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A PROBABILISTIC APPROACH TO NON-LINEAR DIRICHLET PROBLEM

By Masao Nagasawa

(1) Given a continuous strong Feller process (x_t, P_x) on a nice topological state space S (which will be called the base process), an open set D of S , a bounded continuous non-negative function $c(x)$ in D (put $c = 0$ on the complement D^c of D), and bounded continuous functions $q_n(x)$ in D ($q_n = 0$ on D^c) satisfying

$$\sum_{n=0}^{\infty} |q_n(x)| = 1, \quad \text{for } x \in D.$$

Let us consider a non-linear Dirichlet problem, given a bounded measurable function ϕ on the boundary ∂D ,

$$(1) \quad \begin{cases} \mathbf{A}u(x) + c(x) \left(\sum_{n=0}^{\infty} q_n(x) u(x)^n - u(x) \right) = 0, & \text{in } D, \\ u(b) = \phi(b), & \text{on } \partial D, \end{cases}$$

where \mathbf{A} is Dynkin's characteristic operator for the base process (x_t, P_x) .

We will show that solutions (not necessarily unique) of the non-linear Dirichlet problem can be obtained in terms of a branching Markov process under the condition $\|\phi\| \leq 1$.

(2) As is well known in the theory of Markov processes, the unique solution of linear Dirichlet problem

$$(2) \quad \begin{cases} \mathbf{A}u(x) = 0 & \text{in } D, \\ u(b) = \phi(b) & \text{on } \partial D, \\ \lim_{\substack{x \in D \\ x \rightarrow b \in \partial D}} u(x) = \phi(b), & \text{if } b \text{ is regular and } \phi \text{ is} \\ & \text{continuous at } b, \end{cases}$$

is obtained in terms of the base process under the assumption

$$P_x[T < \infty] = 1, \text{ for } x \in \bar{D},$$

where $T = \inf\{t \geq 0; x_t \in \partial D\}$ is the first hitting time to the boundary ∂D . One expression is

$$u(x) = E_x[\phi(x_T)],$$

(cf. e.g. [1] p.32, Theorem 13.1). We have another expression in terms of the stopped process at the boundary

$$\bar{x}_t = x_{t \wedge T}.$$

Let \bar{P}_t be the transition probability of \bar{x}_t , and f be a bounded measurable function on \bar{D} which coincides with ϕ on the boundary. Then

$$(3) \quad u(x) = \lim_{t \rightarrow \infty} \bar{P}_t f(x)$$

gives the same solution. The solution does not depend on the value of f in D . For, since $P_x[T < \infty] = 1$,

$$\begin{aligned} u(x) &= \lim_{t \rightarrow \infty} E_x[f(\bar{x}_t)] = E_x[\lim_{t \rightarrow \infty} f(\bar{x}_t)] \\ &= E_x[\phi(x_T)]. \end{aligned}$$

We will express solutions of (1) in the form of (3) taking the transition probability of a branching Markov process and \hat{P} instead of \bar{P}_t and f (\hat{f} will be defined by (4)).

(3) For simplicity, we assume $q_n(x) \geq 0$, but the same arguments can be carried over for general case.

Let (X_t, P_x) be (\bar{x}_t, c, q_n) -branching Markov process (*) on S , where \bar{x}_t is the stopped process of x_t at ∂D and

$$S = \bigcup_{n=0}^{\infty} \bar{D}^n \cup \{\Delta\}. (**)$$

For a bounded measurable function f on \bar{D} , we define \hat{f} on S by

(*) Cf. [2], [3]. Here, we take $\pi_n(x, dy) = \delta_{(x, \dots, x)}(dy)$, i.e. n -particles created at x start continuously.

(**) \bar{D}^n is the n -fold Cartesian product of \bar{D} , and $\bar{D}^0 = \{\delta\}$ an extra point.

$$(4) \quad \begin{cases} \hat{f}(\mathbf{x}) = f(x_1) \times \dots \times f(x_n), & \text{when } \mathbf{x} = (x_1, \dots, x_n), \\ \hat{f}(\delta) = 1, \\ \hat{f}(\Delta) = 0. \end{cases}$$

If $\|f\| \leq 1$, \hat{f} is bounded on \mathbf{S} .

Let \mathbf{P}_t be the transition probability of the branching Markov process. Taking a bounded measurable function f on \bar{D} with the uniform norm $\|f\| \leq 1$, we assume the existence of the limit

$$(5) \quad u(\mathbf{x}) = \lim_{t \rightarrow \infty} \mathbf{P}_t \hat{f}(\mathbf{x}).$$

(We will discuss the existence of the limit in the next section.)

