circ X_t \cdot 1_{\{T>t\}}$ for all $f \in \mathcal{E}_+$. Now, the existence of the family (μ_t) follows from the following inequalities, if we set $\mu_t = -\frac{d}{dt} \hat{M}_t$:

(a) $\hat{M}_{t+s} \le \hat{M}_t$, $t,s \geqslant 0$;

(b) $\mu U - \hat{M}_t \le \int_0^t \mu P_s ds$, $t \geqslant 0$;

(c) $(\hat{M}_t - \hat{M}_{t+s}) P_u \geqslant \hat{M}_{t+u} - \hat{M}_{t+u+s}$, $t,u,s \geqslant 0$.

So the problem is reduced to make evident these inequalities.

(a) holds, since $M_{\cdot}$ is a decreasing family;

(b) is true because $\hat{M}$ is excessive, what implies

$$\hat{M}_t \geq \mu U P_t = \mu U - \int_0^t \mu P_s ds \quad ;$$

(c) follows from a possible construction for a rêduite :

$\hat{M}_t = \uparrow \lim_k M_t^{(k)}$, where $M_t^{(k)} = \hat{M}^{(k)}_{[t\cdot 2^k]+1}$ and the sequence $\hat{M}_n^{(k)}$ is recursely defined by

$$\hat{M}_0^{(k)} = \mu U \; , \; \hat{M}_{n+1}^{(k)} = \nu U \vee \hat{M}_n^{(k)} P_{2^{-k}} \text{ for } n \geqslant 0 \; .$$

The relation (c) follows in the limit $k \to \infty$ from

$$(\hat{M}_n^{(k)} - \hat{M}_{n+1}^{(k)}) P_{2^{-k}} \geqslant \hat{M}_{n+1}^{(k)} - \hat{M}_{n+2}^{(k)} \; ,$$

what is obviously true.

3) The proof is complete if we show that the second condition in (*) implies the first one, or that $\nu U = \downarrow \lim_{t\to\infty} \hat{M}_t$. Let S be

the stopping time introduced in 1). Because of

$$\mu P_{S\wedge t}U = L_t \geq \hat{M}_t$$

we get, passing to the limit $t \to \infty$,

$$\mu P_S U = \nu U \geq \downarrow \lim_t \hat{M}_t .$$

The converse inequality follows from

$$\hat{M}_t \geq M_t = \lim_{t\to\infty} M_t = \nu U \quad \text{for } t \geq 0.$$

4) ROOT stopping times.

Definition 2. A subset B of $E \times R_+$, which is nearly Borel with respect to the space-time process $t \mapsto (X_t, t)$ is called a barrier, if $(x,t) \in B$ and $t' > t$ implies $(x,t') \in B$ (or, equivalently, if the family of its sections $B_t = \{x: (x,t) \in B\}$ is increasing in t).

Definition 3. If B is a barrier, the time

$$T = \inf\{t : t \geq 0, (X_t, t) \in B\} = \inf\{t: X_t \in B_t, t \geq 0\}$$

is called the ROOT stopping time defined by the barrier B. A stopping time is called simply ROOT stopping time if it is the ROOT time for some barrier.

Theorem 2. Every ROOT stopping time T is of minimal residual expectation (with respect to μP_T).

Proof. Let T be defined by the barrier B. We suppose without loss of generality B to be finely closed for the space-time process, what implies that the sections B_t are finely closed for the original process. Set

$$\nu = \mu P_T \quad \text{and} \quad N_t = \mu U \cdot 1_{\{t<0\}} + \mu P_{T\wedge t} U \cdot 1_{\{t\geq 0\}}, \quad t \in R,$$

$$N(A \times C) = \int N_t(A) \cdot 1_C dt \quad \text{for } A \subset E, \ C \subset R ;$$

Let M and $\hat{M}$ be as in the proof of theorem 1. Then the assertion of theorem 2 is equivalent to $\hat{M} = N$, or

$$\hat{M} \geq N \ , \text{or} \ \hat{M}_t \geq N_t \quad \text{for} \quad t \geq 0 \ .$$

The proof is carried out by proving three auxiliary results.

