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TAYLOR EXPANSION OF A POISSON MEASURE

Wilhelm von Waldenfels

Abstract. Denote by $\mathcal{P}(\mathcal{Q})$ the Poisson measure associated to a positive Radon measure $\mathcal{Q}$ on a locally compact space countable at infinity. If $\mathcal{Q}$ is bounded, $\mathcal{P}(\mathcal{Q})$ can be expressed as a power series in $\mathcal{Q}$. If $\mathcal{Q}$ becomes non-bounded this expansion keeps its sense at least for some $\mathcal{P}(\mathcal{Q})$ -integrable functions (Theorem). These functions can be explicitly characterized (Additional Remark).

A Poisson measure is a generalization of the Poisson process on the real line to arbitrary locally compact spaces countable at infinity. A Poisson process on a finite interval $I \subset \mathbb{R}$ is given by its jumping points $\tau_1, \dots, \tau_N$ in I , where N is a random number. The probability that $N = n$ is equal to $c^n T^n e^{-cT}/n!$, where T is the length of the interval and c is the parameter describing the Poisson process, i.e. the mean frequency of jumping points. Given that the number N of jumping points is equal to n , the n jumping points are distributed independently and uniformly on the interval I . Be $\mathcal{f}(I)$ the topological sum

$$\mathcal{f}(I) = I^0 \cup I^1 \cup I^2 \cup I^3 \cup \dots$$

where $I^0 = \{e\}$, $I^1 = I$, $I^2 = I \times I$, ..., and e is an arbitrary additional point. Be $f \geq 0$ a function on $\mathcal{f}(I)$, whose components $f_n: I^n \rightarrow \mathbb{R}_+$ are Lebesgue-measurable, then $E f(\tau_1, \dots, \tau_N)$ can be calculated and is equal to

$$E f(\tau_1, \dots, \tau_N) = \sum_{n=0}^{\infty} \text{Prob}\{N=n\} \frac{1}{T^n} \int_{I^n} f_n(t_1, \dots, t_n) dt_1 \dots dt_n$$

or

$$E f(\tau_1, \dots, \tau_N) = e^{-cT} \left(f(e) + \sum_{n=1}^{\infty} \frac{c^n}{n!} \int_{I^n} f_n(t_1, \dots, t_n) dt_1 \dots dt_n \right)$$

This formula can easily be extended to any compact space $\mathcal{X}$ and to any positive measure ϱ on $\mathcal{X}$. Be $f \geq 0$ a function on $f(\mathcal{X})$, with the property that $f_n: \mathcal{X}^n \rightarrow \mathbb{R}_+$ is ϱ^n -measurable, then the application of the Poisson measure $p(\varrho)$ on f is defined by

$$(1) \quad \langle p(\varrho), f \rangle = e^{-\varrho(\mathcal{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\otimes n}, f_n \rangle$$

where $\varrho^{\otimes 0} = \delta_e$, the Dirac measure in e the unique point of $\mathcal{X}_0$.

Now $f(\mathcal{X})$ can be interpreted as the free monoid generated by $\mathcal{X}$ with neutral element e , the product being defined by juxtaposition.

$f(\mathcal{X})$ is locally compact containing $\mathcal{X}$ as a compact open subset. The measure ϱ on $\mathcal{X}$ can be interpreted as a measure on $f(\mathcal{X})$. The product in $f(\mathcal{X})$ induces a convolution for measures. The n -th convolution power $\varrho^{\star n}$ of ϱ is exactly $\varrho^{\otimes n}$ carried by $\mathcal{X}^n \subset f(\mathcal{X})$. So the probability measure $p(\varrho)$ can be written

$$\langle p(\varrho), f \rangle = e^{-\varrho(\mathcal{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\star n}, f \rangle$$

or

$$p(\varrho) = e^{-\varrho(\mathcal{X})} \exp_{\star} \varrho$$

$$(2) \quad p(\varrho) = \exp_{\star} a(\varrho)$$

with

$$(2') \quad a(\varrho) = \varrho - \varrho(\mathcal{X}) \delta_e = \int \varrho(dx) (\delta_x - \delta_e).$$

as δ_e is the unit element in the convolution algebra.

$$\text{As } \varrho^{\otimes n}(dx_1, \dots, dx_n) = \varrho(dx_1) \dots \varrho(dx_n)$$

is symmetric in $x_1, \dots, x_n$ only the symmetric part of f_n gives a contribution to the integral. So we can switch as well to $f_c(\mathcal{X})$, the free commutative monoid generated by $\mathcal{X}$. $p(\varrho)$ can be defined

by the same formula as a measure on $f_c(\mathcal{X})$, formulae (2) and (3) hold as well. We denote by $\mathcal{X}_c^k$ the compact open subspace of $f_c(\mathcal{X})$ formed by the monomials of degree k .