(I) $u(\mathbf{x})$ is \mathbf{P}_t -invariant.

$$\text{For, } \mathbf{P}_s u(\mathbf{x}) = \lim_{t \rightarrow \infty} \mathbf{P}_{t+s} \hat{f}(\mathbf{x}) = u(\mathbf{x}).$$

(II) $u(\mathbf{x})$ is multiplicative, i.e., $u(\mathbf{x}) = \hat{u}(\mathbf{x})$.

For, since \mathbf{P}_t satisfies the branching property

$$\mathbf{P}_t \hat{f}(\mathbf{x}) = \widehat{\mathbf{P}_t \hat{f}}(\mathbf{x}),$$

we have

$$u(\mathbf{x}) = \lim_{t \rightarrow \infty} \widehat{\mathbf{P}_t \hat{f}}(\mathbf{x}) = \hat{u}(\mathbf{x}).$$

(III) If \hat{f} belongs to the domain of the weak generator G of \mathbf{P}_t , then f belongs to the domain of the weak generator of $\bar{\mathbf{P}}_t^0$, the transition probability of the killed process of $\bar{\mathbf{x}}_t$ by $\exp(-\int_0^t c(\bar{\mathbf{x}}_s) ds)$.

Proof. $\mathbf{P}_t \hat{f}$ satisfies S-equation; for $x \in \bar{D}$,

$$\mathbf{P}_t \hat{f}(x) = \bar{\mathbf{P}}_t^0 f(x) + \int_0^t ds \int \bar{\mathbf{P}}_s^0(x, dy) c(y) F(y, \mathbf{P}_{t-s} \hat{f}).$$

where $F(x, u) = \sum_{n=0}^{\infty} q_n(x) u(x)^n$. Therefore

$$\frac{P_t \hat{f}(x) - f(x)}{t} = \frac{\bar{P}_t^0 f(x) - f(x)}{t} + \frac{1}{t} \int_0^t ds \bar{P}_s^0 (cF(\cdot, P_{t-s} \hat{f}))(x).$$

The second term of the right hand side converges to $cF(x, f)$ when t tends to zero. Therefore, if the left hand side converges, then so does the first term of the right hand side.

(IV) $u(x)$ defined by (5) belongs to the domain of the weak generator G of P_t and $Gu(x) = 0$.

Since u is P_t -invariant, u belongs to the domain of G , and $Gu(x) = 0$.

Therefore, $u(x)$, $x \in \bar{D}$ belongs to the domain of the weak generator of \bar{P}_t^0 , and by Kac's theorem it belongs to the domain of the weak generator of \bar{P}_t . Thus we have, by (II), (III) and (IV),

PROPOSITION 1. If $u(x)$, $x \in \bar{D}$, defined by (5) exists, then it satisfies

$$Au(x) + c(x) \left\{ \sum_{n=0}^{\infty} q_n(x) u(x)^n - u(x) \right\} = 0 \text{ in } D,$$

and $u(b) = f(b)$, $b \in \partial D$, where A is Dynkin's characteristic operator of the base process.

Remark. Even when $\|f\| \not\leq 1$, if the limit in (5) exists and if $F(\cdot, f)$ is bounded, then (I) \sim (IV) and Proposition 1 hold.

(4) Let τ be the killing time of the base process by $\exp(-\int_0^t c(x_s) ds)$ and T be the first hitting time to the boundary ∂D , and we assume

$$(6) \quad P_x [T < \tau] \geq \epsilon > 0, \text{ for all } x \in \bar{D}.$$

Remark. (6) is satisfied if $E_x [\exp(-\|c\| T)] \geq \epsilon$. For, $P_x [T < \tau] = E_x [\exp(-\int_0^T c(x_s) ds)] \geq E_x [\exp(-\|c\| T)]$.

LEMMA 1. Under the assumption (6)

$P_x [X_t^i \in \partial D \text{ for all } i \text{ or } X_t = \delta \text{ at some } t < \infty, \text{ or the}$
number of particles in } D \text{ tends to } \infty \text{ when } t \rightarrow \infty] = 1,
where } X_t = (X_t^1, \dots, X_t^n(t)).

Proof. Let σ be the first hitting time to \bar{D}^m , and define sequences of Markov times $\{\sigma_n\}$ and $\{\eta_n\}$ by

$$\begin{aligned} \sigma_1 &= \sigma, & \eta_1 &= \sigma_1 + \tau \circ \theta_{\sigma_1}, \\ \sigma_2 &= \eta_1 + \sigma_1 \circ \theta_{\eta_1}, & \eta_2 &= \sigma_2 + \tau \circ \theta_{\sigma_2}, \end{aligned}$$

and so on. Then

$$\begin{aligned} &P_x [X_t \text{ visits } \bar{D}^m \text{ infinitely often}] \\ &= P_x \left[\bigcap_n \{ \sigma_n < +\infty \} \right] \\ &= \lim_{n \rightarrow \infty} P_x [\sigma_n < +\infty] \\ &\leq \lim_{n \rightarrow \infty} (1-\varepsilon)(1-\varepsilon^m)^n = 0, \end{aligned}$$

