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GENERATION OF σ-FIELDS BY STEP PROCESSES

by P.A. Meyer

This paper was drafted during the stay of Frank Knight at Strasbourg University in 1974, as a result of many discussions with him on the subject of his paper " Markov processes with deterministic germ fields"([1]) . We hoped to write more about it, and publish something together - this is why English is used here instead of the customary Alsacian - but we found we had nothing more to add, and Frank Knight didn't accept to put his name below the title there † , so let me sign alone and tell the reader that nothing of this paper could have been written without Knight's contribution.

Let $(E,\underline{\underline{E}})$ be any countably generated measurable space. We say that a process (X_t), defined on some measurable space $(\Omega,\underline{\underline{F}})$ and with values in $(E,\underline{\underline{E}})$ is a step process if for every ω, every t, there exists some ε>0 such that $X_t(\omega)=X_{t+s}(\omega)$ for $0\leqq s<\varepsilon$ (ε may depend on t and ω). That is, X is a right continuous process in the discrete topology of E (though usually $\underline{\underline{E}}$ isn't the Borel field of the discrete topology). Note that the process isn't required to have left limits in the discrete topology : "jumps" are allowed to accumulate from the left, so that the word " step process" is slightly distorted from its usual meaning.

The restriction that $\underline{\underline{E}}$ should be countably generated is basic. We aren't going to comment on this point : without it there isn't one single reasonable word that can be said on the subject.

Our purpose here is to decide whether, given an increasing family $(\underline{\underline{F}}_t)$ of σ-fields on some probability space $(\Omega,\underline{\underline{F}},P)$, we can find a step process (X_t) which generates $(\underline{\underline{F}}_t)$ " after discarding a null set" (a.d.n.s.) . This must be carefully explained : we are allowed to throw away one fixed (measurable) null set once and for all, and on the remaining part of Ω we must have $\underline{\underline{F}}_t=\sigma(X_s,\ s\leq t)$ without any kind of completion relative to P. So P plays a very small role, though it couldn't quite be dispensed with.

Our first remark is the fact that it makes no difference whether we allow our step process to take its values in an arbitrary (countably generated) measurable space $(E,\underline{\underline{E}})$, or just in the interval $[0,1] = \mathbb{I}$. Indeed, denote by (A_n) a sequence of elements of $\underline{\underline{E}}$ which generates $\underline{\underline{E}}$, and by φ the mapping $\Sigma_n\ 3^{-n}I_{A_n}$ from E to $\mathbb{I}$. It is well known that $\underline{\underline{E}}$ is generated by φ . Hence (X_t) generates exactly the same family of σ-fields as the process $(Y_t)=(\varphi(X_t))$ with values in $\mathbb{I}$. On the other hand, if (X_t) is a step process so is (Y_t).

SOME NECESSARY CONDITIONS

First of all, if we have $\underline{\underline{F}}_t=\sigma(X_s,\ s\leq t\ )$ with (X_t) a step process, then by right continuity we have $\underline{\underline{F}}_t=\sigma(X_{rt}\ ,\ r$ rational, $0<r\leq 1)$, hence $\underline{\underline{F}}_t$ <u>is countably generated for every</u> t .

Next, let $\mathbb{Q}$ be the set of rationals in $[0,\infty[$, and let h be a random variable on Ω, measurable with respect to $\underline{\underline{F}}_{t+}$. There exists a function h_n measurable on $\mathbb{E}^{\mathbb{Q}}$ (given its product σ-field) such that $h(\omega) = h_n(X_{r\wedge(t+\frac{1}{n})}(\omega),\ r$ rational). Since X is a step process, we have for n large enough (depending on ω) $X_{r\wedge(t+\frac{1}{n})}(\omega) = X_{r\wedge t}(\omega)$ for <u>all</u> $r\in\mathbb{Q}$: namely, if $X_{\cdot}(\omega)$ is equal to $X_t(\omega)$ on $[t,t+\varepsilon[$, the stopped paths at t and $t+\frac{1}{n}$ are equal for $n>1/\varepsilon$. Therefore