Let $\mathcal{M}(\mathcal{X})$ be the space of all positive measures on $\mathcal{X}$ with the vague topology and let $\mathcal{M}_c(\mathcal{X})$ be the subspace of positive counting measures, i.e. the space of all $\mu \in \mathcal{M}(\mathcal{X})$ of the form

$$\mu = \sum_{j=1}^n \delta_{x_j}$$

$x_j \in \mathcal{X}$, $j=1, \dots, n$ and variable n . Of course $\mathcal{M}_c(\mathcal{X})$ is a submonoid of the additive monoid $\mathcal{M}(\mathcal{X})$. It can be proved that the application

$$(x_1, \dots, x_n) \in f_c(\mathcal{X}) \mapsto \delta_{x_1} + \dots + \delta_{x_n} \in \mathcal{M}_c(\mathcal{X})$$

is a topological isomorphism. So $\rho(\rho)$ can be interpreted, as well, as a measure on $\mathcal{M}_c(\mathcal{X})$ denoted by $\varphi(\rho)$ and $\varphi(\rho)$ is given by

$$(4) \quad \langle \varphi(\rho), f \rangle = e^{-\rho(\mathcal{X})} \left(f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \rho(dx_1) \dots \rho(dx_n) f(\delta_{x_1} + \dots + \delta_{x_n}) \right)$$

$$(5) \quad \varphi(\rho) = \exp_{\star} u(\rho)$$

$$(5') \quad u(\rho) = \int \rho(dx) \left(\delta_{\delta_x} - \delta_0 \right)$$

There 0 is the zero-measure, δ_{δ_x} signifies the Dirac measure on $\mathcal{M}(\mathcal{X})$ in the point $\delta_x \in \mathcal{M}(\mathcal{X})$ and δ_0 the Dirac measure on $\mathcal{M}(\mathcal{X})$ in the point 0 .

As $\mathcal{M}_c(\mathcal{X})$ is a part of the dual of $C(\mathcal{X})$ the space of all continuous real-valued function on $\mathcal{X}$, a Fourier transform for measures on $\mathcal{M}_c(\mathcal{X})$ can be defined. Be $\varphi \in C(\mathcal{X})$, then the Fourier transform of $\varphi(\rho)$ in the point φ is given by the $\varphi(\rho)$ -integral of the function $\mu \in \mathcal{M}_c(\mathcal{X}) \mapsto e^{i \langle \mu, \varphi \rangle}$

So

$$(6) \quad \varphi(\rho)^\wedge(\varphi) = \int \varphi(\rho)(d\mu) e^{i\langle \mu, \varphi \rangle} \\ = \exp \mu(\rho)^\wedge(\varphi)$$

$$(6') \quad \mu(\rho)^\wedge(\varphi) = \int \varphi(dx) (e^{i\varphi(x)} - 1).$$

If $\mathcal{X}$ becomes non-compact and φ a non-bounded measure on $\mathcal{X}$, then formulae (1) - (5) fail, but formula (6) keeps its sense. Consider the space $\mathcal{M}_c(\mathcal{X})$ of all positive counting measures on $\mathcal{X}$, i.e. the space of all measures of the form

$$\sum_{i \in I} \delta_{x_i}$$

where $(x_i)_{i \in I}$ is locally finite: only finitely many of the x_i are contained in a compact subset of $\mathcal{X}$. We assume the vague topology on $\mathcal{M}_c(\mathcal{X})$. Then $\mathcal{M}_c(\mathcal{X})$ can be considered as a part of the dual space of $C_0(\mathcal{X})$, the space of all continuous real-valued functions on $\mathcal{X}$ with compact support. If $\mathcal{X}$ is countable at infinity and φ a positive measure on $\mathcal{X}$, then there exists a unique Radon measure $\varphi(\varphi)$ on $\mathcal{M}_c(\mathcal{X})$ with the Fourier transform (cf. [2], [3])

$$(7) \quad \varphi(\varphi)^\wedge(\varphi) = \exp \int \varphi(dx) (e^{i\varphi(x)} - 1)$$