Proposition 1. For fixed $t \in R_+$ define

$$M_s^t = N_s \cdot 1_{\{s<t\}} + \nu U \cdot 1_{\{s \geq t\}},$$
$$M^t(A \times C) = \int M_s^t(A) \cdot 1_C ds \quad \text{for } A \subset E \ , C \subset R \ ;$$

let $\hat{M}^t$ be the $(\bar{P}_t)$-réduite of M^t and $\hat{M}_s^t$ defined by

$$\hat{M}^t(A \times C) = \int \hat{M}_s^t(A) \cdot 1_C ds \ , \quad \hat{M}_\cdot^t \text{ right continous.}$$

Then the following estimate is true

$$\hat{M}_{t+s}^t \leq \hat{M}_{t+s} + (N_t - \hat{M}_t) P_s \ , \quad s \geq 0 \ .$$

Proof. The measure $\tilde{N}$, defined by

$$\tilde{N}(A \times C) = \int_C (N_s \cdot 1_{\{s<t\}} + (\hat{M}_s + (N_t - \hat{M}_t) P_{s-t}) \cdot 1_{\{s \geq t\}}) ds$$

is $(\bar{P}_t)$-excessive and greater than M^t ; hence it is also greater than $\hat{M}^t$. The proposition follows by "desintegration" .

The following lemma is of some interest in itself and sounds rather plausible; in the special case $B = A \times R_+$ for some $A \subset E$ it is exactly the statement of the theorem.

Lemma. Let A be a finely closed subset of E and $D = D_A$ (the first hitting time to A). If S is any stopping time satisfying $\mu P_S U \geq \mu P_D U$ then $S \leq D$ a.s. (P^μ).

Proof. The inequality $\mu P_S U \geq \mu P_D U$ implies under our general assumptions that there exists a stopping time D' with

$$D' \geq S \quad \text{and} \quad \mu P_{D'} = \mu P_D \ .$$

(If necessary one has to enlarge for this the basic probability space; but the wanted result $S \leq D$ holds in Ω if it is true in the enlarged space.) Since $X_D \in A$ a.s. μP_D and hence

$\mu P_{D'}$ is carried by A ; this means $X_{D'} \in A$ a.s. and therefore $D' \geq D$ by definition of D . From equality of the potentials $\mu P_D U = \mu P_{D'} U$ follows that $D' = D$ a.s. and so $D \geq S$ a.s.

Proposition 2. $\lim\limits_{t \to 0} t^{-1} \langle \mu U - \hat{M}_t, 1 \rangle \leq \lim\limits_{t \to 0} t^{-1} \langle \mu U - N_t, 1 \rangle$.

Proof. Apply the lemma with $A = B_0$ (section of B at $t = 0$) and S some stopping time of m.r.e. with respect to $\nu = \mu P_T$. (The assumptions of the lemma are satisfied, because of $D \geq T$, what implies $\mu P_D U \leq \mu P_T U = \mu P_S U$.) The lemma gives us $D \geq S$ and therefore $\hat{M}_t = \mu P_{S \wedge t} U \geq \mu P_{D \wedge t} U$ for all $t \geq 0$. So it suffices to prove

$$\lim_t t^{-1} \langle \mu U - \mu P_{D \wedge t} U, 1 \rangle \leq \lim_t t^{-1} \langle \mu U - N_t, 1 \rangle.$$

But this follows from

$$\lim_t t^{-1} \langle N_t - \mu P_{D \wedge t} U, 1 \rangle = \lim_t t^{-1} \mathcal{E}^{\mu}(D \wedge t - T \wedge t) \leq$$

$$\lim_t P^{\mu}(T \leq t,\ D > 0) = P^{\mu}(T = 0,\ D > 0) = P^{\mu}(X_0 \in B_0,\ X_0 \notin B_0) = 0$$

($\{D > 0\} = \{X_0 \notin B_0\}$ because B_0 is finely closed .)