$$h(\omega) = \lim_{n\to\infty} h_n(\ X_{r\wedge t}(\omega),\ r \text{ rational }\ )$$

is $\underline{\underline{F}}_t$-measurable. <u>Hence</u> $\underline{\underline{F}}_t=\underline{\underline{F}}_{t+}$ <u>for any</u> t. A nice consequence is the fact that we don't need to distinguish between strict sense and wide sense stopping times T, nor between the corresponding σ-fields $\underline{\underline{F}}_T$ and $\underline{\underline{F}}_{T+}$.

Let T be a stopping time. The σ-field $\underline{\underline{F}}_{T-}$ is generated by $\underline{\underline{F}}_0$ and all events $A\cap\{s<T\}$, $A\in\underline{\underline{F}}_s$. We may write as above $I_A=h(X_{r\wedge s}$, r rational). Then $I_{A\cap\{s<T\}} = I_{\{s<T\}}h(X_{r\wedge s\wedge T}$, r rational), and we see that $\underline{\underline{F}}_{T-}$ is contained in the σ-field generated by T and the stopped process $X_t^T=X_{t\wedge T}$. Since $\underline{\underline{F}}_{T+}$ is equal to $\bigcap_n \underline{\underline{F}}_{(T+\frac{1}{n})-}$, we deduce from it that for every $\underline{\underline{F}}_{T+}$-measurable r.v. h and every n there exists a function h_n on $\mathbb{R}\times\mathbb{R}^{\mathbb{Q}}$ such that $h(\omega)=h_n(T(\omega),X_r^{T+1/n}(\omega)$, r rat$^{\ell}$) As above, the stopped path at $T+1/n$ being equal to the stopped path at T for n large, we deduce from it that h is measurable with respect to $\sigma(T,X_t^T$, $t\in\mathbb{R}_+)$. Hence $\underline{\underline{F}}_{T+}\subset\sigma(T,$ X_t^T, $t\in\mathbb{R}_+)$, equality follows obviously , and we have proved that $\underline{\underline{F}}_{T+}=\underline{\underline{F}}_T$ *is countably generated for every* T , a significant stengthening of our first result.

We are going now to prove that the two necessary conditions we have underlined are " almost " sufficient.

SUFFICIENCY OF THE CONDITIONS A.D.N.S.

Let $(\Omega,\underline{\underline{F}})$ be a measurable space with an increasing family of σ-fields $\underline{\underline{F}}_t$, and a probability law P. Since we are interested only in $(\underline{\underline{F}}_t)$, we may assume that $\underline{\underline{F}}=\underline{\underline{F}}_\infty$ and collapse the atoms of $\underline{\underline{F}}$ into points. Our basic assumption is that P is carried by some $\Omega_0\subset\Omega$ such that the induced σ-field $\underline{\underline{F}}|_{\Omega_0}$ is Blackwell - since $\underline{\underline{F}}=\underline{\underline{F}}_\infty$ will be assumed to be countably generated, this means that P is an abstract Radon measure in the sense of Yen [2]. Our aim is

THEOREM 1. *Assume* 1) $\underline{\underline{F}}_t=\underline{\underline{F}}_{t+}$ *for every* t , 2) *for any stopping time* T $\underline{\underline{F}}_T$ *is countably generated*. *Then the family is generated by a step process after discarding a null set*.

We may assume that $(\Omega,\underline{\underline{F}})$ itself is a Blackwell space by discarding $\Omega\backslash\Omega_0$. This is a trivial step : the interesting things happen on a Blackwell space. According to standard results, we may assume that Ω

is an analytic (Suslin) subset of the interval $\mathbf{I}=[0,1]$, and $\underline{\underline{F}}$ is the Borel field of Ω. See for inst. [3], II .25 p.79.