Further investigation shows that formula (2) may keep its sense as well. This can be seen by writing (2) in a more explicit way

$$\begin{aligned} \langle \varphi(\varphi), f \rangle &= f(e) + \int \varphi(dx) (f(x) - f(e)) \\ &+ \frac{1}{2!} \iint \varphi(dx_1) \varphi(dx_2) (f(x_1, x_2) - f(x_1) - f(x_2) + f(e)) \\ &+ \frac{1}{3!} \iiint \varphi(dx_1) \varphi(dx_2) \varphi(dx_3) (f(x_1, x_2, x_3) - f(x_1, x_2) \\ &\quad - f(x_1, x_3) - f(x_2, x_3) + f(x_1) + f(x_2) + f(x_3) - f(e)) \\ &+ \dots \end{aligned}$$

In fact, the following theorem holds.

Theorem: Assume $\mathcal{X}$ to be a locally compact space countable at infinity and ϱ a positive Radon measure on $\mathcal{X}$. Let f be a function on $\mathcal{M}_c(\mathcal{X})$ with the property: The functions

$$(8) \quad \begin{aligned} g_0(e) &= f(0) \\ g_1(x) &= f(d_x) - f(0) \\ g_2(x_1, x_2) &= f(d_{x_1} + d_{x_2}) - f(d_{x_1}) - f(d_{x_2}) + f(0) \\ &\vdots \\ g_n(x_1, \dots, x_n) &= \sum_{I \subset \{1, 2, \dots, n\}} (-1)^{n-|I|} f\left(\sum_{i \in I} d_{x_i}\right) \\ &\vdots \end{aligned}$$

are $\varrho^{\otimes n}$ -measurable on $\mathcal{X}^n$ and

$$(9) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\otimes n}, |g_n| \rangle < \infty$$

Denote by μ_K the restriction of $\mu \in \mathcal{M}_c(K)$ to a compact subspace $K \subset \mathcal{X}$ and suppose that $f(\mu_K) \rightarrow f(\mu)$ in $\mathcal{Y}(\varrho)$ -measure for $K \uparrow \mathcal{X}$ (that is the case if e.g. f is vaguely continuous). Then f is $\mathcal{Y}(\varrho)$ -integrable and

$$(10) \quad \langle \mathcal{Y}(\varrho), f \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\otimes n}, g_n \rangle$$

In order to understand the theorem let us investigate the connection between f and the function $g_n, n=0, 1, 2, \dots$

One finds

$$\begin{aligned} f(0) &= g_0(e) \\ f(d_x) &= g_0(e) + g_1(x) \\ f(d_{x_1} + d_{x_2}) &= g_0(e) + g_1(x_1) + g_1(x_2) + g_2(x_1, x_2) \\ &\vdots \\ f(d_{x_1} + \dots + d_{x_n}) &= \sum_{k=0}^n \sum_{i_1 < i_2 < \dots < i_k} g_k(x_{i_1}, \dots, x_{i_k}). \end{aligned}$$

Taking into account that the functions $g_k(x_1, \dots, x_k)$ are symmetric in their arguments $x_1, \dots, x_k$ observe

$$\begin{aligned}\sum g_1(x_i) &= \langle \mu, g \rangle \\ \sum_{i < j} g_2(x_i, x_j) &= \frac{1}{2} \iint \mu(d\xi_1) \mu(d\xi_2) g_2(\xi_1, \xi_2) \\ &\quad - \frac{1}{2} \int \mu(d\xi) g(\xi, \xi) \\ \sum_{i < j < k} g_3(x_i, x_j, x_k) &= \frac{1}{6} \iiint \mu(d\xi_1) \mu(d\xi_2) \mu(d\xi_3) g_3(\xi_1, \xi_2, \xi_3) \\ &\quad - \frac{1}{2} \iint \mu(d\xi_1) \mu(d\xi_2) g_3(\xi_1, \xi_1, \xi_2) + \frac{1}{3} \int \mu(d\xi) g_3(\xi, \xi, \xi),\end{aligned}$$

for $\mu = \delta_{x_1} + \dots + \delta_{x_n}$.