Only a notational generalization of Proposition 2 is

Proposition 2'. For all $t \in R_+$ one has

$$\lim_{h \to 0} h^{-1} \langle N_t - \hat{M}^t_{t+h}, 1 \rangle \leq \lim_{h \to 0} h^{-1} \langle N_t - N_{t+h}, 1 \rangle .$$

Proof of the theorem (continuation): We consider the two functions on R_+ , m and n , defined as

$$m(t) = \langle \mu U - \hat{M}_t, 1 \rangle , \quad n(t) = \langle \mu U - N_t, 1 \rangle$$

and show, that $m \leq n$. This will complete the proof.

Since both functions are Lipschitz-continuous and $m(0) = n(0)$, it is sufficient to compare the right derivatives :

$$\left. \frac{d^+ n}{ds} \right|_{s=t} = \lim_{h \to 0} h^{-1} \langle N_t - N_{t+h}, 1 \rangle \geq \lim_h h^{-1} \langle N_t - \hat{M}^t_{t+h}, 1 \rangle \geq$$

$$\lim_h h^{-1}\langle N_t - \widehat{M}_{t+h} - (N_t - \widehat{M}_t)P_h, 1\rangle \geq \lim_h h^{-1}\langle \widehat{M}_t - \widehat{M}_{t+h}, 1\rangle = $$
$$= \left.\frac{d^+m}{ds}\right|_{s=t}$$

where the first inequality holds **by Prop.2′** and the last by Prop.1 . So the theorem is proved.

<u>Corollary</u>. <u>(Uniqueness of the ROOT stopping time). Let T be a ROOT stopping time and S of m.r.e with respect to μP_T .Then one has $S = T$ a.s. (P^μ).</u>

<u>Proof</u>. Since T is also of m.r.e. with respect to μP_T by the theorem, we have for all $t \geq 0$

$$\mu P_{S\wedge t}U = \mu P_{T\wedge t}U \ .$$

It follows

$$\mathcal{E}^\mu \int_0^{S\wedge t} f\circ X_u du = \mathcal{E}^\mu \int_0^{T\wedge t} f\circ X_u du \ , \ f \in \mathcal{E}_+, \ t \geq 0$$
$$\mathcal{E}^\mu \int_0^{S} F(X_u,u)du = \mathcal{E}^\mu \int_0^{T} F(X_u,u)du \ , \ F \geq 0 \text{ on } E \times R_+ \ .$$

Now we apply the lemma to the space-time process (because T is the first hitting time to a finely closed set) and obtain $S \leq T$ a.s. (P^μ). But since $\mu P_S U = \mu P_T U$ one has $S = T$.

<u>Remark.</u> If S and T are stopping times , $\mu P_S = \mu P_T = \nu$, and T is of m.r.e. then obviously S is of m.r.e., too, if for some strictly positive $f \in \mathcal{E}_+$ and all $t \in R_+$

$$\langle \mu P_{T\wedge t}U - \mu P_T U, f\rangle = \langle \mu P_{S\wedge t}U - \mu P_S U, f\rangle < \infty$$

This holds, in particular,with f = 1 if

$\mathcal{E}^\mu S^2 = 2\int_0^\infty dt(\mathcal{E}^\mu \int_{S\wedge t}^{S} du) = 2\int_0^\infty dt(\mathcal{E}^\mu \int_{T\wedge t}^{T} du) = \mathcal{E}^\mu T^2$,because for each t the inequality $\mathcal{E}^\mu \int_{S\wedge t}^{S} du \geq \mathcal{E}^\mu \int_{T\wedge t}^{T} du$ is true by the m.r.e. property of T . So we get from the corollary the final result : <u>if a ROOT time T with $\mu P_T = \nu$ exists and $\mathcal{E}^\mu T^2 < \infty$, then any time S with $\mu P_S = \nu$ and $\mathcal{E}^\mu S^2 = \mathcal{E}^\mu T^2$ is</u>

equal to T a.s. (P^μ).

5) The converse problem.

Under general hypotheses the following theorem does not yield very strong estimates for a stopping time of m.r.e. It is easy to construct examples of a deterministic process and a measure ν where the upper and lower estimate for a m.r.e. time with respect to ν are $+\infty$ and 0, respectively. The corollary, however, makes sure that for a nontrivial class of processes the ROOT times are exactly the times of m.r.e.