It is now convenient - though not essential - to use a remark of Yen [4] . Let (r_n) be an enumeration of the rationals (including 0) and let H_{nm} be a sequence generating $\underline{\underline{F}}_{r_n-}$, understanding $\underline{\underline{F}}_{0-}$ as $\underline{\underline{F}}_0$. Let also $(n,m)\mapsto a(n,m)$ be a 1-1 mapping from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$. We set

$$A_t(\omega) = \Sigma_{n,m}\, 3^{-a(n,m)} I_{[r_n,\infty]}(t) I_{H_{nm}}(\omega) \quad (0\leqq t\leqq\infty)$$

This is a right continuous, increasing process. On the other hand, the σ-field on $[0,\infty]\times\Omega$ generated by $(t,\omega)\mapsto A_t(\omega)$ is also generated by the sets $[r_n,\infty]\times H_{nm}$, so it is just the <u>predictable</u> σ-field. Let T be any stopping time, and Y be a $\underline{\underline{F}}_{T-}$-measurable random variable . There exists a predictable process $(Z_t)_{0\leqq t\leqq+\infty}$ such that $Y=Z_T$ ([3], IV.67 b), p.199). Since the predictable σ-field is generated by A, we may find a Borel function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that $Z=\varphi(A)$, hence $\varphi(A_T)$ and $\underline{\underline{F}}_{T-}$ <u>is generated by</u> A_T <u>for any stopping time</u> T. This includes the fact that A_0 generates $\underline{\underline{F}}_0$.

Next, we need two lemmas :

LEMMA 1. <u>Let</u> ω <u>and</u> ω' <u>belong to the same atom of</u> $\underline{\underline{F}}_t$. <u>Then they belong to the same atom of</u> $\underline{\underline{F}}_{t+\varepsilon}$ <u>for some</u> $\varepsilon>0$.

Assume the contrary : for any n there exists $L_n \in \underline{\underline{F}}_{t+1/n}$ such that $\omega\in L_n$, $\omega'\notin L_n$. Setting $L = \liminf_n L_n$ we have $L\in\underline{\underline{F}}_{t+}$, $\omega\in L$, $\omega'\notin L$. Since $\underline{\underline{F}}_t=\underline{\underline{F}}_{t+}$, ω and ω' do not belong to the same atom of $\underline{\underline{F}}_t$.

LEMMA 2. <u>There exists a measurable mapping</u> $h : (\Omega,\underline{\underline{F}}_0)$ <u>to</u> $(\Omega,\underline{\underline{F}})$ <u>such that</u> , <u>for</u> P-<u>a.e.</u> ω, $h(\omega)$ <u>belongs to the same atom of</u> $\underline{\underline{F}}_0$ <u>as</u> ω .

Let Q be the image measure $A_0(P)$ on $\mathbb{R}$. Imbed Ω as an analytic subset of $\mathbf{I}$ and denote by G the graph of A_0 in $\mathbb{R}\times\mathbf{I}$, that is the set of all $(r,w)\in\mathbb{R}\times\mathbf{I}$ such that $w\in W$, $r=A_0(w)$. One easily checks that G is an analytic set in $\mathbb{R}\times\mathbb{I}$. According to a standard section theorem ([3], III.44, p.102) there exists a borel set $K\subset\mathbb{R}$ carrying Q, and a Borel mapping k from K to $\mathbb{I}$ such that $(r,k(r))\in G$ for $r\in K$. Choose $\omega_0 \in\Omega$ and set

$h(\omega)=\omega_0$ if $A_0(\omega)\notin K$, an event which belongs to $\underline{\underline{F}}_0$ and has probability 0 ,

$h(\omega)= k(A_0(\omega))$ otherwise.

Then h is $\underline{\underline{F}}_0$-measurable, and we have P-a.s. that $A_0(\omega)=A_0(h(\omega))$, meaning that ω and $h(\omega)$ belong to the same atom of $\underline{\underline{F}}_0$.