This leads to the assumption that any such sum can be expressed by μ .

We begin with a well-known lemma from elementary algebra.

Lemma 1 (Newton). Let $\mathcal{P}[x_1, \dots, x_n]$ be the ring of polynomials in n commutative indeterminates over the rational numbers. Then the symmetric functions

$$\sigma_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} \cdots x_{i_k}$$

can be expressed as polynomials with rational coefficients of the power sums

$$\Delta_k = \sum_{j=1}^n x_j^k$$

These polynomials are independent of the number n of indeterminates and are given by the formal power series

$$1 + \sigma_1 \xi + \sigma_2 \xi^2 + \sigma_3 \xi^3 + \dots = \exp \left[\Delta_1 \xi - \Delta_2 \xi^2/2 + \Delta_3 \xi^3/3 - \dots \right].$$

Proof. We give the proof as it is very short and not very known.

One has $(1 + x_1 \xi)(1 + x_2 \xi) \cdots (1 + x_n \xi) = 1 + \sigma_1 \xi + \sigma_2 \xi^2 + \dots + \sigma_n \xi^n$

and $1 + x_i \xi = \exp \log(1 + x_i \xi).$

So

$$\begin{aligned}
 & 1 + \epsilon_1 \xi + \epsilon_2 \xi^2 + \dots \\
 &= \exp \sum_{i=1}^n \log (1 + x_i \xi) \\
 &= \exp \sum_{i=1}^n \sum_{k=1}^{\infty} (-1)^{k-1} \xi^k x_i^k / k \\
 &= \exp \sum_{k=1}^{\infty} (-1)^{k-1} \xi^k \Delta_k / k.
 \end{aligned}$$

We recall the definition of $f_c(\mathcal{X}) = \sum_{k=0}^{\infty} \mathcal{X}_c^k$ the free commutative monoid generated by $\mathcal{X}$. If $\mathcal{X}$ is locally compact, $f_c(\mathcal{X})$ is locally compact, too. Any measure μ on $\mathcal{X}$ can be considered as a measure on $f_c(\mathcal{X})$. The convolution powers $\mu^{\star n} = \mu^n$ of μ are measures on $\mathcal{X}_c^n$.

Denote the restriction to $\mathcal{X}_c^n$ of a function g on $f_c(\mathcal{X})$ by g_n , then

$$\langle \mu^n, g \rangle = \int \dots \int \mu(dx_1) \dots \mu(dx_n) g_n(x_1, \dots, x_n)$$

Another measure on $f_c(\mathcal{X})$ carried by $\mathcal{X}_c^n$ and related to μ is

$$\Delta_n(\mu) : \langle \Delta_n(\mu), g \rangle = \int \mu(dx) g_n(x, \dots, x).$$

We define now a third measure $\mu^{(n)}$ on $f_c(\mathcal{X})$ carried by $\mathcal{X}_c^n$ by the formal power series

$$1 + \mu^{(1)} \xi + \mu^{(2)} \xi^2 + \dots = \exp_{\star} (\Delta_1(\mu) \xi - \Delta_2(\mu) \xi^2/2 + \Delta_3(\mu) \xi^3/3! + \dots)$$

Lemma 2. If $\mu = \delta_{x_1} + \dots + \delta_{x_n}$ and if g is a function on $\mathcal{X}_c^k$ then

$$\langle \mu^{(k)}, g \rangle = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} g(x_{i_1}, \dots, x_{i_k}).$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X}$. The application $x_i \mapsto \delta_{x_i}$ can be extended to a homomorphism from $\mathcal{P}[x_1, \dots, x_n]$ into the convolution algebra of measures on $f_c(\mathcal{X})$. The image of $x_1 + \dots + x_n$

is μ and the image of $\Delta_k = \sum x_i^{(k)}$ is $\sum (\delta_{x_i})^k = \Delta_k(\mu)$

as

$$\langle \sum (\delta_{x_i})^k, g \rangle = \sum_{i=1}^n g(x_i, \dots, x_i) = \langle \Delta_k(\mu), g \rangle$$

By lemma 1 the image of $\sum_{i_1 < i_2 < \dots < i_k} x_{i_1} \dots x_{i_k}$ is $\mu^{(k)}$. This proves lemma 2.