Theorem 3. If T is of m.r.e. with respect to μP_T then there exists a barrier B (with sections B_t) such that

$$\inf\{t: t \geq 0,\ X_t \in B_{t+}\} \leq T \leq \inf\{t: t \geq 0,\ X_t \in B_t\} \quad \text{a.s.}(P^\mu),$$

where $B_{t+} = \bigcap_{s>t} B_s$.

Corollary. If the one-point sets are regular for the process (X_t) any time T of m.r.e with respect to μP_T is a ROOT time.

Proof of the corollary. We apply the theorem to a given T of m.r.e.; we assume that B is finely closed in space-time (passage to the fine closure of B does not change the upper bound for T, whereas the lower bound can only decrease). If we show that $B_{s+} \subset B_s$ then we see that both estimates agree and that T is defined by the barrier B.

Now let x be in B_{s+}; this means $(x,h) \in B$ for all $s < h$. The regularity of the set $\{x\}$ implies that almost all paths of the space-time process starting from (x,s) in an arbitrary small time intervall hit the set $\{(x,h): h > s\}$ and hence B. Since B is finely closed we have $(x,s) \in B$ or $x \in B_s$.

Proof of the theorem. Let T be of m.r.e. with respect to μP_T; define the measures λ_t and μ_t for $t \in R_+$ by

$$\langle \lambda_t, f \rangle = \mathcal{E}^{\mu} f\circ X_T \cdot 1_{\{T<t\}}, \quad \langle \mu_t, f \rangle = \mathcal{E}^{\mu} f\circ X_t \cdot 1_{\{T \geq t\}}, \quad f \in \mathcal{E}_+.$$

(The idea of the proof is to show that λ_t and μ_t are something like disjoint; choose B_t as a carrier of λ_t which is not charged by μ_t and B as the set with sections B_t; B is a barrier since λ_t is increasing and μ_t decreasing in t.) First we prove two propositions.

Proposition 3. Let $t \in R_+$ be fixed; for any stopping time S the measures $f \mapsto \mathcal{E}^{\mu} f\circ X_S \cdot 1_{\{t \leq S < T\}}$ and λ_t are disjoint.

Proof. 1) If the assertion is wrong then in a suitably enlarged probability space there exists a stopping time S with

$$0 \neq \mu P_S \leq \lambda_t, \quad t \leq S < T \quad \text{in } \{S < \infty\}$$

and a time H with

$$\mu P_H = \mu P_S, \quad H = T < t \quad \text{in } \{H < \infty\};$$

finally, by our interpretation of balayage order, a time K with

$$K \geq H, \quad \mathcal{E}^{\mu} \int_H^K f\circ X_u \cdot 1_{\{H<t\}} du = \mathcal{E}^{\mu} \int_S^T f\circ X_u \cdot 1_{\{S<\infty\}} du, \quad f \in \mathcal{E}_+.$$

Here the set $A := \{H < K\} \subseteq \{H < t\}$ is disjoint to $B := \{S < T\} = \{S < \infty\}$, since $T < t$ on A and $T > t$ on B; $P^{\mu}(A)$ and $P^{\mu}(B)$ are strictly positive. We define a new stopping time T' by setting

$$T' = K \text{ on } A, \quad T' = S \text{ on } B, \quad T' = T \text{ elsewhere}.$$

2) We shall show that $\mu P_T = \mu P_{T'}$ and $\mu P_{T'\wedge t} U \lneqq \mu P_{T\wedge t} U$, what contradicts the m.r.e. property of T. Let us suppose $f \in \mathcal{E}_+$ is μU-integrable; then one has

$$\langle \mu P_T U - \mu P_{T'} U, f \rangle = \mathcal{E}^{\mu}\left(\int_0^{T'} f\circ X_u du - \int_0^T f\circ X_u du \right) =$$

$= \mathcal{E}^{\mu}(1_A \cdot \int_H^K f \circ X_u du - 1_B \cdot \int_S^T f \circ X_u du) = 0$;

if in addition f is strictly positive we have

$\langle \mu P_{T \wedge t} U - \mu P_{T' \wedge t} U , f \rangle = \mathcal{E}^{\mu}(\int_0^{T' \wedge t} f \circ X_u du - \int_0^{T \wedge t} f \circ X_u du) =$