because

$$\begin{aligned} P_x [\sigma_1 < +\infty] &\leq 1 - P_x [T < \tau] \leq 1 - \varepsilon, \\ P_x [\sigma_2 < +\infty] &= E_x [P_{X_{\sigma_1}} [\tau + \sigma_1 \circ \theta_{\tau} < +\infty]; \sigma_1 < +\infty] \\ &\leq E_x \left[\left(1 - \prod_{i=1}^m P_{X_{\sigma_1}^i} [T < \tau] \right); \sigma_1 < +\infty \right] \\ &\leq (1-\varepsilon)(1-\varepsilon^m), \end{aligned}$$

and so on. Thus we have the lemma.

As a corollary of Lemma 1, we have

PROPOSITION 2. Given a measurable function } \phi \text{ on the}
boundary } \partial D \text{ with } \|\phi\| \leq 1, \text{ set}

$$(7) \quad f = \begin{cases} \phi & \text{on } \partial D \\ 0 & \text{in } D. \end{cases}$$

Then,

$$(8) \quad u(x) = \lim_{t \rightarrow \infty} \mathbf{E}_x[\widehat{f}(X_t)], \quad x \in \bar{D}, \text{ exists.}$$

PROPOSITION 3. Given ϕ and define f as in Proposition 2,
then u defined by (8) satisfies

$$(9) \quad \lim_{\substack{x \in D \\ x \rightarrow b \in \partial D}} u(x) = \phi(b),$$

if b is a regular point of the boundary ∂D and if ϕ is
continuous at b . (*)

Proof.

$$|u(x) - \phi(b)| \leq \lim_{t \rightarrow \infty} \mathbf{E}_x[|\widehat{f}(X_t) - \phi(b)|].$$

Put $B = \{X_t \text{ hits first to the boundary before branching}\}$.

$$\begin{aligned} I &= \lim_{t \rightarrow \infty} \mathbf{E}_x[|\widehat{f}(X_t) - \phi(b)|; B] \\ &= \lim_{t \rightarrow \infty} \mathbf{E}_x[|f(\bar{x}_t) - \phi(b)|; B], \end{aligned}$$

because $X_t = \bar{x}_t = x_{t \wedge T}$ on B ;

$$\begin{aligned} &\leq \mathbf{E}_x[|\phi(x_T) - \phi(b)|; T < \tau] \\ &\leq \mathbf{E}_x[|\phi(x_T) - \phi(b)|], \end{aligned}$$

where \mathbf{E}_x is the expectation with respect to the base process.
 If b is regular and if ϕ is continuous at b , then there
 exists a neighbourhood U_b of b and

$$(10) \quad \mathbf{E}_x[|\phi(x_T) - \phi(b)|] < \epsilon \text{ for all } x \in U_b,$$

(cf. e.g. [1] p.32, Theorem 13.1). Thus we have $I < \epsilon$.

$$\begin{aligned} II &= \lim_{t \rightarrow \infty} \mathbf{E}_x[|\widehat{f}(X_t) - \phi(b)|; B^c] \\ &\leq 2\|\phi\| \mathbf{P}_x[B^c] \leq 2\mathbf{P}_x[T \geq \tau] \leq 2(1 - \mathbf{P}_x[T < \tau]) \\ &\leq 2(1 - \mathbf{P}_x[T < s < \tau]), \text{ for any } s > 0. \end{aligned}$$

(*) The regularity is for the base process.

$$\begin{aligned} P_x[T < s < \tau] &= P_x[\exp(-\int_0^s c(x_s) ds), T < s] \\ &\geq \exp(-\|c\|s) P_x[T < s]. \end{aligned}$$

Take s sufficiently small so that $\exp(-\|c\|s) \geq 1 - \epsilon$. Since $P_x[T < s]$ is lower semicontinuous in x (cf. e.g. [1] p.28 Lemma 13.2) and $P_b[T < s] = 1$ because b is regular, there exists a neighbourhood U'_b of b such that

$$P_x[T < s] \geq 1 - \epsilon, \quad \text{for all } x \in U'_b.$$

Therefore

$$P_x[T < s < \tau] \geq (1-\epsilon)^2 > 1 - 2\epsilon.$$

Thus we have $II < 4\epsilon$, and

$$|u(x) - \phi(b)| < 5\epsilon, \quad \text{for all } x \in U_b \cap U'_b.$$

Since ϵ is arbitrary, (9) is proved.

Remark. We assumed $\|\phi\| \leq 1$ in Proposition 3. However, if ϕ is bounded and if the limit exists in (8), then (9) is valid.