Lemma 3. If μ is a counting measure, then $\mu^{(k)}$ is a positive measure on $\mathcal{X}$.

Proof. If $g \geq 0$ of compact support, then $\langle \mu^{(k)}, g \rangle = \langle \mu_K^{(k)}, g \rangle$, if K is compact and contains the support of g . As μ_K is a finite counting measure, lemma 2 applies.

An immediate consequence of lemma 2 is

Lemma 4. On the assumptions of the theorem if μ is a finite counting measure

$$f(\mu) = g_0(e) + \langle \mu^{(1)}, g \rangle + \langle \mu^{(2)}, g \rangle + \dots$$

If φ is a bounded measure on $\mathcal{X}$, then $\varphi(\rho)$ can be defined as in (5) and (5'). If $K \subset \mathcal{X}$ is compact and μ a positive measure on $\mathcal{X}$, its restriction to K will be denoted by μ_K . The measure can be considered as a bounded measure on $\mathcal{X}$.

Lemma 5. For any compact $K \subset \mathcal{X}$ the mapping $\mu \mapsto \mu_K$ is $\varphi(\rho)$ -measurable and the image of $\varphi(\rho)$ is equal to $\varphi(\rho_K)$.

Proof. We show at first that the mapping is measurable. Let $\mathcal{U}$ be an open neighborhood of K and let ψ be a continuous function $\mathcal{X} \rightarrow [0, 1]$ with compact support in $\mathcal{U}$ such that $\psi = 1$ on K . Then $\mu \mapsto \mu\psi$ is continuous and $\mu\psi = \mu_K$ if $\mu(\mathcal{U}-K) = 0$. But $\varphi(\rho) \{ \mu : \mu(\mathcal{U}-K) = 0 \} = \exp(-\rho(\mathcal{U}-K))$. So $\mu \mapsto \mu_K$ is continuous on the closed subset of all μ with $\mu(\mathcal{U}-K) = 0$, whose $\varphi(\rho)$ -measure approximates 1 if $\rho(\mathcal{U}-K)$ goes to zero.

The Fourier transform of the image is

$$\begin{aligned} \int \varphi(p)(d\mu) e^{i\langle \mu_K, \varphi \rangle} &= \int \varphi(p)(d\mu) e^{i\langle \mu, \varphi_K \rangle} \\ &= \exp \langle p, e^{i\varphi_K} - 1 \rangle = \exp \langle p_K, e^{i\varphi} - 1 \rangle \\ &= \varphi(p_K)^\wedge(\varphi). \end{aligned}$$

This proves the lemma.

Lemma 6. If g is a p^k -integrable function on $\mathcal{X}_c^k$, then for $\varphi(p)$ -almost every μ the function g is $\mu^{(k)}$ -integrable. The function $\mu \mapsto \langle \mu^{(k)}, g \rangle$ is $\varphi(p)$ -integrable and

$$\int \varphi(p)(d\mu) \langle \mu^{(k)}, g \rangle = \frac{1}{k!} \langle p^k, g \rangle.$$

Proof. Assume a continuous function $\varphi \geq 0$ on $\mathcal{X}_c^k$ whose support is contained in K_c^k where $K \subset \mathcal{X}$ compact. Then $\mu \in \mathcal{M}_c(\mathcal{X}) \mapsto \langle \mu^{(k)}, \varphi \rangle$ is continuous and ≥ 0 ,

$$\begin{aligned} \int \varphi(p)(d\mu) \langle \mu^{(k)}, \varphi \rangle &= \int \varphi(p)(d\mu) \langle \mu_K^{(k)}, \varphi \rangle \\ &= e^{-p(K)} \sum_{n \geq k} \frac{1}{n!} \int \dots \int p(dx_1) \dots p(dx_n) \sum_{i_1 < i_2 < \dots < i_k} \varphi(x_{i_1}, \dots, x_{i_k}) \\ &= \frac{1}{k!} \langle p^{(k)}, \varphi \rangle. \end{aligned}$$

This formula extends to any continuous φ of compact support.