We can now start the construction. Consider the random variable

$$T(\omega) = \inf \{ r \text{ rational} : A_r(\omega)\neq A_r(h(\omega)) \}$$

This is obviously a wide sense stopping time, hence a strict sense stopping time since the family $(\underline{\underline{F}}_t)$ is right continuous. According to lemma 2 we a.s. have $A_0(\omega)=A_0(h(\omega))$, and according to lemma 1 $A_r(\omega)= A_r(h(\omega))$ for r small enough, that is $T(\omega)>0$. Set $T(\omega)=u$. Since $A_r(\omega)=A_r(h(\omega))$ for $r<u$, ω and $h(\omega)$ belong to the same atom of $\underline{\underline{F}}_{u-}$, and this in turn implies $A_u(\omega)=A_u(h(\omega))$. So we have - now without assuming $T(\omega)>0$

$$A_T(\omega) = A_0(\omega)I_{\{T(\omega)=0\}} + A_{T(\omega)}(h(\omega))I_{\{T(\omega)>0\}}$$

Since h is $\underline{\underline{F}}_0$-measurable, A_T belongs to the σ-field generated by T and $\underline{\underline{F}}_0$. On the other hand, A_T generates $\underline{\underline{F}}_{T-}$ and we have proved

LEMMA 3. <u>There exists a stopping time</u> T <u>such that</u> T>0 <u>a.s.</u>, <u>and that</u> $\underline{\underline{F}}_{T-}$ <u>is generated by</u> T <u>and</u> $\underline{\underline{F}}_0$.

This signifies intuitively that " nothing happens between 0 and T" : T is the "first jump time after 0" . This will be the starting point of a transfinite induction.

Set $\underline{\underline{G}}_t=\underline{\underline{F}}_{T+t}$. Then it is well known that a r.v. S is a stopping time (a wide sense stopping time) of $(\underline{\underline{G}}_t)$ if and only if T+S is a stopping time (a wide sense s.t.) of $(\underline{\underline{F}}_t)$. Hence stopping times and wide sense stopping times of $(\underline{\underline{G}}_t)$ are the same, implying that $\underline{\underline{G}}_t = \underline{\underline{G}}_{t+}$. On the other hand, if S is a stopping time of $(\underline{\underline{G}}_t)$, $\underline{\underline{G}}_S = \underline{\underline{F}}_{T+S}$ is countably generated, and finally $(\underline{\underline{G}}_t)$ has the same properties à $(\underline{\underline{F}}_t)$. The argument that led to the construction of T can be applied again, and proceeding inductively we get the following result.

LEMMA 4. <u>There exists a family</u> (T_α) <u>of stopping times, indexed by the countable ordinal, such that</u>

i) $T_0=0$,

ii) $\alpha<\beta \Rightarrow T_\alpha \leqq T_\beta$,

iii) <u>if</u> β <u>is a limit ordinal,</u> $T_\beta = \sup_{\alpha<\beta} T_\alpha$,

iv) $T_{\alpha+1} > T_\alpha$ <u>a.s. on</u> $\{T_\alpha < \infty\}$,

v) $\underline{\underline{F}}_{T_{\alpha+1}-}$ <u>is generated by</u> $\underline{\underline{F}}_{T_\alpha}$ <u>and</u> $T_{\alpha+1}$.

