

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

KAI LAI CHUNG

## Note on last exit decomposition

*Séminaire de probabilités (Strasbourg)*, tome 8 (1974), p. 20-21

[http://www.numdam.org/item?id=SPS\\_1974\\_\\_8\\_\\_20\\_0](http://www.numdam.org/item?id=SPS_1974__8__20_0)

© Springer-Verlag, Berlin Heidelberg New York, 1974, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

NOTE ON LAST EXIT DECOMPOSITION

K.L.Chung

This will show how to derive the last exit decomposition in the boundary theory for Markov chains as set forth in [1] and [2]. The lectures given in Strasbourg were cut short toward the end as indicated by the last sentence on p.87 of [2].

One must now begin there by going to pp.161-2 of [1]. Here are the two formulas for the chain starting at a boundary atom  $a_0$  :

$$(I) P\{X(t)=j : t < \beta_{n+1} \mid \left. \begin{matrix} Z_0, \dots, Z_{n-1}, Z_n = a \\ \beta_0, \dots, \beta_{n-1}, \beta_n = s \end{matrix} \right\} = \rho_j^a(t-s), \quad 0 \leq s \leq t ;$$

$$(II) P\{X(t)=j\} = \sum_{n=0}^{\infty} \frac{1}{a_1 \dots a_n} \int_0^t \rho_j^{a_n}(t-s) d[F^{a_0 a_1} * \dots * F^{a_{n-1} a_n}](s) .$$

The  $\beta_n$ 's are the successive boundary switching time and may be assumed to be strictly increasing when all atoms are sticky,  $Z_n$  is the new atom switched to at  $\beta_n$ ,  $\rho^a$  is the entrance law from a and before another switching,  $F^{ab}$  is the distribution of the switching time from a to b ( namely that of the switching time when the chain starts at a, provided that the next switch is to b, see p.53 of [2] ). Formulas (I) and (II) are easy consequences of the moderate Markov property applied at boundary hitting times.

Given any  $t$ , let  $n$  be determined by  $\beta_n \leq t < \beta_{n+1}$ . Let  $H_n^b$  be defined as follows :

$$H_n^b(ds) = P\{\beta_n \in ds ; X(\beta_n+) = b\}$$

( this depends also on the starting point  $a_0$  of the chain, since P itself depends on it ). Then (II) may be rewritten as

$$P\{X(t)=j\} = \sum_n \int_0^t H_n^b(ds) \rho_j^b(t-s) .$$

Now use (1) on p.62 of [2] :

$$(III) \quad \rho_j^b(t) = \int_0^t \eta_j^b(t-s) E^b(ds)$$

to get the preceding probability to be

$$\sum_n \sum_b \int_0^t (H_n^b * E^b)(ds) \eta_j^b(t-s) .$$

This is it. What we did above is simply this : suppose the  $n$ -th switch is to b, then consider the last exit time from b before the next

switch. The total time up to then has the distribution  $\sum_n (H_n^b * E^b)$ . But the post-last-exit-from b process has the entrance law  $\eta^b$  regardless of what has happened before, as indicated on pp. 75-6 of [1]. One can also see this in two steps : first up to the switching time to b, then between this and the last exit time from b, using (III) above.

This is the analogue of the strong Markov property for a last exit time, proved in a general setting by Pittenger-Shih . It is not even necessary for the analytic derivation in the above. Symbolically, all we do is to lump together the first three factors in the complete decomposition formula ( p.72 of [2] )

$$L_i(\lambda)[I-F(\lambda)]^{-1}E(\lambda)\eta_j(\lambda)$$

to get the last exit decomposition, just as we get the first entrance decomposition by lumping together the last three factors. The Laplace transforms are used merely for convenience and do not smear over anything, because there are explicit formulas like (I) and (II). The latter can be used to avoid the inversion of the matrix  $I-F(\lambda)$  above, as shown by Lamb (Zeitschr. fur Wahrsch. vol.19 (213-224), 1971 ).

- [1] Boundary theory for Markov chains II. Acta Math. vol.115, pp. 111-163 (1966).  
 [2] Lectures on Boundary Theory for Markov chains. Annals of Math. Studies 65, Princeton University Press, 1970.