If $\varphi \geq 0$ is lower semi-continuous, there exists a net

$$\varphi_i \in C_0(\mathcal{X}), \varphi_i \uparrow \varphi.$$

So

$$0 \leq \langle \mu^{(k)}, \varphi_i \rangle \uparrow \langle \mu^{(k)}, \varphi \rangle$$

$$\int \varphi(p)(d\mu) \langle \mu^{(k)}, \varphi_i \rangle \uparrow \int \varphi(p)(d\mu) \langle \mu^{(k)}, \varphi \rangle$$

$$\langle p^k, \varphi_i \rangle \uparrow \langle p^k, \varphi \rangle.$$

So $\mu \mapsto \langle \mu^{(k)}, \varphi \rangle$ is lower semi-continuous, its $\mathcal{Y}(\rho)$ -integral is $1/k! \langle \rho^k, \varphi \rangle$ and φ is $\mu^{(k)}$ -integrable $\mathcal{Y}(\rho)$ -a.e. if $\langle \rho^k, \varphi \rangle < \infty$.

Assume now that $\varphi \geq 0$ is a $\mathcal{S}$ -null function. Then there exists a sequence of lower semi-continuous functions $\varphi_n \downarrow \tilde{\varphi} \geq \varphi$ such that $\langle \rho^k, \varphi_n \rangle \downarrow 0$. For $\mathcal{Y}(\rho)$ -almost every μ the functions $\varphi_1, \varphi_2, \dots$ are $\mu^{(k)}$ -integrable and $\langle \mu^{(k)}, \varphi_n \rangle \downarrow \langle \mu^{(k)}, \tilde{\varphi} \rangle$. Therefore $\int \mathcal{Y}(\rho)(d\mu) \langle \mu^{(k)}, \varphi_n \rangle \downarrow \int \mathcal{Y}(\rho)(d\mu) \langle \mu^{(k)}, \tilde{\varphi} \rangle$ and $\tilde{\varphi}$ and φ are $\mu^{(k)}$ -null functions for a.e. μ .

Assume finally $\varphi \in L^1(\rho^k)$. Then there exists a sequence $\varphi_n \in C_0(\mathcal{X})$, $\varphi_n \rightarrow \varphi$ $\mathcal{S}^k$ -a.e. and $|\varphi_n| \leq \psi$ where ψ is lower semi-continuous and integrable. Then φ_n converges to φ $\mu^{(k)}$ -a.e. for almost all μ . As $|\varphi_n| \leq \psi$ and ψ is $\mu^{(k)}$ -integrable a.e., the function φ is $\mu^{(k)}$ -integrable a.e. and $\langle \mu^{(k)}, \varphi_n \rangle \rightarrow \langle \mu^{(k)}, \varphi \rangle$ q.e.. The theorem of Lebesgue yields the end of the proof.

Proposition. Assume a sequence $g_k, k=0,1,2,\dots$ of ρ^k -integrable functions on $\mathcal{X}_c^k$ such that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \langle \rho^k, |g_k| \rangle < \infty.$$

Then the function

$$f(\mu) = \sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

is $\mathcal{Y}(\rho)$ -almost everywhere defined and

$$(11) \quad \int \mathcal{Y}(\rho)(d\mu) f(\mu) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \rho^k, g_k \rangle.$$

Proof. Immediate.

Proof of the theorem. By the assumption of the theorem and the proposition

$$\tilde{f}(\mu) = \sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

is $\mathcal{G}(\rho)$ -integrable and its integral is given by (11). By lemma 4 one has for any compact $K \subset \mathcal{X}$

$$f(\mu_K) = \tilde{f}(\mu_K)$$

If $K \uparrow \mathcal{X}$ which can be done by a sequence as $\mathcal{X}$ is countable at infinity, $f(\mu_K) \rightarrow f(\mu)$ i. m. by assumption and $\tilde{f}(\mu_K) \rightarrow \tilde{f}(\mu)$ $\mathcal{G}(\rho)$ -almost everywhere. Hence $f(\mu) = \tilde{f}(\mu)$ $\mathcal{G}(\rho)$ -a.e.

This proves the theorem.

Additional remark to the theorem. The function $f(\mu)$ is $\mathcal{G}(\rho)$ -a.e. equal to the function

$$\sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

and $f(\mu_K)$ converges to $f(\mu)$ $\mathcal{G}(\rho)$ -almost everywhere.

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