According to a well known result, there exists an ordinal δ such that $T_\delta=+\infty$ a.s. ([3], 0.8, p.6 and errata sheet). By going to the next limit ordinal if necessary, we may assume δ is a limit ordinal. We now discard the null sets $\{T_\delta<\infty\}$, and $\{T_\alpha=T_{\alpha+1}<\infty\}$ for $\alpha<\delta$. We denote by D the countable set $\{\alpha: \alpha<\delta\}$ and denote by E the product space $D\times\mathbb{R}$ with the obvious σ-field (this measurable space is isomorphic to $\mathbb{R}$). For each α, let f_α be a real valued random variable generating $\underline{\underline{F}}_{T_\alpha}$. We define the following step process (X_t) with values in E

$$X_t = (\alpha, f_\alpha) \text{ if } T_\alpha \leqq t < T_{\alpha+1}$$

and show it generates the family $(\underline{\underline{F}}_t)$ on the set that hasn't been discarded. For simplicity we change notation, Ω being reduced to the remaining "good" set. Let $\underline{\underline{G}}_t$ be $\sigma(X_s, s\leqq t)$.

We prove $\underline{\underline{G}}_t \subset \underline{\underline{F}}_t$, that is, (X_t) is adapted. Let h be any measurable function on E, and h_α be the Borel function on $\mathbb{R}$ defined by $h_\alpha(x) = h(\alpha,x)$. Then we have

$$h\circ X_t = \Sigma_{\alpha<\delta}\, h_\alpha \circ f_\alpha I_{\{T_\alpha \leqq t\}} I_{\{t<T_{\alpha+1}\}}$$

and $\{t<T_{\alpha+1}\}$ belongs to $\underline{\underline{F}}_t$, as does $h_\alpha \circ f_\alpha I_{\{T_\alpha \leq t\}}$ since f_α is $\underline{\underline{F}}_{T_\alpha}$-measurable.

Conversely, the stopping times T_α are stopping times of $(\underline{\underline{G}}_t)$, since they are the successive jump times of (X_t). The random variable f_α is $\underline{\underline{G}}_{T_\alpha}$-measurable, since $X_{T_\alpha}=(\alpha,f_\alpha)$, hence $\underline{\underline{F}}_{T_\alpha}\subset\underline{\underline{G}}_{T_\alpha}$. On the other hand, for any stopping time T of $(\underline{\underline{F}}_t)$ such that $T_\alpha \leq T \leq T_{\alpha+1}$, $\underline{\underline{F}}_{T-}$ is generated by $\underline{\underline{F}}_{T_\alpha}$ and T (the argument has been given above for $\alpha=0$ in the proof of lemma 3, the key idea of which was the fact that $A_{\bullet}$ doesn't increase between 0 and T_1. It can be extended to any α).
Let L be an element of $\underline{\underline{F}}_{t-}$. In order to prove $L\in\underline{\underline{G}}_{t-}$ we need only show $L\cap\{T_\alpha \leq t \leq T_{\alpha+1}\}\in\underline{\underline{G}}_{t-}$ for every α . Call this event M, set $T=T_\alpha \vee (T_{\alpha+1}\wedge t)$. We have $\{t\leq T_{\alpha+1}\} = \{T_{\alpha+1}<t\}^c \in\underline{\underline{F}}_{t-}$, $\{T_\alpha<t\}\in\underline{\underline{F}}_{t-}$, hence $M\in\underline{\underline{F}}_{t-}$, and $M\cap\{t\leq T\}\in\underline{\underline{F}}_{T-}$. Now this is just M, so M belongs to the σ-field generated by $\underline{\underline{F}}_{T_\alpha}$ and T, therefore to that generated by $\underline{\underline{G}}_{T_\alpha}$ and T. On the other hand, M is contained in $\{T_\alpha<t\}\in\underline{\underline{G}}_t$, and the intersection of this set with any element of $\underline{\underline{G}}_{T_\alpha}$, or of $\sigma(T)$, is $\underline{\underline{G}}_t$-measurable.
Finally, we have proved $\underline{\underline{F}}_{t-}\subset\underline{\underline{G}}_{t-}$, taking right limits $\underline{\underline{F}}_{t+}\subset\underline{\underline{G}}_{t+}$, and since both families are right continuous $\underline{\underline{F}}_t\subset \underline{\underline{G}}_t$. This concludes the proof of theorem 1.

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