$= \mathcal{E}^{\mu}(1_A \cdot \int_{H \wedge t}^{K \wedge t} f \circ X_u du - 1_B \cdot \int_{S \wedge t}^{T \wedge t} f \circ X_u du) =$ (since $S \wedge t = T \wedge t = t$ on B) $= \mathcal{E}^{\mu}(1_A \cdot \int_H^{K \wedge t} f \circ X_u du) > 0$, since $H < K \wedge t$ on A and $P^{\mu}(A) > 0$.

<u>Proposition 4</u>. Let ρ, ν be finite measures on E and W a stopping time, for every stopping time S (even in a enlarged probability space) the measure $f \mapsto \mathcal{E}^{\rho} f \circ X_S \cdot 1_{\{S < W\}}$ be disjoint to ν . Then there exists a finely closed carrier A of ν for which $W \leq D_A$ a.s. (P^{ρ}) .

<u>Proof</u>. 1) Let $A_s \subset E$ be a finely closed set for which

(*) $\nu_{\infty}^{s} U := \text{Réd } (\nu - s \cdot \rho) U = (\nu - s\rho) H_{A_s} U$

for each fixed $s \in (0,1)$.

Since ν_{∞}^{s} is carried by A_s and ν_{∞}^{s} tends to ν as s tends to 0 , ν is carried by $A' := \bigcup_{s \text{ rational}} A_s$ and hence by A , the fine closure of A'. Since $D_A = D_{A'} = \inf \{D_{A_s} : s \text{ rational}\}$ it is sufficient to prove that $W \leq D_{A_s}$, for every $s \in (0,1)$.

2) Fix s and choose a time R so that $\bar{\nu}^s := \nu - \nu_{\infty}^{s} = s \cdot \rho P_R$; it follows $\nu H_{A_s} - \nu_{\infty}^{s} = s \cdot \rho P_R H_{A_s}$, hence, in virtue of (*), $s \cdot \rho P_R H_{A_s} = s \cdot \rho H_{A_s}$. This leads to $\rho P_R U \geq \rho H_{A_s} U$ and finally by our lemma to $R \leq D_{A_s}$ a.s. (P^{ρ}).

3) To conclude the proof we show that $W \leq R$ a.s.

From $\nu \geq s \cdot \rho P_R$ it follows

$$s^{-1} \cdot \langle \nu , f \rangle \geq \mathcal{E}^{\rho} f \circ X_R \cdot 1_{\{R < W\}} , \ f \in \mathcal{E}_+$$

and, by hypothesis on W ,

$$\mathcal{E}^{\rho} f\circ X_R \cdot 1_{\{R<W\}} = 0 \text{ for all } f\in\mathcal{E}_+, \text{ i.e. } W \leq R \text{ a.s.}(P^{\rho}).$$

Proof of the theorem (continuation). From the propositions we get by setting $\rho = \mu_t$, $\nu = \lambda_t$:

there exists a finely closed set A_t such that

(i) A_t carries λ_t , (ii) $T \leq t + D_{A_t}\circ\theta_t$ a.s. (P^{μ}).

If one defines B by $B := \bigcup_{t \text{ rational}} A_t \times [t,\infty)$ then B is a barrier and (ii) implies

$$T \leq \inf\{t : t \geq 0, X_t \in B_t\} \text{ a.s.}$$

Conversely, by (i) we have for almost all (P^{μ}) $\omega\in\Omega$:

$$X_{T(\omega)}(\omega) \in A_t \text{ for all } t \text{ rational, } t > T(\omega)$$

and, because of $A_t \subset B_t$ for t rational,

$$X_{T(\omega)}(\omega) \in B_{u+} \text{ if } T(\omega) = u \ ;$$

this means $T \geq \inf\{t : t \geq 0, X_t \in B_{t+}\}$

as was to be proved.

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Hermann Rost

D 69 Heidelberg (West Germany)

Institut für Angewandte Mathematik

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