Thus we have

THEOREM. Under the assumption (6), there exists

$$u(x) = \lim_{t \rightarrow \infty} E_x[\hat{f}(X_t)], \quad x \in \bar{D},$$

where f is defined by (7) for a given ϕ on ∂D ($\|\phi\| \leq 1$), and u is a solution of non-linear Dirichlet problem (1) satisfying the boundary limit property (9).

We proved Theorem in the case of $q_n \geq 0$. When q_n is not non-negative, we can prove the theorem using the branching Markov process with sign (cf. [3],[4]) instead of usual branching Markov process.

Moreover, there is no difficulty to generalize the result to the system

$$\begin{cases} \mathbf{A}_i u_i + c_i \left\{ \sum_{n_1=0}^{\infty} \dots \sum_{n_k=0}^{\infty} q_{n_1 \dots n_k}^i (u_1)^{n_1} \dots (u_k)^{n_k} - u_i \right\} = 0 \\ \quad \text{in } D, \text{ for } i = 1, 2, \dots, k, \\ u_i(b) = \phi_i(b) \text{ on } \partial D, \end{cases}$$

where $\sum \dots \sum q_{n_1 \dots n_k}^i(x) = 1$, $x \in D$ ($= 0$ outside D).

To do this, what we need is just to introduce an appropriate branching Markov processes (cf. [3] pp.505-507).

(5) Instead of (7), let us take

$$(11) \quad f = \begin{cases} \phi & \text{on } \partial D, \\ g & \text{in } D, \end{cases}$$

as an initial value, where g is a measurable function in D with $\|g\| \leq 1$. When $\|g\| < 1$, the limit

$$(12) \quad u(x) = \lim_{t \rightarrow \infty} \mathbf{E}_x [\hat{f}(\mathbf{X}_t)]$$

exists and it does not depend on the choice of the initial value g in D . Let n_t^D be the number of particles in D at t . By lemma 1,

$$(13) \quad u(x) = \lim_{t \rightarrow \infty} \mathbf{E}_x [\hat{f}(\mathbf{X}_t); X_s^i \in \partial D \text{ for all } i \text{ or } \mathbf{X}_s = \delta \\ \text{at some } s < \infty] \\ + \lim_{t \rightarrow \infty} \mathbf{E}_x [\hat{f}(\mathbf{X}_t); n_s^D \uparrow \infty \text{ when } s \uparrow \infty],$$

where the second term is equal to zero when $\|g\| < 1$ and the first term does not depend on g .

In general, the limit in (12) depends on the choice of the initial value g in D if

$$(14) \quad \mathbf{P}_x [n_t^D \uparrow \infty \text{ when } t \uparrow \infty] > 0$$

at some point x_0 in D . For example, taking $\phi \equiv 1$ on the boundary for simplicity, if we take $f_1 \equiv 1$ on \bar{D} , then

$$u_1(x) = \lim_{t \rightarrow \infty} \mathbf{E}_x[\hat{f}_1(\mathbf{x}_t)] = 1, \text{ for all } x \in \bar{D}, (*)$$

while if we take $f_0 = 1$ on ∂D ($= 0$ in D), then

$$u_0(x) = \lim_{t \rightarrow \infty} \mathbf{E}_x[\hat{f}_0(\mathbf{x}_t)]$$

takes value less than one at $x_0 \in D$, because of (13) and (14). Actually, $u_0(x)$ is the extinction probability of particles from D (cf. [5], [6]).

Remark. In order to express the stochastic solution, defined in (5) or (8), in terms of "the first hitting time to the boundary", we must introduce a vector of hitting times of every branches of the branching Markov process. When $x_t^i \in \partial D$ for all i or $\mathbf{x}_t = \delta$ at some $t < \infty$, let T_i be the first hitting time of x_t^i to the boundary, where $\mathbf{x}_t = (x_t^1, \dots, x_t^n(t))$. Then put

$$\mathbf{T} = (T_1, T_2, \dots, T_n),$$

(under the assumption, the total number of particles is finite, say, n). When the number of particles in D tends to infinity, let's put $\mathbf{T} = \infty$. Let us call \mathbf{T} the first hitting time of the branching Markov process to the boundary. Then we have

$$(15) \quad \begin{aligned} u(x) &= \lim_{t \rightarrow \infty} \mathbf{E}_x[\hat{f}(\mathbf{x}_t)] \\ &= \mathbf{E}_x[\hat{f}(\mathbf{x}_{\mathbf{T}}); \mathbf{T} < \infty] = \mathbf{E}_x[\hat{\phi}(\mathbf{x}_{\mathbf{T}}); \mathbf{T} < \infty], \end{aligned}$$

where

$$\mathbf{x}_{\mathbf{T}} = (x_{T_1}^1, x_{T_2}^2, \dots, x_{T_n}^n).$$

(*) We assume here that the branching Markov process does not explode in finite time. When explosion occurs, $u_1(x) = 1$ - explosion probability.

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